Geometrization of 3-manifolds with Heegaard genus two

Vivien Easson

Junior Geometry Seminar, Oxford

13th November 2002

Contents

1 Introduction

Our aim today[1] is to take a particular class of closed 3-dimensional manifolds, explain why they all have a decomposition into geometric pieces, and give some conditions which help determine what the geometries of these pieces are.

The existence of the decomposition follows from a result of William Thurston known as the Symmetry Theorem. To apply this, we have to show the existence of an involution - a 2-fold symmetry - on the manifolds in question.

We are dealing with three main issues: topology, geometry and symmetry. These correspond to three theorems or statements that we will discuss: the Dehn-Lickorish representation of surface homeomorphisms, the 3-manifold geometrization conjecture, and Thurston's Symmetry Theorem. We will use these three to deduce our main result, which is

Theorem 1 (Genus two 3-manifolds are geometrizable)
Every closed, orientable manifold with a Heegaard splitting of genus two (or less) has a geometric decomposition.

The three results we will use are:

Theorem (Dehn-Lickorish, attempted proof 1910, proved 1962)
Any homeomorphism $h : F \to F$ of a compact, orientable surface F, which is the identity on ∂F or orientation-preserving if $\partial F = \emptyset$, is isotopic to a composition of Dehn twists.

Conjecture (Thurston, 1976)
Does every compact orientable 3-manifold have a geometric decomposition?

Theorem (Thurston, 1982)
Suppose that M is a closed, orientable, irreducible 3-manifold which admits an action by a finite group G of orientation-preserving diffeomorphisms such that some non-trivial element has a fixed point set of dimension one. Then M admits a geometric decomposition preserved by the group action.

[1] These notes were originally put together for a postgraduate seminar at Oxford University in November 2002. This version dates from May 2004.

2 *Topology*: Heegaard genus

2.1 Basic definitions

A manifold is said to be *closed* if it is compact and has empty boundary, and *orientable* if it has an atlas of charts whose transition maps are orientation-preserving. We will always consider our manifolds to be compact and orientable, and normally they will also be closed.

Every compact 3-manifold possesses a unique smooth structure and a unique piecewise linear structure, so that any two 3-manifolds are homeomorphic if and only if they are diffeomorphic; such a manifold may also be finitely triangulated by tetrahedra. A thickened neighbourhood of the 1-skeleton of a triangulation and its complement form a decomposition of the manifold into two solid thickened graphs, separated by a surface.

This shows that every closed orientable 3-manifold has a *Heegaard splitting* of some finite genus g; namely that there exists a closed, orientable surface of genus g which separates the manifold into two solid handlebodies, each with g holes. For a given manifold M, the minimum value of g is known as the *Heegaard genus* of M.

The value of the Heegaard genus is unknown for most 3-manifolds M. There is a complete classification of those of genus 0 or 1, and we will show that genus 2 manifolds must behave well, but after that not much is known. The classifications for low genus are as follows.

Proposition 2 (Genus zero splittings)
If a 3-manifold M has a splitting of genus zero, it is homeomorphic to S^3.

For, if we glue together two solid balls by an orientation-reversing homeomorphism along their boundary, the gluing map is isotopic to the identity; this gives a homeomorphism to S^3. Recall that an isotopy is a homotopy for which every section of its image is a homeomorphism.

Proposition 3 (Genus one splittings)
If a 3-manifold M has a splitting of genus one, it is either homeomorphic to a lens space or to $S^1 \times S^2$.

A lens space $L_{p,q}$ is a 3-manifold formed as a quotient of S^3 (considered as the unit sphere of $\mathbb{C}^2$) by the cyclic group $\mathbb{Z}_p$ of isometries generated by $(z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi iq/p} z_2)$ for coprime integers p and q. The splitting of genus one has

Heegaard surface $|z_1|^2 = |z_2|^2 = \frac{1}{2}$ and divides $L_{p,q}$ into two solid tori given by $|z_1|^2 \leq \frac{1}{2}$ and $|z_1|^2 \geq \frac{1}{2}$.

The genus one splitting of $S^1 \times S^2$ is even easier to see: take two solid tori and glue them by the identity map along their boundaries (reversing the orientation of one of the handlebodies). Analysis of all possible torus gluing maps shows that the manifolds listed are the only ones with genus one splittings.

A genus two splitting, for example, is defined by the images of the boundaries of a complete set of meridian discs under the gluing map, as shown.

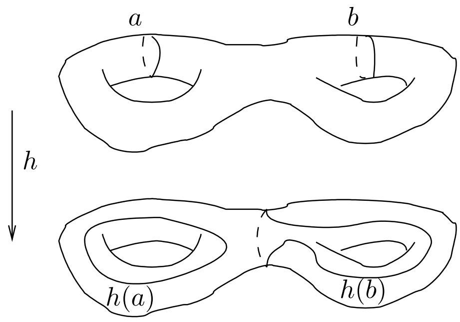

Figure 1: Defining a genus two splitting

More background concerning 3-manifolds and Heegaard splittings may be found in books of Hempel [7], Stillwell [13] and Thurston [14], and also in an excellent survey paper of Martin Scharlemann [10].

2.2 The Weierstrass involution τ and Dehn twists

Let $\mathcal{F}_2$ be the closed, orientable surface of genus 2, standardly embedded in $\mathbb{R}^3$. Then, taking an axis of rotation as shown, the Weierstrass involution τ is represented by a rotation of angle π about this axis. Thus τ is an isometry of order two.

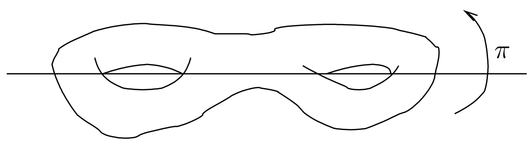

Figure 2: The Weierstrass involution τ

Note that this involution may be defined for a surface of any genus. However, it can only be extended to an arbitrary Heegaard splitting if the genus is at most two.

We also need to define a Dehn twist about a simple closed curve.

Definition 4
Let $t : \mathcal{F}_g \to \mathcal{F}_g$ be a homeomorphism from a surface $\mathcal{F}_g$ to itself. Then we say that t is a Dehn twist about a if t is isotopic to a homeomorphism t' such that there exists a simple non-separating closed curve a and an annular neighbourhood $N_\epsilon(a)$ of a with t' the identity on $\mathcal{F}_g \setminus N_\epsilon(a)$ and t' acting as a twist of 2π of one component of $\partial N_\epsilon(a)$ relative to the other.

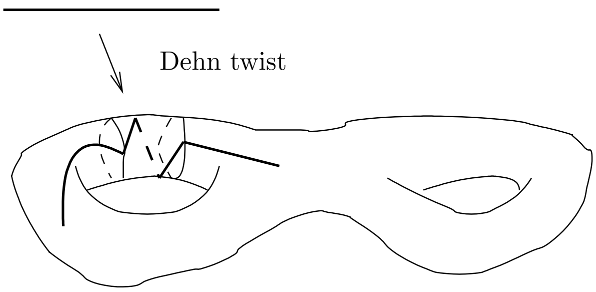

Figure 3: Dehn twist about a simple closed curve

2.3 Extending the involution to genus 2 manifolds

It is easy to see that τ extends to a homeomorphism of a handlebody which is contained as the bounded component of the complement of the standardly embedded surface $\mathcal{F}_2$ in $\mathbb{R}^3$. What is hard is to show that τ extends across the other handlebody, glued by some arbitrary homeomorphism h of $\mathcal{F}_2$. To show this, we need the following result.

Proposition 5 (up to isotopy, τ commutes with h)
Any diffeomorphism $h : \mathcal{F}_2 \to \mathcal{F}_2$ is isotopic to $h' : \mathcal{F}_2 \to \mathcal{F}_2$ such that $\tau h' = h'\tau$. Equivalently, the hyperelliptic involution τ is central in the mapping class group of $\mathcal{F}_2$.

Corollary 6 (τ extends to both handlebodies)
Let $M = H_1 \cup_{\mathcal{F}_2} H_2$ be a manifold represented by a Heegaard splitting of genus 2 and with gluing map h. Then $\tau : \mathcal{F}_2 \to \mathcal{F}_2$ extends to an involution τ_M of M.

Proof. Changing the gluing homeomorphism of a splitting to an isotopic homeomorphism does not alter the manifold. Thus, using the proposition, we may assume that h commutes with τ. We can extend τ to either handlebody H_i by considering the standard embedding of H_i in $\mathbb{R}^3$, symmetrically placed about the axis of τ, a rotation of π. These extensions agree on the Heegaard surface and hence comprise τ_M. For, if x is a point on ∂H_1, and y is its image under h on ∂H_2, we have

$$\tau_{H_2}(y) = \tau_M(h(x)) = (\tau \circ h)(x) = (h \circ \tau)(x) = h(\tau_{H_1}(x))$$

Thus τ extends to an involution τ_M on all of M. □

The proposition may be proven as follows. First we show that τ takes every simple closed curve on $\mathcal{F}_2$ to an isotopic curve (possibly reversing orientation). This shows that τ commutes with Dehn twists (since orientation is irrelevant when defining these). We then apply the Dehn-Lickorish theorem to complete the proof.

Proof of Proposition 5. Fix a hyperbolic structure on $\mathcal{F}_2$, and a Fuchsian group $G \leq PSL(2, \mathbb{R})$ corresponding to it. We sketch a proof that τ sends every simple closed geodesic of $\mathcal{F}_2$ to itself, possibly reversing orientation. Full details may be found in a paper of Hass and Susskind [6].

Let γ be an oriented simple closed curve on $\mathcal{F}_2$, separating the surface into two surfaces of genus one with a disc removed, say T_1 and T_2. Lift to the universal cover $\mathbb{H}^2$, choosing a connected component $\tilde{\gamma}$ of the preimage of γ and similarly the lifts $\tilde{T_1}$ and $\tilde{T_2}$ adjacent to $\tilde{\gamma}$.

Let g be a generator for the stabilizer of $\tilde{\gamma}$ in G. It can be shown that there is a transformation h with $h^2 = g$, which projects to a conformal involution on each of T_1 and T_2 separately, and hence on $\mathcal{F}_2$. For, if we cut T_i open along a simple closed curve α disjoint from γ, there exists an involution of T_i swapping the copies of α and preserving γ: its preimage in $\mathbb{H}^2$ is the Möbius transformation of order two with fixed point the midpoint of the common perpendicular to chosen lifts of the images of α.

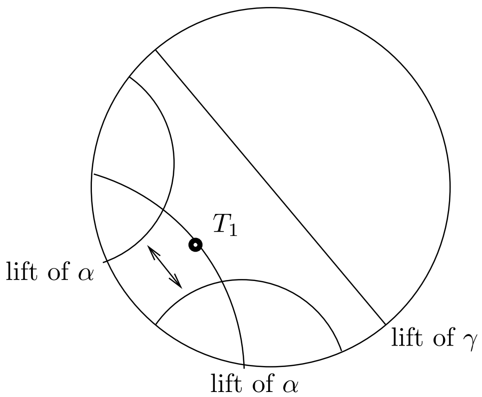

Figure 4: The elliptic Möbius transformation of order two

Moreover, this involution has three fixed points on each of T_1 and T_2, and so is the *unique* hyperelliptic involution on $\mathcal{F}_2$, i.e. the order two conformal automorphism fixing 6 points. The involution preserves α and γ by construction.

In particular, since the involution constructed is unique, it is independent of our choice of dividing curve γ, and so preserves every such dividing geodesic curve. In addition, if α is any non-separating simple closed curve, we can choose a dividing curve γ' disjoint from it, and apply the construction to show that the involution also preserves α.

Since every isotopy class contains a geodesic, τ sends every simple closed curve to an isotopic curve. Thus τ commutes with all Dehn twists. To complete the proof of the proposition, we now turn to the first of our theorems: the Dehn-Lickorish theorem.

2.4 The Dehn-Lickorish Theorem

Theorem 7 (Dehn-Lickorish, attempted proof 1910, proved 1962)) *Any homeomorphism $h : F \to F$ of a compact, orientable surface F, which is the identity on ∂F or orientation-preserving if $\partial F = \emptyset$, is isotopic to a composition of Dehn twists.*

Proof. Write t_a to denote the Dehn twist about a simple closed curve a on F. Note that t_a^{-1} is also a Dehn twist.

Suppose that α and β are non-separating simple closed curves on F. Consider $\alpha \cap \beta$. If this contains a single point, the composition $t_a \circ t_b$ sends α (via $\alpha\beta$) to β. If it is empty, we can find a third non-separating simple closed curve γ such that γ intersects each of α and β in a single point, ensuring that there exists a composition of Dehn twists (and homeomorphisms isotopic to the identity) under which α is sent to β (via γ).

If $|\alpha \cap \beta| \geq 2$, we can ensure it is a finite integer by a small isotopy (putting it into general position) and then use induction on this integer n by finding a curve γ which intersects α at most once and β at most $n-1$ times. Thus we again find a composition of Dehn twists and identity-isotopic homeomorphisms under which α is sent to β.

Now consider a complete set of meridian discs for F, say $m_1, \ldots, m_g$. Since m_1 and $h(m_1)$ are non-separating simple closed curves on F, the above argument shows that there exists a homeomorphism $f_1 : F \to F$ taking $h(m_1)$ to m_1, such that f_1 is a composition of Dehn twists and homeomorphisms isotopic to the identity.

We may assume that the orientations of $f_1(h(m_1))$ and m_1 coincide, for if not then we may take a curve β intersecting m_1 in exactly one point, so that $(t_\beta \circ t_{m_1} \circ t_\beta)^2$ takes m_1 to m_1^{-1}, and modify f_1 by this composition of Dehn twists.

We may also assume that $f_1 \circ h$ is the identity on m_1, since there exists some map isotopic to f_1 with this property. Cut F along m_1 to get some new bounded surface F_1. Then, since the orientations of $f_1(h(m_1))$ and m_1 coincide, $f_1 \circ h$ is the identity on ∂F_1.

Repeating for all the meridian discs, we obtain a homeomorphism $\phi = f_g \circ \ldots \circ f_1 \circ h$ of the disc F_g with g holes, acting as the identity on ∂F_g. But the group of homeomorphisms of F_g is isomorphic to the group of isotopy classes of thickened braids, hence generated by the generators of the pure braid group and a primitive twist around each braid, as shown.

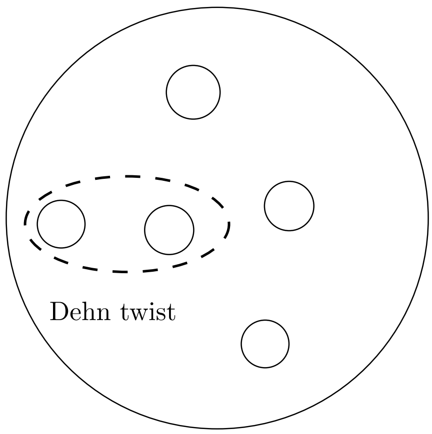

Figure 5: A generator of the pure braid group

Each of these generators corresponds to a Dehn twist about some simple closed curve in F_g. Therefore ϕ is a composition of Dehn twists and homeomorphisms isotopic to the identity. Since each f_i is also of this form, so is our original homeomorphism h. $\square$

We may now conclude the proof of Proposition 5. Take an arbitrary diffeomorphism $h : F_2 \to F_2$. Since τ commutes with every Dehn twist, and by the Dehn-Lickorish Theorem h is isotopic to a composition of Dehn twists h', we have $\tau h' = h' \tau$ as required. $\square$

3 *Geometry*: geometrization of 3-manifolds

3.1 Geometric decomposition

Let M be a compact orientable 3-manifold. An essential 2-sphere in M is an embedded copy of S^2 which does not bound a ball D^3 in M. Kneser showed (1929) that M has a unique prime decomposition into manifolds with no essential 2-spheres and copies of $S^1 \times S^2$. This prime decomposition is obtained by cutting along a maximal collection of disjoint embedded 2-spheres.

Manifolds with no essential 2-spheres are said to be *irreducible*; $S^1 \times S^2$ is the only reducible prime 3-manifold. If M has a prime decomposition into manifolds $M_1, \ldots, M_n$, we say that M is the *connected sum* of these manifolds and write $M = M_1 \# M_2 \# \ldots \# M_n$.

In the late 1970s, Jaco-Shalen and Johannson published details of a similar canonical decomposition along essential tori (and annuli), known as the JSJ-decomposition. Background material describing this decomposition may be found in a survey paper of Neumann and Swarup [9].

If we apply both of these decompositions to M we will get a number of pieces, each of which will be irreducible and atoroidal - plus perhaps some pieces of the form $S^1 \times S^2$. We say that M has a geometric decomposition if we can put a complete geometric structure on each of these pieces.

3.2 The eight three-dimensional geometries

In two dimensions there are three possible geometries: simply connected manifolds from which we may create a compact quotient surface corresponding to a discrete subgroup of the isometry group. These are the constant curvature geometries $S^2, \mathbb{E}^2$ and $\mathbb{H}^2$. By the complex analysis Uniformization Theorem of the nineteenth century, every complete geometric structure is formed as a discrete quotient of one of these constant curvature manifolds.

In three dimensions it makes sense to widen the notion of geometry to include simply connected 3-manifolds which have a homogeneous but not necessarily isotropic Riemannian metric. That is, the metric looks the same at every point but not necessarily the same in every direction from that point.

Thurston showed that there are precisely eight such model geometries in three dimensions which may be exhibited by a compact 3-manifold.

Of these, three geometries are constant curvature: $S^3, \mathbb{E}^3$ and $\mathbb{H}^3$. Two are direct products: $S^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^1$. The remaining three are twisted products, known as nilgeometry, $\widetilde{SL_2(\mathbb{R})}$ and solvegeometry. Peter Scott's 1983 paper [11] gives further details about the properties and behaviour of these three-dimensional geometries.

It turns out that six of these geometries only occur in 3-manifolds known as Seifert fibre spaces, classified by Herbert Seifert in 1933 [12]. These manifolds are mostly well understood: for our purposes they can be thought of as circle bundles over an orbifold, a surface with finitely many rational singular points.

The remaining geometries are solvegeometry and hyperbolic geometry. The former is modelled only by 3-manifolds possessing a foliation by tori, which is again quite a restrictive condition. Thus the latter is the most common geometry: most geometric 3-manifolds are hyperbolic.

3.3 Thurston's geometrization conjecture

From this conclusion William Thurston made a further conjecture: that every 3-manifold which could have a complete hyperbolic structure, should have one. In particular, every irreducible atoroidal 3-manifold with infinite fundamental group should be a discrete compact quotient of $\mathbb{H}^3$. In terms of a geometric decomposition as described above, his conjecture reduces to the following.

Conjecture 8 (Thurston's Geometrization Conjecture)
Every compact, orientable 3-manifold has a geometric decomposition.

This is known to hold for various classes of 3-manifolds, the most important being those with an incompressible surface (in which every loop bounding a disc in the ambient manifold bounds one in the surface), known as Haken manifolds. This approach proceeds by a delicate induction on a hierarchy obtained by repeatedly splitting a Haken manifold along an incompressible surface until a collection of disjoint 3-balls is obtained. A summary of the proof by John Morgan may be found in [8].

The conjecture naturally splits up into three cases: where the fundamental group is finite, where it is infinite and contains a $\mathbb{Z} \times \mathbb{Z}$ subgroup, and where it is infinite but does not contain such a subgroup.

The first case subsumes the Poincaré conjecture: any closed, simply-connected 3-manifold is homeomorphic to S^3; it is still open. The second case has been resolved in the last few years: any such manifold either contains an essential torus (and hence splits further) or is a Seifert fibre space (and hence possesses a SFS geometry). The third case is still open: it needs to be shown that all such manifolds are hyperbolic.

4 *Symmetry*: using the Orbifold Theorem

4.1 Thurston's Symmetry Theorem

Let us see if we can relate Heegaard splittings, the Weierstrass involution and geometrization. The Heegaard surface is certainly not incompressible, and we cannot use the techniques used for Haken manifolds to obtain a proof of geometrization by induction. It is the existence of the involution that will guide us to an approach.

The key is the third of our theorems, the Symmetry Theorem. This is actually a special case of a much stronger and important theorem known as the Orbifold Theorem, claimed by Thurston in 1981 and versions of which have been given proofs only recently. The major work has been done by two separate groups: see either [4] for the Cooper-Hodgson-Kerckhoff approach or [1], [2] and [3] for the work of Boileau-Leeb-Porti. The proofs are long, technical and in any case well beyond the scope of our discussion.

We restate the particular case we are interested in.

Theorem 9 (Thurston's Symmetry Theorem)
Suppose that M is a closed, orientable, irreducible 3-manifold which admits an action by a finite group G of orientation-preserving diffeomorphisms such that some non-trivial element has a fixed point set of dimension one. Then M admits a geometric decomposition preserved by the group action.

4.2 Application to irreducible genus two manifolds

We now have all the results needed to apply this theorem. We showed that the Weierstrass half-revolution involution τ of a genus two surface extends to an involution of any manifold M with a Heegaard splitting of genus two. The involution generates a group of finite order two, and is a non-trivial element of that group.

Its fixed point set consists of three arcs in each handlebody, linking up (after isotopy) to give three linked circles in the manifold. This is one-dimensional and so M satisfies the conditions of the group action. We know that M is closed and orientable; if it is also irreducible, we may apply the Symmetry Theorem to deduce that it has a geometric decomposition.

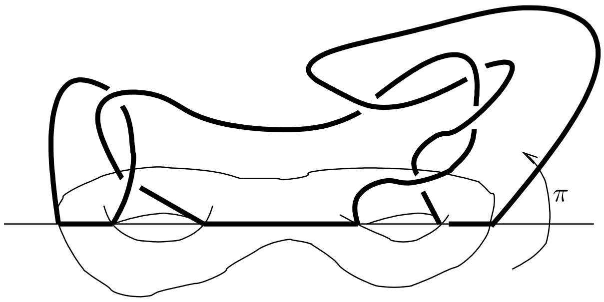

Figure 6: Sample fixed point set of τ

Therefore every irreducible 3-manifold with Heegaard genus two is geometrizable in the sense of Thurston's geometrization conjecture.

4.3 Genus two manifolds: the reducible case

We may also deal directly with the easier case of reducible genus two manifolds, as follows. A reducible manifold is either $S^1 \times S^2$ or it contains an essential 2-sphere S. Since $S^1 \times S^2$ has geometry modelled on $S^2 \times \mathbb{E}^1$, we restrict our attention to the latter case.

Let M be a reducible manifold of Heegaard genus two. A result of Haken [5] shows that there is a choice of essential 2-sphere S so that it intersects the Heegaard surface in a single circle and each handlebody in an essential disc. If M has a separating 2-sphere then S can also be chosen to be separating.

If this sphere S separates M into two disjoint pieces, it must then also separate each handlebody into two solid tori. Thus M is a connected sum of two manifolds as shown, each with a Heegaard splitting of genus one.

If M contains no separating essential 2-sphere, it must contain a non-separating one. In particular it must have a $S^1 \times S^2$ summand in its prime decomposition, which we remove. Then $M = M_1 \# (S^1 \times S^2)$ for some manifold M_1. But the sphere may be assumed to bisect a 1-handle in each handlebody of the genus two splitting for M, and thus M_1 has a genus one Heegaard splitting.

In particular, any reducible manifold M with a Heegaard splitting of genus two is expressible as the connected sum of two manifolds with Heegaard splittings of genus one. These are all geometric: $S^1 \times S^2$ is modelled on $S^2 \times \mathbb{E}^1$, and all lens spaces are modelled on S^3. Thus M has a geometric decomposition.

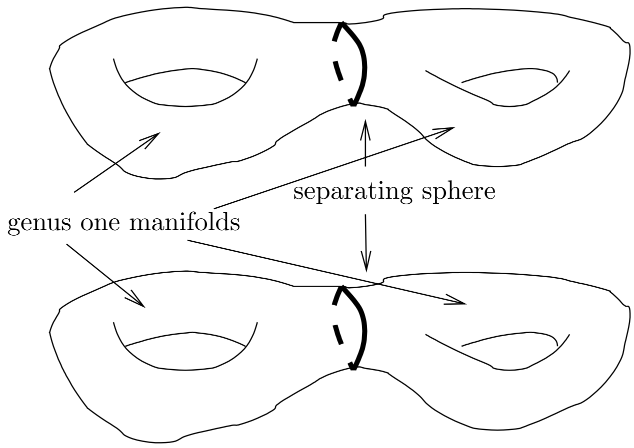

Figure 7: Obtaining splittings of genus one

Putting this with the results of the previous section, we deduce Theorem 1: every closed, orientable 3-manifold with a Heegaard splitting of genus two has a geometric decomposition.

References

[1] Michael Boileau, Bernhard Leeb, and Joan Porti, *Geometrization of 3–dimensional orbifolds, Part I*, Preprint (2000), 1–34, arXiv:math.GT/0010185.

[2] ______, *Geometrization of 3–dimensional orbifolds, Part II*, Preprint (2000), 1–50, arXiv:math.GT/0010184.

[3] Michael Boileau and Joan Porti, *Geometrization of 3–orbifolds of cyclic type*, Astérisque **272** (2001), 1–208.

[4] Daryl Cooper, Craig D. Hodgson, and Steven P. Kerckhoff, *Three-dimensional orbifolds and cone manifolds*, MSJ Memoirs **5** (2000).

[5] Wolfgang Haken, Some results on surfaces in 3–manifolds, 34–98, Prentice-Hall, Englewood Cliffs, New Jersey, 1968, pp. 34–98.

[6] Andrew Hass and Perry Susskind, *The geometry of the hyperelliptic involution in genus two*, Proc. Amer. Math. Soc. **105** (1989), 159–165.

[7] John Hempel, *3–manifolds*, Ann. of Math. Stud. 86, Princeton University Press, 1976.

[8] John W. Morgan and Hyman Bass (eds.), *The Smith Conjecture*, Pure and Applied Mathematics Series, Academic Press Inc., London, 1984.

[9] Walter D. Neumann and Gadde A. Swarup, *Canonical decompositions of 3–manifolds*, Geom. Topol. **1** (1997), 21–40.

[10] Martin Scharlemann, *Heegaard splittings of compact 3–manifolds*, Handbook of Geometric Topology (Robert J. Daverman and Richard B. Sher, eds.), North Holland, Amsterdam, 2002, arXiv:math.GT/0007144, pp. 921–953.

[11] Peter Scott, *The geometries of 3–manifolds*, Bull. London Math. Soc. **15** (1983), 401–487.

[12] Herbert Seifert, *Topologie dreidimensionaler gefasterter Räume*, Acta. Math. **60** (1933), 147–238.

[13] John Stillwell, *Classical topology and combinatorial group theory*, 2nd ed., Graduate Texts in Mathematics, 72, Springer-Verlag, New York, 1993.

[14] William P. Thurston, *Three-dimensional geometry and topology, volume i*, Princeton University Press, edited by Silvio Levy, 1997.