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3-MANIFOLDS WITH HEEGAARD GENUS AT MOST TWO

By

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Chapter 1

Introduction

An n-dimensional manifold is a T_2-separable metric topological space $\mathcal{M}$ such that every point $x \in \mathcal{M}$ has a neighbourhood $U_x \subseteq \mathcal{M}$ homeomorphic to the n-dimensional Euclidean space E^n. The homeomorphism problem for compact, connected n-manifolds is due to Henri Poincaré [30] and it belongs among the most studied problems since the beginnings of 20-th century.

For manifolds of dimensions one and two is the classification well-known :

> There is (up to homeomorphism) a *unique* compact connected 1-dimensional manifold - a circle given by the set of solutions of the equation $x^2 + y^2 = 1, [x, y] \in E^2$.
>
> There are (up to homeomorphism) *two* infinite classes of compact connected 2-dimensional manifolds (surfaces). Every 2-manifold is uniquely determined by its orientability and Euler characteristics.

The Classification theorem for dimension two is included in basic courses of algebraic topology [18, 23] or discrete mathematics. Its proof is based on "cut-and-glue" operations and the result is nicely expressed by the well-known Euler-Poincaré equation:

$$v - e + f = \chi, \text{ where } \chi = \begin{cases} 2 - 2g, & S \text{ is orientable,} \\ 2 - \hat{g}, & S \text{ is non-orientable.} \end{cases}$$

where v, e, f are the respective numbers of vertices, edges and faces of a cellular decomposition of a surface S with Euler characteristic χ. The number g ($\hat{g}$) is called the genus (non-orientable genus) of S. It follows that the Euler characteristic can be obtained from any cellular decomposition of S in an easy way. To determine the orientability of S it is sufficient to decide whether the dual of the barycentric subdivision of a 2-cell decomposition of S is bipartite or not.

An extension of the classification to higher dimensions was the problem, which has naturally arisen in the beginnings of modern topology. Finally, it transpired that for dimensions n higher than three the classification of n-manifolds by means of finite number of invariants and finite number of identities relating them cannot be achieved due to a theorem of Markov [23, p. 1]. The classification problem for the dimension $n = 3$ remains open till nowadays.

In what follows we shall restrict our considerations to orientable compact connected 3-manifolds exclusively. In further explanation we often omit the adjectives compact connected orientable. A particular, but fundamental instance of classification problem was formulated by Henri Poincaré in 1904 [30] in the following form:

> ”Is there a 3-manifold with trivial homology group, which is non-homeomorphic to the 3-dimensional sphere?”

Early after the formulation of the previous problem Poincaré himself found some (counter-) examples. One of the Poincaré examples, namely the Poincaré dodecahedral space, will appear in our analysis as well (for details see Section 4.7.1). Another instance of a non-trivial ”homology sphere” appears in the celebrated example of Montesinos [4] showing that a 3-manifold can be defined as a branched cover over S^3 in two essentially different ways, i.e. the respective knots differ essentially. The existence of counter-examples led Poincaré to transform the previous problem to the following form:

> ”Is every simply-connected 3-manifold (with trivial fundamental group), homeomorphic to the 3-dimensional sphere?”

This problem was attacked during all the 20-th century using a lot of different methods. Studies around the Poincaré conjecture contributed to many branches of mathematics such as the theory of knots and Riemannian manifolds. Note that the Poincaré problem belongs among so-called ”Millennium problems” claimed by Clay Mathematical Institute.

The most promising approach to the classification problem is related with a more general question:

> How much the structure of a 3-manifold is determined by its fundamental group?

Generally, the structure of a 3-manifold cannot be deduced from its fundamental group. Lens spaces $\mathcal{L}(p,q)$, p a prime and $1 \leq q \leq \left[\frac{p}{2}\right]$, form a family of mutually distinct 3-manifolds with the cyclic fundamental group $\mathbb{Z}_p$ of order p. The idea to attack the classification can be explained in two steps:

1. Find a proper family of (binary) operations such that each 3-manifold has with respect to the chosen operations a unique decomposition into decomposable 3-manifolds.

2. Assuming that the Poincaré problem is solved proving that the only simply-connected 3-manifold is S^3, we represent an indecomposable 3-manifold $\mathcal{M}$ as a quotient of either S^3 or of E^3 by its fundamental group $\pi_1(\mathcal{M})$.

As concerns part one, it turned out* that the connected sum and the Johanson-Jaco-Shalen torus decomposition is a complete set of useful operations. A monograph by Hempel [19] is an useful source of information about fundamental groups of 3-manifolds and their connected sums. A recent work of Grigori Perelman gives us a hope that the assertion of Poincaré can be proved in the affirmative as a particular instance of the celebrated Thurston's Geometrisation Conjecture. Assuming the Poincaré Conjecture is true, the classification of prime 3-manifolds transfers to a problem to describe different free actions of certain groups acting on S^3 or E^3, respectively. A 3-manifold is then expressed as a quotient of the universal cover S^3 or E^3 by he fundamental group. The appropriate groups and actions on S^3 (the elliptic case), were classified by Milnor [26]. An effective way to describe the actions of fundamental groups on E^3 consists in endowing E^3 with an appropriate geometry and interpreting the fundamental group as a group of isometries with respect to the chosen geometry. Thurston conjectured that altogether it is sufficient to use 8 geometries, seven of them associated with E^3 and the spherical geometry attached to S^3. Perelman's attempt to prove the Poincaré Conjecture, as a first step to confirm the Thuston's Geometrisation Conjecture, is under detailed discussion of hundreds of mathematicians around the world and it looks to be complete for this time†. The homeomorphism problem was studied in thousands of papers and books written by hundreds

*it took a long time

†May 2005

of authors, so it is impossible to complete the survey on the problem including all the aspects. In the remaining part of the introduction we focus on one particular technique related to the results proved in this thesis. The results coming from the classical approach to the classification problem used in this thesis are nicely surveyed in the seminar talk of Vivien Easson [9] (see Appendix). More information on geometrisation of 3-manifolds can be found in Thurston's book [33].

Our method is based on a work of Mario Pezzana, who developed the theory of crystallisations [29] in early 70-th of the 20-th century. The concept of Pezzana's theory extends to any dimension. One of the advantages of the Pezzana's approach is a possibility to generalise some invariants defined for 3-manifolds (such as Heegaard genus, for instance) to manifolds of any dimension [2]. A crystallisation is a certain combinatorial representation of a given 3-manifold. Since every orientable 3-manifold $\mathcal{M}$ admits a triangulation [28], $\mathcal{M}$ can be replaced by its underlying simplicial complex $S(\mathcal{M})$. Take the first barycentric subdivision of $S(\mathcal{M})$ and construct its dual graph $\Gamma(\mathcal{M})$. Since each tetrahedron of $S(\mathcal{M})$ has four triangles, $\Gamma(\mathcal{M})$ is a 4-valent graph. It is well-known that $\mathcal{M}$ is orientable if and only if $\Gamma(\mathcal{M})$ is bipartite. Moreover, there is an induced edge-colouring of $\Gamma(\mathcal{M})$ by four colours. A converse statement was investigated by Pezzana who proved [29] that a bipartite 4-valent 4-edge-coloured graph Γ represents an orientable 3-manifold if and only if the subgraph H of Γ induced by a subset of edges coloured by any triple of colours is planar. If H is planar and connected for any triple of colours, Γ is called a crystallisation. Since a 3-manifold is fully described by its crystallisation, the homeomorphism problem transfers to a problem on crystallisations (see Chapter 3 for details). This way the dimension three is reduced to dimension one and topological objects are replaced by combinatorial ones. Unfortunately, the same 3-manifold can be represented by infinitely many crystallisations. As we shall see later, the combinatorial formulation of the homeomorphism problem remains non-trivial. Ferri and Gagliardi proved [12] that the homeomorphism relation between 3-manifolds is in the Pezzana's theory reflected by the dipole-move equivalence on bipartite 4-valent 4-edge-coloured graphs.

Dipole-moves are roughly said 3-dimensional equivalents of "cut-and-glue" operations well-known from the dimension two. Using a sequence of dipole-moves one can transform a crystallisation $\Gamma = \Gamma(\mathcal{M})$ representing a manifold $\mathcal{M}$ to another one, say Γ'. Ferri and Gagliardi's theorem states that one can construct a finite sequence of dipole-moves transforming Γ onto Γ' if and only if the represented 3-manifolds are homeomorphic. In spite of generality of dipole-moves, these operations cannot be used for classification of 3-manifolds, in a straight way, due to the complexity of the dipole-move relation. The following decision problem arises:

> Decide whether two bipartite 4-valent 4-edge-coloured connected graphs Γ, Γ' are, or are not, dipole-move equivalent.

Since no bound on the number of dipole-moves transforming Γ onto Γ' in terms of (Γ, Γ') is known it is not clear whether the above problem is an algorithmically solvable problem.

Every 3-manifold yields a "canonical form" of a crystallisation which can be described by an ordered $2(g+1)$-tuple of integers, called an admissible $2(g+1)$-tuple. The integer g bounds the Heegaard genus of the represented 3-manifold. The above result of Casali and Grasseli [7] is our entry point in the topic. Colour-preserving graph isomorphisms induce an equivalence relation on the set of admissible $2(g+1)$-tuples called $\mathcal{H}$-equivalence [6]. The $\mathcal{H}$-equivalence divides the set of admissible $2(g+1)$-tuples into infinitely many equivalence classes – orbits of the action of the group H of transformations of admissible $2(g+1)$-tuples. Clearly, the H-equivalence is a refinement of the restriction of the dipole-move equivalence on the set of graphs associated with admissible $2(g+1)$-tuples. For $g = 2$, Graselli, Mulazzani and Nedela introduced in [17] a special operation σ on admissible 6-tuples giving rise to an equivalence relation on the set of admissible 6-tuples. The operation σ corresponds to a special sequence of dipole-moves which transforms the crystallisation represented by an admissible 6-tuple to another one, non-equivalent in the sense of H-equivalence. Orbits of the group

$G = \langle H, \sigma \rangle$ define a new equivalence relation, called $\mathcal{G}$-equivalence. Two $\mathcal{G}$-equivalent 6-tuples are dipole move equivalent, hence $\mathcal{G}$-equivalence can be viewed as an approximation of the dipole-move equivalence on the set of admissible 6-tuples.

Establishing the combinatorial representation of 3-manifolds and equivalences between crystallisations a possibility to automatise the classification process of 3-manifolds arises. Recall that 3-manifolds up to genus two are coded by admissible 6-tuples of integers. Further, the sum of first three items of an admissible 6-tuple is exactly half of number of vertices of respective crystallisation. This number is called the complexity of an admissible 6-tuple. One can generate a list of admissible 6-tuples up to given complexity to describe 3-manifolds coded by "small" crystallisations. A first attempt to complete a catalogue of admissible 6-tuples is due to Casali [6]. Maximal complexity of 6-tuples considered in this catalogue was 21. This catalogue contains representatives of equivalence classes of H-equivalence. A disadvantage of the catalogue of M. R. Casali is that the same 3-manifold is represented repeatedly. Our aim is to derive a catalogue of representatives of 3-manifolds with Heegaard genus two distinguished up to homeomorphism. An explicit formulation of the problem reads as follows.

> Derive a catalogue of 3-manifolds with Heegaard genus two, distinguished up to homeomorphism, represented by admissible 6-tuples of complexity at most 21.

First step is to study the action of the group G on the set of admissible 6-tuples. We show that replacing $\mathcal{H}$-equivalence by $\mathcal{G}$-equivalence the Casali's catalogue of representatives can be considerably reduced. Moreover, we show that the problem to recognise whether two admissible 6-tuples of integers are $\mathcal{G}$-equivalent can be effectively solved by a computer. As a result a catalogue of representatives of $\mathcal{G}$-classes up to complexity 21, similar to Casali's, is completed. This way a reduction of the catalogue of Casali [6] with respect to $\mathcal{G}$-equivalence is obtained. The above mentioned results are included in Chapter 3 of our thesis.

Unfortunately, the catalogue of representatives of $\mathcal{G}$-classes of admissible 6-tuples may contain (and actually contains) 6-tuples representing the same 3-manifolds. In order to get a catalogue of representatives with respect to dipole-move equivalence we study (in Chapter 4) some invariants of crystallisations represented by admissible 6-tuples. The most important such an invariant is the fundamental group of the represented 3-manifold. Crystallisation theory gives us an elegant algorithm for deriving a fundamental group from a crystallisation. This algorithm is due to Carlo Gagliardi [14]. One can examine any crystallisation using Gagliardi's algorithm and gather the presentation of certain finitely presented group which is isomorphic to the fundamental group $\pi_1(\mathcal{M})$ of the represented 3-manifold. Further, we use a well-known result stating that the first homology group $H_1(\mathcal{M})$ is an abelianisation of fundamental group of a 3-manifold $\pi_1(\mathcal{M})$. Hence we derive first homology groups for every 6-tuple in the catalogue. Obtaining the homology groups we divide the 6-tuples into the homology classes. Further step is to examine each homology class to prove (or disprove) mutual isomorphisms between the fundamental groups associated with the 6-tuples. Hence we get isomorphism classes of fundamental groups of represented 3-manifolds up to genus two. In order to identify homeomorphism classes of represented 3-manifolds we have to determine the classes of dipole-move equivalence on crystallisations with the same fundamental groups. In few cases when needed it was done by Carlo Gagliardi and Paola Bandieri by using the software called DUKE developed at Modena University. Final results give us some evidence of a conjecture states that prime 3-manifolds with Heegaard genus two are determined by their fundamental groups. The main goal of this thesis is the following classification result which in a condensed form reads as follows.

Theorem. *There are **78** prime 3-manifolds of genus two of complexity* ≤ 21. *Among them, there are **39** elliptic 3-manifolds, **4** Euclidean 3-manifolds and **35** other 3-manifolds with infinite fundamental groups.*

1.1 Content of the thesis

Main aim of this thesis is to study crystallisations representing 3-manifolds with Heegaard genus at most two. In order to attack the classification problem for this family of 3-manifolds we will deal with the following particular subproblems.

P1: *Decide whether two admissible 6-tuples f_1 and f_2 are $\mathcal{G}$-equivalent. Complete the catalogue of representatives of $\mathcal{G}$-equivalence classes up to complexity 21.*

P2: *Determine the isomorphism classes of fundamental groups of 3-manifolds in the catalogue of representatives of $\mathcal{G}$-classes.*

P3: *Show that two 6-tuples from the catalogue with the same associated fundamental group are dipole-move equivalent.*

P4: *Interpret the obtained results in crystallisations.*

Complete solution of Problem **P1** can be found in our paper [21]. This paper forms a base of Chapter 3, "Combinatorial approach". Our analysis of $\mathcal{G}$-equivalence results in an effective algorithm solving this problem. The algorithm is used to derive the catalogue of minimal representatives of $\mathcal{G}$-classes up to complexity 21. Moreover, the classification of crystallisations of regular genus at most one is derived. This way a new, combinatorial proof of the well-known classification of 3-manifolds with Heegaard genus at most one can be obtained. A classification of 3-manifolds of genus two is in general not known. The main object of our study are crystallisations of 3-manifolds of genus two.

In the process of creation of the starting catalogue of admissible 6-tuples we also derive the relators of fundamental groups given by the respective 6-tuples. In Chapter 4 "Analysis of isomorphism classes of fundamental groups" we take the reduced catalogue of $\mathcal{G}$-representatives obtained in the previous step and continue by the analysis of the fundamental groups. We first derive the respective homology groups and divide the list of admissible 6-tuples into the homology classes. The cardinality of each homology class is relatively small. In what follows we use the software GAP [16] to determine the isomorphism classes of fundamental groups. Isomorphism classes of finite fundamental groups are determined with some exceptions in this way completely. In the case of infinite fundamental groups we first compute some invariants such as low index subgroups. For the couples of presentations which cannot be distinguished by the invariants we determine isomorphisms ad hoc by finding respective images of generators and checking that the relators are preserved. This way we solve Problem **P2**.

The statement of the main theorem follows from the combination of results stated in Chapters 3 and 4. The interpretation of our results heavily depends on classical results on 3-manifolds presented in Chapter 2, "Preliminaries". Using these results and with a help from software DUKE we can successfully solve Problem **P3**. The classification may continue by examining geometries of the 3-manifolds in the sense of Thurston solving Problem **P4**. Some hints together with a discussion on open problems can be found in Chapter 5, "Concluding remarks". Developed software is described in Chapter 6, "Software notes". All results including derived catalogues of 6-tuples, catalogues of the associated fundamental groups fills up the Appendix. Moreover, a hard-copy of the seminar talk by Vivien Easson to give the reader more information on geometrisation of 3-manifolds with genus two is included.

Chapter 2

Preliminaries

In this chapter we introduce some classical results of topology of 3-manifolds. These results stands as a base of all of our following considerations. As we shall see, we mostly will not apply them in a straight way, but the completion of results of the thesis is not possible without using a content of this chapter. Besides basic definitions such as the manifold, simplicial complex etc., we also sketch some deeper results such as a geometrisation.

We will denote *n-dimensional Euclidean space* by E^n, the *unit ball* $\{x \in E^n : \|x\| \leq 1\}$ by B^n, and the *unit sphere* $\{x \in E^n : \|x\| = 1\}$ by S^{n-1}. We will call a space homeomorphic to B^n (S^{n-1}) a *n-cell* (*(n − 1)-sphere*).

Definition 1 *[19] A topological 3-manifold is a separable metric space each of whose points has an open neighbourhood homeomorphic to E^3.*

In what follows all the considered 3-manifolds will, unless otherwise stated, assumed to be compact, connected and orientable. Representing 3-manifolds via simplicial complexes is a classical method used in algebraic topology. Every 3-manifold can be triangulated [28]. Let us note that a particular triangulation need not to be ”nice” in the sense of ”gluing homeomorphisms”. In our case we will study simplicial complexes which grows from 3-manifolds in a natural way and gluing homeomorphisms are uniformised.

Simplicial complex. We will view a *simplicial complex* as a locally finite collection K of closed simplexes in E^3 satisfying

i) if $\sigma \in K$ and τ is a sub-simplex of σ, then $\tau \in K$,

ii) if $\sigma, \tau \in K$, then $\sigma \cap \tau$ is a face of both σ and of τ.

Every compact connected n-manifold, $n \leq 3$, can be expressed as a simplicial complex containing a finite set of simplices of dimension n [28]. For instance, a compact connected surface can be triangulated. However, we can form a triangulation of a surface by infinitely many ways. For example, we can choose a point in a triangle of a given triangulation, connect it with the vertices of that triangle and form a new triangulation of the same surface. A general problem is to decide, whether two different simplicial complexes represent the same n-manifold. A crystallisation theory gives us a tool to distinguish the representing simplicial complexes in the set of all possible cases.

Fundamental group. An important invariant of a topological space X is the *fundamental group* $\pi_1(X)$ of X. Elements of this group are homotopy classes of closed curves based at a point x_0. The product of two classes $[f], [g]$ is given by $[f] \circ [g] = [fg]$. It is known that in a piecewise-linear space the fundamental group does not depend on the choice of the base point. For more details see [23, Chap. 2].

Free group. Let F be a group, X be a nonempty set, and $\sigma : X \to F$ a function. Then (F, σ) is said to be *free* on X if to each function α from X to a group G there corresponds a unique homomorphism $\beta : F \to G$ such that $\alpha = \sigma\beta$ [31]. We talk about the set X as about the set of generators and about the group F as about *free group*. The cardinality $|X|$ we will call the *rank* of free group F. In what follows, we will deal only with groups with finite rank.

Free presentation and finitely presented group. Every group is an image of a free group. The epimorphism π of a free group F to G is called a *free presentation* of the group G. Elements of the kernel of this epimorphism $Ker\ \pi \cong R \trianglelefteq F$ are called the *relators of the presentation*. Groups given by finite set of generators and by finite set of relators are called *finitely presented groups*. All considered groups, we deal with, are finitely presented.

Free product of groups. Let there be given a nonempty set of groups $\{G_l;\ l \in \Lambda\}$. By a *free product* of the G_l we mean a group G and a collection of homomorphisms $\iota_l : G_l \to G$ with the following mapping property. Given a set of homomorphisms $\phi_l : G_l \to H$, there is unique homomorphism $\phi : G \to H$ such that $\iota_l\phi = \phi_l$ [31].

Homology group. A *commutator* $[a, b]$ of elements a, b of the group G is an element $[a, b] = aba^{-1}b^{-1}$. The *derived subgroup* $G' \trianglelefteq G$ generated by all commutators is known to be normal. The factor G/G' is called an *abelianisation* of the group G. Abelianisation of the fundamental group $\pi_1(\mathcal{M})$ of a 3-manifold $\mathcal{M}$ is called the *homology group* $H_1(\mathcal{M})$.

Connected sum. The *connected sum* of two 3-manifolds $\mathcal{M}$ and $\mathcal{N}$, denoted $\mathcal{M}\#\mathcal{N}$, is formed by cutting contractible parts $D_1 \subset \mathcal{M}$ and $D_2 \subset \mathcal{N}$ from both manifolds and gluing them along the boundaries $\partial D_1 \subset \mathcal{M}$ and $\partial D_2 \subset \mathcal{N}$ together following a homeomorphism $\partial D_1 \to \partial D_2$ [23].

Connected sum is a well-defined associative and commutative operation in the category of oriented 3-manifolds and orientation preserving homeomorphisms [19, p. 24].

Theorem 2.1 (Van Kampen) *[23, p. 91] [19, p. 25] Let $\mathcal{M}$,$\mathcal{N}$ be 3-manifolds. Let $\mathcal{X} \cong \mathcal{M} \cup \mathcal{N}$. If $\mathcal{M} \cap \mathcal{N}$ is simply connected, then $\pi_1(\mathcal{X})$ is the free product of the groups $\pi_1(\mathcal{M})$ and $\pi_1(\mathcal{N})$ with respect to homomorphisms $\psi_1 : \pi_1(\mathcal{M}) \to \pi_1(\mathcal{X})$ and $\psi_2 : \pi_1(\mathcal{N}) \to \pi_1(\mathcal{X})$ induced by inclusions*

The above theorem deals with a particular case of a more general statement establishing that the fundamental group of a connected sum of two topological spaces is a free product of fundamental groups of the factors. Generally the reverse implication is not true. However, in the particular case of compact, connected 3-manifolds we have the following theorem.

Theorem 2.2 (Grushko) *[19, p. 66] Let $\mathcal{M}$ be a compact, connected 3-manifold. If $\pi_1(\mathcal{M}) \cong G_1 * G_2$ then $\mathcal{M} = \mathcal{M}_1\#\mathcal{M}_2$ where $\pi_1(\mathcal{M}_i) \cong G_i$, $i = 1, 2$.*

Definition 2 *[19, p. 27] A 3-manifold $\mathcal{M}$ is a prime if $\mathcal{M} = \mathcal{M}_1\#\mathcal{M}_2$ implies one of $\mathcal{M}_1, \mathcal{M}_2$ to be the S^3.*

Definition 3 *[19, p. 28] A 3-manifold is irreducible if each S^2 in $\mathcal{M}$ bounds a 3-cell in $\mathcal{M}$.*

Irreducible 3-manifolds are prime. As a partial converse we have the following lemma.

Lemma 2.3 [19, p. 28] The only prime reducible 3-manifold is $S^1 \times S^2$.

Theorem 2.4 (Milnor) *[19, p. 31,p. 35][27] Each compact, connected and orientable* 3*-manifold* $\mathcal{M}$ *can be expressed as a connected sum* $\mathcal{M} = \mathcal{M}_1 \# \mathcal{M}_2 \# \ldots \# \mathcal{M}_n$ *of finite number of prime factors. The decomposition is unique up to reordering of factors in the category of oriented* 3*-manifolds.*

One of the difficulties in study of the structure of prime 3-manifolds is a possibility of existence of non-trivial simply-connected prime manifolds. Poincaré asserted that a manifold with trivial homology (group) is simply connected and therefore a sphere: "...est simplement c'est-a dire homeómorphe à l'hypersphere" [30]. Shortly thereafter he discovered an example of a non-simply connected homology 3-sphere*. However the following is still unsettled.

Conjecture 2.5 (Poincaré Conjecture) *[19, p. 26] Each closed, connected, simply connected* 3*-manifold is homeomorphic to* S^3*.*

Using the term "homotopy n-sphere" for an n-manifold homotopy equivalent to S^n, we have

Theorem 2.6 *[19, p. 26] A* 3*-manifold* $\mathcal{M}$ *is a homotopy* 3*-sphere if and only if* $\mathcal{M}$ *is closed, connected, simply connected* 3*-manifold.*

As we shall see for 3-manifolds up to genus two the Poincaré Conjecture holds.

Definition 4 (Heegaard) *[19, p. 15] A* 3*-manifold* $\mathcal{M}$ *with boundary which contains a collection* $\{D_1, D_2, \ldots, D_n\}$ *of pairwise disjoint, properly embedded* 2*-cells such that the result of cutting* $\mathcal{M}$ *along* $\bigcup D_i$ *is a* 3*-cell is called a cube with n-handles, or alternatively, handlebody.*

By Van Kampen's theorem $\pi_1(\mathcal{M})$ *is a free group of rank* n*.*

A *Heegaard splitting* of a closed, connected and orientable 3-manifold $\mathcal{M}$ is a pair (V_1, V_2) where V_i is a cube with handles (i=1,2), $M = V_1 \cup V_2$, and $V_1 \cap V_2 = \partial V_1 = \partial V_2$.

We note that the boundary of a cube with n-handles, V, is compact, connected surface of Euler characteristic $2 - 2n$ which is orientable if and only if V is orientable. Thus for a Heegaard splitting (V_1, V_2) of a 3-manifold $\mathcal{M}$, V_1 and V_2 have the same number of handles and both are orientable since $\mathcal{M}$ is orientable. [19, p. 17]

Theorem 2.7 *[19, p. 17] Each* 3*-manifold* $\mathcal{M}$ *has a Heegaard Splitting.*

Definition 5 *Heegaard genus of* $\mathcal{M}$ *is minimum of genera of Heegaard splittings of* $\mathcal{M}$*, where the minimum is taken through all Heegaard splittings of* $\mathcal{M}$*.*

Proposition 2.8 *[9] If a* 3*-manifold has a splitting of genus zero, it is homeomorphic to* S^3*.*

A *lens space* $\mathcal{L}(p, q)$ is a 3-manifold formed as a quotient of S^3 (considered as the unit sphere of $\mathbb{C}^2$) by the cyclic group $\mathbb{Z}_p$ of isometries generated by $(z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$ for coprime integers p and q. The splitting of genus one has Heegaard surface $|z_1|^2 = |z_2|^2 = \frac{1}{2}$ and divides $\mathcal{L}(p, q)$ into two solid tori given by $|z_1|^2 \leq \frac{1}{2}$ and $|z_1|^2 \geq \frac{1}{2}$ [9]. Let us remark that lens spaces are irreducible 3-manifolds [19, p. 28].

The genus one splitting of $S^1 \times S^2$ is even easier to see – take two solid tori and glue them by the identity map along their boundaries reversing the orientation of one of the handlebodies. Analysis of all possible torus gluing maps shows that the manifolds listed are the only ones with genus one splittings [9][19, pp. 20 – 23].

*see subsection Chapter 4, subsection "Homology class $H_1 = 1$" for some counterexamples

Proposition 2.9 (Hempel) *[9][19, pp. 20 – 23] If 3-manifold $\mathcal{M}$ has a splitting of genus one, it is either homeomorphic to a lens space or to $S^1 \times S^2$.*

Generally, to understand the structure of 3-manifolds the following approach was used; using certain set of operations each 3-manifold is decomposed into prime 3-manifolds with respect to these operations. It happened that two operations turned to be useful. First is the connected sum, the other one is the so called Johanson-Jaco-Shalen decomposition [33].

Let us consider a prime 3-manifold $\mathcal{M}$ with respect to these two operations. Clearly $\mathcal{M} \approx \widetilde{\mathcal{M}}/\pi_1(\mathcal{M})$, where $\widetilde{\mathcal{M}}$ is the universal cover. A goal is that the action of the fundamental group on the universal cover have a geometric meaning. Thurston [33] discovered that there are *eight* types of geometries, including the well-known elliptic on S^3 and Euclidean on E^3, which can be attached with $\widetilde{\mathcal{M}}$. The action of the fundamental group $\pi(\mathcal{M})$ on $\widetilde{\mathcal{M}}$ is interpreted as the action of a group of "geometric transformations". In this case $\mathcal{M}$ is said to be *geometric*. A 3-manifold is said to have a *geometric decomposition* if it decomposes into prime 3-manifolds $\mathcal{M}_i$ such that each $\mathcal{M}_i$ is geometric.

A celebrated Thurston's conjecture reads as follows.

Conjecture 2.10 (Thurston) *Every compact connected orientable 3-manifold has a geometric decomposition.*

More information on the geometrisation can be found in Appendix written by Vivien Easson and in the book of William P. Thurston [33].

As concerns genus two 3-manifolds, their structure is more complicated compared to the 3-manifolds of genus at most one. Every 3-manifold up to Heegaard genus two admits so-called Weierstrass involution. Combining this fact with Thurston's Symmetry Theorem [33] one can prove the following two statements.

Theorem 2.11 *[9] Every irreducible 3-manifold with Heegaard genus two is geometrisable.*

Theorem 2.12 *[9] Any reducible manifold $\mathcal{M}$ with Heegaard splitting of genus two is expressible as a connected sum of two 3-manifolds with Heegaard splittings of genus one. These are lens spaces or $S^1 \times S^2$*

The above two theorems imply that a genus two 3-manifold has a geometric decomposition is sense of Thurston. Another consequence reads as follows.

Theorem 2.13 *Let $\mathcal{M}$ be a compact connected orientable 3-manifold of genus two. Let the fundamental group of $\mathcal{M}$ be finite. Then the universal cover of $\mathcal{M}$ is the Poincaré sphere S^3.*

Proof. Assume, on the contrary, that the universal cover $\widetilde{M}$ is not S^3 but it is simple connected. By its definition it is a counterexample to the Poincaré Conjecture. By Theorem 2.11 this is impossible. □

Note that Milnor [26] described all the groups which acts on S^3 freely.

Corrolary 2.14 *Let $\mathcal{M}$ be a compact, connected, orientable 3-manifold of genus at most two. Let $\mathcal{M}$ has finite cyclic fundamental group of order p. Then $\mathcal{M}$ is homeomorphic to a lens space $\mathcal{L}(p, q)$, for some $q \in \mathbb{N}$.*

Theorems 2.12, 2.13 and 2.14 are of crucial importance. They can be used to simplify the most of processes we will do in creating the catalogue of representatives of 3-manifolds. It follows that we can immediately exclude all the 6-tuples admitting fundamental groups which are cyclic or which are free products of cyclic groups. Only prime 3-manifolds with Heegaard genus two seems to be really interesting.

Chapter 3

Combinatorial approach

In this chapter we will study combinatorial methods to attack the homeomorphism problem for 3-manifolds up to genus two. A theorem of Pezzana [29] implies that every closed connected 3-manifold can be represented by a particular 4-edge-coloured graph called a crystallisation. The concept of crystallisations plays a crucial role in our next considerations.

There is a well defined equivalence relation on the set of crystallisations representing 3-manifolds called dipole-move equivalence [12]. This equivalence allows us to decide which crystallisations represent the same 3-manifold. Unfortunately, the straight use of dipole-moves to solve the above homeomorphism problem seems to be intractable. In fact, no limit for the number of dipole-moves needed to decide whether two crystallisations determine the same compact connected 3-manifold is known.

Let $\mathcal{M}$ be a 3-manifold. It is well known [2, 29] that a simplicial complex $S(\mathcal{M})$ representing $\mathcal{M}$ can be given by a crystallisation $\Gamma(\mathcal{M})$. Vertices of $\Gamma(\mathcal{M})$ represent simplices of $S(\mathcal{M})$ of dimension 3 and edges of $\Gamma(\mathcal{M})$ represent "gluing" of maximal simplexes of the complex $S(\mathcal{M})$ in subsimplices of dimension 2. Given crystallisation (4-valent 4-edge-coloured graph) $\Gamma(\mathcal{M})$ can be embedded into an orientable surface. The regular genus of $\mathcal{M}$ is the minimal genus of an orientable surface into which $\Gamma(\mathcal{M})$ embeds in a particular way described in the next section, where the minimum is taken through all representations $\Gamma(\mathcal{M})$ of $\mathcal{M}$. It is proved that the regular genus of $\mathcal{M}$ bounds the Heegaard genus of the 3-manifold $\mathcal{M}$ [15].

In this chapter we will also classify the crystallisations representing 3-manifolds with Heegaard genus zero and one. The 3-manifolds of Heegaard genus two becomes the first interesting class because a classification is not known for this class. Following [7] we represent a 3-manifold of genus at most two as a vector of integers of length six and consider certain equivalence relations defined on 6-tuples and preserving the associated 3-manifold of genus two. The 6-tuples code some particular crystallisations naturally embedded into a surface of genus two. The first considered equivalence on the set of 6-tuples is introduced in [6] and it is called $\mathcal{H}$-equivalence. This equivalence is induced by colour-preserving graph isomorphisms between the respective crystallisations. In [17] other equivalence on the set of 6-tuples is defined. This equivalence is called $\mathcal{G}$-equivalence and it extends the $\mathcal{H}$-equivalence. If two 6-tuples f and g are $\mathcal{G}$-equivalent then the respective crystallisations are dipole-move equivalent. Thus if f and g are $\mathcal{G}$-equivalent 6-tuples then they represent homeomorphic 3-manifolds of genus at most two*. Hence the $\mathcal{G}$-equivalence provides an approximation of the dipole-move equivalence. We show that there exists a quick algorithm to decide whether two

*at most two, as we shall see

6-tuples are $\mathcal{G}$-equivalent. This gives us a possibility to create a catalogue of $\mathcal{G}$-representatives of $\mathcal{G}$-classes of crystallisation up to given number of vertices. In what follows we first summarise some needed results, prove a classification of 3-manifolds with Heegaard genus at most one and study the properties of $\mathcal{G}$-equivalence proving that simple algorithm solving it exists. This algorithm is used to derive a list of minimal representatives of $\mathcal{G}$-classes which will be used in further analysis.

3.1 From a 3-manifold to a 4-edge-coloured graph

Each orientable 3-manifold $\mathcal{M}$ can be represented by a bipartite 4-edge-coloured graph [2]. Let T be any simplicial triangulation of $\mathcal{M}$ and T' be its first barycentric subdivision. Each vertex $\hat{\omega}$, which is the barycentre of the simplex ω of T is labelled by the dimension of ω. Take the dual graph Γ of T' and if **uv** is an edge and $\{i, j, k\}$ are the colours of respective triangle in T use the colour complementary to $\{i, j, k\}$ to colour the edge **uv**. The labelling of vertices of T induces a decomposition of the tetrahedrons of T into two classes distinguished by orientation, where adjacent tetrahedrons belong to different classes. Thus Γ is bipartite. The dual graph Γ of T', together with the edge-colouring ν, is a 4-edge-coloured graph, representing T. Conversely, given bipartite 4-edge-coloured graph one can construct an associated 3-dimensional complex T. However, in general, T need not to be homeomorphic to a 3-manifold.

Let us remark that the above representation of a 3-manifold may not be optimal in sense of the size. In fact, many of the 3-manifolds considered in this thesis can be represented by a much smaller simplicial complexes [5]. On the other hand, the representation (by means of bipartite 4-edge-coloured graphs) we are going to use gives us an insight into the structure of a represented 3-manifold. Moreover, as we shall see later, using the representation by means of bipartite 4-edge-coloured graphs the homeomorphism relation on 3-manifolds can be described in a combinatorial way (see Theorem 3.2). Representation by general simplicial complexes yields no such a nice equivalence relation.

Definition 6 (n-edge-coloured graph) *Let $\Gamma = \{V(\Gamma), E(\Gamma)\}$ be a bipartite n-valent graph and let there exist a mapping $\nu : E(\Gamma) \to \Delta_n$, $1 \leq n \leq 4$ such that for all pairs of incident edges $f, g \in E(\Gamma) : \nu(f) \neq \nu(g)$. This mapping is called a graph colouring and the graph Γ a n-edge-coloured graph.*

Residual graph. Given 4-edge-coloured graph Γ, the k-edge coloured graph $(0 < k < 4)$ is called a *residual graph* $\Gamma_{\mathcal{B}}$, where $\mathcal{B} \subset \Delta_4$. We get the residual $\Gamma_{\mathcal{B}}$ from Γ_{Δ_4} by deleting the edges coloured by colours from $\mathcal{B}$. We often use the symbol $\Gamma_{\hat{c}}$ for residual graph created by deleting the edges coloured by the colour c.

Remark. We use the symbol Γ instead of exact description Γ_{Δ_4} for 4-edge-coloured graphs. In what follows we oftenly replace "4-edge-coloured graph" by "4-coloured graph".

Colour-compatible embedding. Let Γ be a bipartite 4-edge-coloured graph. Since the colouring is regular a factor induced by two colours is a disjoint union of bicoloured cycles. There are six such bicoloured 2-factors. We may form a 2-cell embedding of Γ by taking a subset consisting of four bicoloured 2-factors and gluing a 2-cell to each bicoloured cycle in each chosen bicoloured 2-factor. The obtained embedding of Γ will be called *colour-compatible embedding*. Given 4-edge-coloured graph Γ, there are three colour-compatible embeddings of Γ, each one is by definition orientable.

Regular genus. Given 4-edge-coloured graph Γ the minimum of the genera of colour-compatible embeddings of Γ will be called a *regular genus* $g(\Gamma)$ of Γ. *Regular genus of a 3-manifold $\mathcal{M}$* is the minimum of $g(\Gamma)$, where the minimum is taken through all bipartite 4-edge-coloured graphs Γ

representing $\mathcal{M}$. Shortly we talk about the genus of $\mathcal{M}$. It is known that regular genus of $\mathcal{M}$ is equal to the Heegaard genus of $\mathcal{M}$ [15].

Dipole-move. Let Γ be a 4-coloured graph and let Θ is a subgraph of Γ consisting of vertices **x**,**y** joined by h edges ($1 \leq h \leq 3$) coloured by colours $c_1, \ldots, c_h$. If **x** and **y** are in two different components of graph $\Gamma_{\Delta_4 - \{c_1,\ldots,c_h\}}$ induced by the set of complementary colours $\Delta_4 - \{c_1, \ldots, c_h\}$ then the subgraph Θ will be called a *dipole of type h*.

There is a well defined operator over the set of 4-coloured graphs [12] called *dipole-move*. Note that a dipole-move can be defined for $(n+1)$-coloured, connected and bipartite graphs (n-manifolds) generally.

Construction 3.1 (Elementary dipole-move) If Θ is a dipole of type h in Γ_{Δ_4} coloured by colours $\{c_1, \ldots, c_h\}$ we define a dipole-move as follows (see Fig. 3.1):

a) *Cutting of* Θ

- remove edges and vertices of Θ
- glue "hanging" edges of graph Γ_{Δ_4} of the same colour

b) *Adding of* Θ as an inverse to cutting

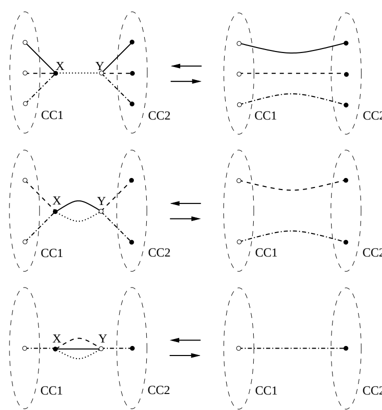

Figure 3.1: Dipole-moves

Main result of [12] states that graphs Γ and Γ' represent isomorphic 3-manifolds if and only if there is a finite sequence of dipole-moves transforming Γ to Γ'. Hence the "homeomorphism problem" reduces to the problem to decide whether two 4-edge-coloured graphs are "dipole-move equivalent".

Theorem 3.2 (Ferri and Gagliardi) *[12] Two 4-edge-coloured graphs Γ and Γ' represent the same 3-manifold if and only if Γ transforms into Γ' by applying a finite sequence of dipole-moves.*

Definition 7 *Let Γ and Γ' be 4-edge-coloured graphs. We say that Γ and Γ' are dipole-move equivalent if Γ transforms into Γ' by a finite sequence of dipole-moves.*

Crystallisations. As already mentioned, not all 4-coloured bipartite graphs represent compact, connected and orientable 3-manifolds. A 4-coloured graph Γ is said to be *contracted* if for each colour $c \in \Delta_4$ the residual graph $\Gamma_{\hat{c}}$ is connected. Contracted 4-coloured graphs are called *crystallisations*. The following characterisation theorem was proved by Pezzana.

Theorem 3.3 (Pezzana) *[11] Let Γ be a crystallisation. Then Γ represents a compact, connected and orientable 3-manifold if and only if the residual graph $\Gamma_{\hat{c}}$ with the induced colouration admits a colour-compatible embedding into the 2-sphere S^2 for each $c \in \Delta_4$.*

3.2 Crystallisations of regular genus at most one

Lemma 3.4 Let Γ be a 4-valent bipartite multi-graph embedded into a surface of Euler characteristic χ. Let f_i be the number of faces of length i. Then

$$2f_2 = \sum_{i>4}(i-4)f_i + 4\chi,$$

where v is the number of vertices, e is the number of edges and f is the number of faces of Γ.

Proof. By Euler-Poincaré theorem we have

$$v - e + f = \chi.$$

Since Γ is 4-valent $e = 2v$, moreover $2e = \sum_i if_i, i \in \mathcal{N}$. Inserting these equations we get

$$4f - \sum_i if_i = 4\chi$$

Using $f = \sum_i f_i$ we obtain

$$\sum_i (4-i)f_i = \sum_{i<4}(4-i)f_i + \sum_{i>4}(4-i)f_i = 4\chi.$$

Since Γ is bipartite

$$2f_2 + \sum_{i>4}(4-i)f_i = 4\chi$$

and we are done. □

Corrolary 3.5 *If $\chi = 2$ then $f_2 > 0$. If $\chi = 0$ then either every face is 4-gonal, or $f_2 > 0$.*

Definition 8 *Let $\iota : \Gamma \hookrightarrow S$ be a colour-compatible embedding of a crystallisation Γ into a surface S. The embedding ι is reduced if and only if Γ is a crystallisation and no 2-gonal face can be cancelled using dipole-move operation.*

Proposition 3.6 *Every bipartite 4-edge-coloured graph of regular genus 0 is dipole-move equivalent to the 4-dipole.*

Proof. Let $i : \Gamma \hookrightarrow S^0$ be a colour-compatible embedding. We may assume, that Γ is reduced. If it is not, we can reduce it cutting digonal faces preserving the planarity of the embedding. By Corollary 3.5 i contains a face F of length two. Denote the vertices incident with F by $\mathbf{u}$, $\mathbf{v}$ and assume its edges are coloured by 0,1. There exists a 2-3 face, say O, incident with vertex $\mathbf{u}$. Since Γ is reduced, the face O is incident with vertex $\mathbf{v}$, as well. Moreover, there is a 0-3 face B, which is incident with F through the edge coloured by 0. Denote by e and g the edges coloured by 3 incident with $\mathbf{u}$ and $\mathbf{v}$, respectively, see Figure 3.2. Let $e \neq g$. Then $B \cup O \cup \{e, g\}$ is a separating

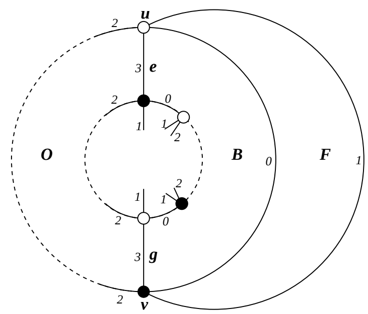

Figure 3.2: Colour compatible embedding of 4-valent graph into S^0

open cylinder. It follows, that $\Gamma \setminus \{e, g\}$ is a disconnected graph. Hence Γ is not contracted, a contradiction. Finally, let $e = g$. Then we have 3-dipole with edges coloured by 1-0-3. Since Γ is reduced, there is a 2-edge joining $\mathbf{u}$ and $\mathbf{v}$. It follows, that Γ is the 4-dipole. $\square$

By a *shifted toroidal rectangular grid* of type $(k \times n; m)$ we mean a graph arising from a $k \times n$ rectangular grid by identifying the opposite horizontal an vertical sides. The vertical sides are identified with shift $m \geq 0$. Figure 3.3 shows the shifted (toroidal) rectangular grid of type $(2 \times 10; 4)$. By its definition every shifted rectangular grid forms a toroidal map of type $(4, 4)$. Altshuler and in a more general framework Thomassen have proved the converse implication [1, 32].

Proposition 3.7 (Altshuler [1]) *Every toroidal map of type* $(4, 4)$ *is given by a shifted rectangular grid of some type.*

Proposition 3.8 *A crystallisation of regular genus one is isomorphic either to a shifted rectangular grid of type* $(2 \times n; m)$*, for some* $m, n \in \mathbb{N}$ *and* $(m, n) = 2$ *or it is dipole-move equivalent to one of the two exceptional graphs depicted on Fig. 3.9.*

Proof. Let $\Gamma \hookrightarrow T$ be a colour-compatible embedding of a bipartite 4-edge-coloured graph Γ. By Corollary 3.5 either each face if 4-gonal or there exists a face of length two.
Case I: $\Gamma \hookrightarrow T$ is of type $(4, 4)$. By Proposition 3.7, Γ is a shifted rectangular grid of type $(k \times n; m)$. It remains to prove that $k = 2$ and $(m, n) = 2$. Without loss of generality we assume that the vertical edges form alternating 1-3 cycles of length n, while the horizontal edges form alternating 0-2 cycles. It follows that both k, n are even and so $k \geq 2$. Assume $k > 2$. Take a horizontal 0-2 path of length 3 and let e and f be its initial and terminal edge. The edges e, f have the same colour, say 0. Since $k > 2$, $e \neq f$. Take the set C of all horizontal edges parallel with e (and f). The set C is a cut-set separating the 1-2-3-subgraph into two disjoint connectivity components, a contradiction. Hence $k = 2$. Let $\mathcal{R} =(\mathbf{v}_0, \mathbf{v}_1 \ldots \mathbf{v}_{n-1})$ be a vertical 1-3 cycle. Then a horizontal 0-2 cycle $\mathcal{C}$ takes consecutively vertices $\mathbf{v}_0$,$\mathbf{v}_m$, $\mathbf{v}_{2m} \ldots$ Since $g(\Gamma) = 1$ then $m \neq 0$. It follows that $|\mathcal{R} \cap \mathcal{C}| = n/(m, n)$. Thus the number of the connectivity components of the 0-1-2-subgraph is $(n, m)/2$. Since Γ is contracted, $(n, m)/2 = 1$.
Case II. There are faces of length 2. We may assume that the digonal faces cannot be cancelled by using respective dipole-moves.

Claim 1. For every digonal face D there exists a face F of size > 4 such that $\bar{D} \cup \bar{F}$ contains a non-contractible cycle.

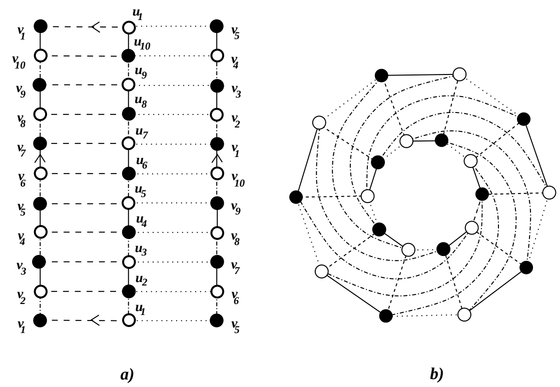

Figure 3.3: Shifted rectangular grid of type $(2 \times 10; 4)$ and its another drawing

Let $\mathbf{u}, \mathbf{v}$ be vertices coincident to D. Let D be coloured by 0-1. Let $F_{\mathbf{u}}$, $F_{\mathbf{v}}$ be 2-3 faces incident to $\mathbf{u}$ and $\mathbf{v}$, respectively. Since the graph is contracted, $F_{\mathbf{u}} = F_{\mathbf{v}} = F$. The face F is of size ≥ 4, otherwise the whole graph is a 4-dipole. If F is of size 4 then **u-v** are consecutive on the boundary of F and we have 3-dipole, a contradiction. Assuming $\bar{D} \cup \bar{F}$ bounds a disk we derive a that Γ is a 4-dipole in the same way as in spherical case (see Proposition 3.6).

Assume two digonal faces D_i incident with vertices $\mathbf{u}_i$, $\mathbf{v}_i$, $i = 1, 2$ are attached to the same face F as claimed in Claim 1. We say that D_1 and D_2 are attached in a *crossing position* if the boundary cycle ∂F can be expressed in a form $\partial F = (W_1\mathbf{u}_1W_2\mathbf{u}_2W_3\mathbf{v}_1W_4\mathbf{v}_2)$ where $W_i, i = 1, 2, 3, 4$ are some paths.

Claim 2. Two digonal faces D_1 and D_2 attached to the same face F of size > 4 are attached to F in a crossing position.

Assume the digons D_1 and D_2 are in a non-crossing position. Then $\partial F = (W_1\mathbf{u}_1W_2\mathbf{v}_1W_3\mathbf{u}_2W_4\mathbf{v}_2)$. Without loss of generality we may assume that ∂D_1 and ∂D_2 are coloured by 0-1 and that ∂F is a 2-3 cycle. Let e, f, g, h be the four edges coloured by 3 and incident with $\mathbf{u}_1$, $\mathbf{v}_1$, $\mathbf{u}_2$, $\mathbf{v}_2$, respectively. Then $\{e, f, g, h\}$ is a cut-set separating the 0-1-2 subgraph into two components, a contradiction.

Claim 3. $f_2 \leq 2$.

Assume, for the contrary that there are three distinct digonal faces D_1, D_2, D_3 and let F_1, F_2, F_3 be the respective faces attached to D_1, D_2 and D_3.

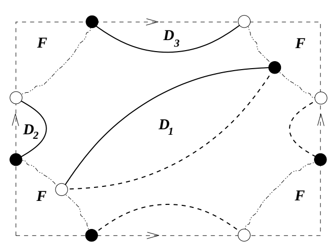

Figure 3.4: Three dipoles incident with the same face

Sub-case $\mathbf{F_2 = F_3 = F}$. Assume D_2 and D_3 are attached to the same face F of length > 4. By Claim 2 they are attached in a crossing position (see Fig. 3.6). It follows that D_1 is attached to F

as well and by Claim 2 it is in a crossing position with both D_2, D_3 (see Fig. 3.4). It follows that the 1-2-3 subgraph of Γ contains a subdivision of $K_{3,3}$, a contradiction with Theorem 3.3.

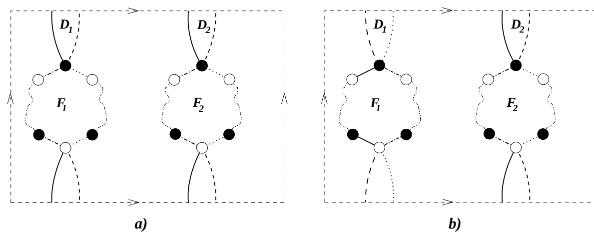

Figure 3.5: Two dipoles share at least one colour in common

Sub-case $\mathbf{F_1 \neq F_2, F_2 \neq F_3, F_1 \neq F_3}$. (See Fig. 3.5) Clearly, we can choose two of D_1, D_2, D_3 such that they are coloured by at most three colours, say the colour 3 is not used. Assume the chosen dipoles are D_1, D_2. We take the four edges e, f, g, h coloured by 3 in F_1 and F_2 incident to the vertices of the dipoles D_1, D_2. Then $\Gamma \setminus \{e, f, g, h\}$ is disconnected, a contradiction.

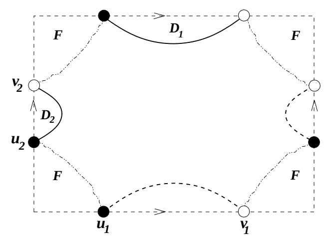

Figure 3.6: Two digons attached in a crossing position to F

Using Claim 3 and Lemma 3.4 we deduce that for faces of length > 4 the following holds: Either there is only one 8-gon or at most two 6-gons in Γ. Assume there is one 8-gonal face F in Γ. By Lemma 3.4 there are exactly two digonal faces D_1, D_2 attached to F in vertices $\mathbf{u}_1$, $\mathbf{v}_1$, $\mathbf{u}_2$, $\mathbf{v}_2$. By Claim 2 D_1 and D_2 are attached to F in a crossing position, see Fig 3.6. It follows that $\partial F = (W_1 \mathbf{u}_1 W_2 \mathbf{u}_2 W_3 \mathbf{v}_1 W_4 \mathbf{v}_2)$ up to relabelling. One can see, that no matter how D_1 and D_2 are attached to F, two of the paths W_i are of even length and two are of odd length. At least one of the paths of odd length is formed just by an edge e, otherwise the sum $|W_1| + |W_2| + |W_3| + |W_4| \geq 10$, a contradiction. One can see that the edge e extends to an alternating path P of length three based on vertices $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{v}_1$, $\mathbf{v}_2$, forming a part of boundary cycle of a face of length > 4, a contradiction.

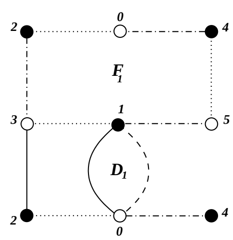

Figure 3.7: Basic graph from which non-simple genus one crystallisation are constructed

Let us discuss the second possibility. We have two digonal faces D_1, D_2 attached to two hexagons F_1, F_2 respectively. All the other faces are 4-gonal. Since we have exactly two 6-gonal faces, there is a 4-gonal face incident to the two vertices of D_1 as depicted on Fig 3.7. Let us call this initial subgraph $\mathcal{B}$. In what follows we use the assignment of vertices introduced on the Fig. 3.7. Let us extend $\mathcal{B}$ by two vertices, say black **x** and white **y**, such that the cycle **0-4-y-x-5-1** bounds the 6-gonal face F_2. Since D_2 is attached to a couple of antipodal vertices of F_2, D_2 is incident with **4** and **5**. In particular we have a 4-gonal face bounded by cycle **4-y-x-5**. Assuming $\mathbf{x} = \mathbf{2}$ and $\mathbf{y} = \mathbf{3}$ we get a graph Γ on Fig. 3.9a. If this is not the case we are forced to append a cylindric stripe consisting of two 4-gonal faces bounded by cycles **x-y-z-w** and **x-w-z-y**, respectively, where **z** and **w** are some vertices. Hence the edges **02**, **13**, **yz** and **xw** are coloured by the same colour and separate Γ into two connectivity components, a contradiction. Assume $F_2 = F_1$ meaning D_2 is incident with vertices **5** and $\mathbf{x} = \mathbf{2}$. Then $\mathbf{y} = \mathbf{3}$ and we get an embedding depicted on Fig 3.8a. However, it is easy to see that this embedding is not colour-compatible.

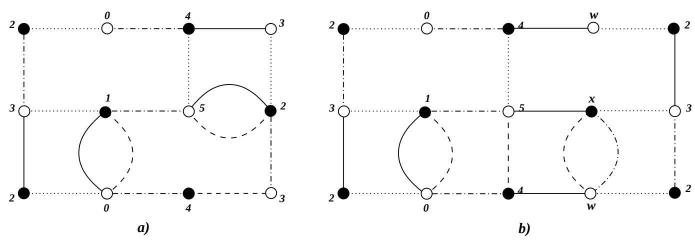

Figure 3.8: Disallowed constructions of embeddings

On the other hand, assume that we complete the base graph $\mathcal{B}$ by 4-gonal face **0-4-5-1**. Denote the resulting configuration $\tilde{\mathcal{B}}$. Colour-compatible embedding forces us to continue a construction. Let us append a stripe of 4-gonal faces with boundary cycles **4-5-x-w** and **5-4-w-x**. Let c be the colour of edges **02** and **13**. Then the edges **02**, **13** together with the edges coloured by c, incident to **w** and **x** form a monochromatic edge-cut, a contradiction.

We have proved that the embedding contains two incident configurations, $\tilde{\mathcal{B}}_1$, $\tilde{\mathcal{B}}_2$, isomorphic (up to recolouring of edges) to $\tilde{\mathcal{B}}$. These two configurations can be glued together in two ways. Since all the other faces are 4-gonal by the above argument there are no other faces. We end with two embeddings depicted on figures Fig. 3.8b and Fig. 3.9b. The first one is not admissible since the underlying graph is not regularly 4-edge-coloured. The second one satisfies all the requirements. □

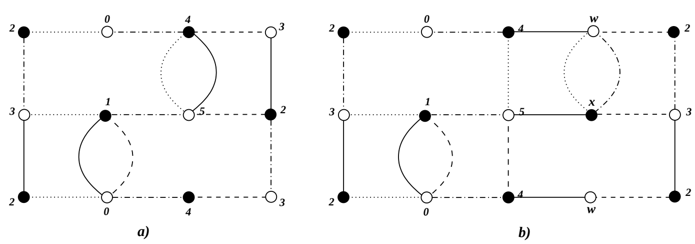

Figure 3.9: Non-simple reduced crystallisations of genus one

3.3 Crystallisations of genus two

Let n be a non-negative integer. It follows from [7] that each (closed) 3-manifold can be represented by a crystallisation Γ which structure can be coded by a $2(n+1)$-tuple of integers satisfying certain conditions. Let $\widetilde{\mathcal{F}}_n$ be a set of $2(n+1)$-tuples of non-negative integers:

$$f = (h_0, h_1 \dots h_n, q_0, q_1, \dots q_n),\ \ h_i, q_i \in \mathbb{N} \cup \{0\},$$

satisfying the following axioms:

(i) $\forall i \in \mathbb{Z}_{n+1} : h_i > 0$,

(ii) all h_i has the same parity,

(iii) $\forall i \in \mathbb{Z}_{n+1} : 0 \leq q_i < h_{i-1} + h_i = 2l_i$,

(iv) all q_i has the same parity.

Remark. From here all operations with numbers q_i will be considered modulo $2l_i$, and according to (iii), q_i will be always the least non negative integer of the class.

Now let us define the set $V(f)$ for a $2(n+1)$-tuple $f \in \widetilde{\mathcal{F}_n}$:

$$V(f) = \bigcup_{i \in \mathbb{Z}_{n+1}} \{i\} \times \mathbb{Z}_{2l_i}$$

and the following involutory permutations on $V(f)$:

$$\alpha_0(i,j) = (i, j + (-1)^j),$$
$$\alpha_1(i,j) = (i, j - (-1)^j),$$
$$\alpha_2(i,j) = \begin{cases} (i+1, 2l_{i+1} - j - 1); & 0 < j < h_i \\ (i-1, 2l_i \quad - j - 1); & h_i \leq j < 2l_i \end{cases},$$
$$\alpha_3(i,j) = \rho \circ \alpha_2 \circ \rho^{-1},$$

where $\rho : V(f) \to V(f)$ is a bijection defined by the rule

$$\rho(i,j) = (i, j + q_i).$$

Now let $f \in \widetilde{\mathcal{F}_n}$ and satisfies the following conditions:

(v) $\forall i \in \mathbb{Z}_{n+1} : h_i + q_i$ is odd, h_i and q_i have different parity,

(vi) the group $\langle \alpha_2, \alpha_3 \rangle$ has exactly three orbits.

Definition 9 (Admissible 6-tuple) *The elements of the set $\mathcal{F}_n \subset \widetilde{\mathcal{F}}_n$ satisfying conditions (i) – (vi) will be called admissible $2(n+1)$-tuples.*

Construction 3.9 (Graph $\Gamma(f)$ associated to $2(n+1)$-tuple) Given $2(n+1)$-tuple f we define the associated graph $\Gamma(f)$ as follows. Let $V = V(f)$ be the set of vertices of $\Gamma(f)$. Then the permutations $\alpha_0, \alpha_1, \alpha_2$ and α_3 define the decomposition of the edge set into four colours, the orbits of α_i form the edges of Γ coloured by i, for $i = 0, 1, 2, 3$. Observe that the subgraphs $\Gamma_{\hat{c}}$, $c \in \Delta_4$ induced by the respective sets of colours are isomorphic planar graphs.

Vice-versa let Γ be a 4-coloured graph with a bicoloured 2-factor containing circles of even length $C_0, C_1, C_2, \ldots, C_g$ coloured by colours 0 and 1. Other edges coloured by colours 2 and 3 join vertices of Γ such that the induced subgraphs $\Gamma_{\hat{3}}$ and $\Gamma_{\hat{2}}$ are planar and isomorphic. Now, let us code the graph by the $2(n+1)$-tuple $f = (h_0, h_1, \ldots, h_n, q_0, q_1, \ldots, q_n)$. The first n items code the numbers of edges coloured by 2 (3) joining the circles C_{i-1} and C_i, $(i = 0, 1, 2, \ldots, n)$ of Γ_{Δ_4}. Clearly, the planar subgraphs $\Gamma_{\hat{3}} \simeq \Gamma_{\hat{2}}$ of Γ are uniquely determined by the integers $h_0, h_1, h_2, \ldots, h_n$. Then $\Gamma(f)$ arises by gluing $\Gamma_{\hat{3}}$ with $\Gamma_{\hat{2}}$ in the cycles $C_0, C_1, C_2 \ldots, C_n$ coloured by 0 and 1. The integers $q_0, q_1, q_2 \ldots, q_n$ determine the rotations of cycles $C_0, C_1, C_2 \ldots, C_n$ in $\Gamma_{\hat{2}}$ before the gluing is done. In this way we get an embedding of $\Gamma(n)$ into the surface of genus n. For example, the graph coded by 6-tuple `(3,1,3,2,2,0)` is embedded into the surface of genus 2 (see Fig. 3.10).

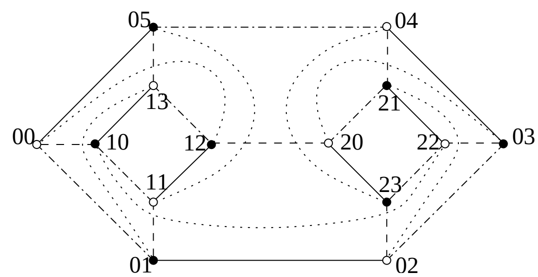

Figure 3.10: The graph represented by 6-tuple $(3, 1, 3; 2, 2, 0)$

The conditions (i) – (vi) come in part from the interpretation while in part they are forced by the requirement that Γ represents a orientable compact connected 3-manifold.

Theorem 3.10 *[7] For every compact connected* 3*-manifold* $\mathcal{M}$ *of genus* g *there exists* $f \in \mathcal{F}_g$ *such that* $\Gamma(f)$ *represents* $\mathcal{M}$.

Agreement. Since we will deal only with manifolds coded by $2(n+1)$-tuples, we will shortly write the manifold coded by $2(n+1)$-tuple f as $\mathcal{M}(f)$.

Lemma 3.11 Let n be a non-negative integer and let $\mathcal{M}$ be a 3-manifold represented by a graph $\Gamma(f)$ given by a $2(n+1)$-tuple $f \in \mathcal{F}_n$. Then the Heegaard genus $g(\mathcal{M})$ of $\mathcal{M}$ is

$$g(\mathcal{M}) \leq n.$$

Proof. To prove the result we show that the regular genus of Γ is at most n. Set g be the genus of the surface determined by the rotation of colours $\rho = (0312)$. Recall that faces of the embedding $\Gamma \hookrightarrow S_g$ are bounded by 0-3, 3-1, 1-2 and 2-0 coloured cycles in Γ. One can easily check that the above set of cycles consists of four faces which are $2(n+1)$-gons and all the other faces are 4-gons. Thus the number of faces is

$$f = 4 + 2\sum_{i=0}^{n}(h_i - 1) = 2 - 2n + 2\sum_{i=0}^{n} h_i.$$

Clearly, the number of vertices is $v = 2\sum_{i=0}^{n} h_i$ and number of edges is $e = 4\sum_{i=0}^{n} h_i$. Inserting these numbers into Euler-Poincaré formula we get

$$v - e + f = 2 - 2n.$$

Hence, $g = n$, and $g(\mathcal{M}) \leq g(\Gamma) \leq n$. □

Agreement. From now we consider $n = g$, so that we talk about $2(g+1)$-tuples and similarly about the set of admissibles as $\mathcal{F}_g$.

Definition 10 (Frames) *[6] Let $\Gamma(f), f \in \mathcal{F}_2$ be a crystallisation of 3-manifold of genus two. Let $\Gamma(f)$ be simple. Then $\Gamma(f)$ is a frame iff following conditions hold:*

a) $h_i > 0;\ i \in \mathbb{Z}_3$,

b) for each $(i,j) \in V(f)$, (i,j) and one of $(i,j+1)$,$(i,j-1)$ does not belong to the same component of $\{2,3\}$-factor.

It is known, that frames are the only irreducible crystallisations of 3-manifolds of genus two. In contrary, if Γ is not a frame, some cancelling dipole-moves can be applied.

Definition 11 (Complexity of $2(g+1)$-tuple) *Let $f \in \mathcal{F}_g$. The number*

$$z(f) = \sum_{i=0}^{g} h_i$$

is called the complexity of f.

Construction 3.12 (Presentation of the fundamental group $\pi_1(f)$) Let $\Gamma(f)$ be a crystallisation of a 3-manifold given by an admissible $2(g+1)$-tuple.
Let $G = \langle x_0, x_1, \ldots, x_g \mid R_0, R_1, \ldots, R_g \rangle$ be a $(g+1)$-generator group. Set $S = \{x_0, x_1, \ldots, x_g\}$ to be set of the cycles of the $\{0,1\}$-factor of $\Gamma(f)$. Hence we have a mapping $\beta : V(\Gamma) \to S$ given by inclusion. By the definition, the $\{2,3\}$-factor has $g+1$ cycles $W_0, W_1, W_2, \ldots, W_g$. The relator R_i is defined by $W_i = (\mathbf{v}_{0,i}\mathbf{v}_{1,i}\mathbf{v}_{2,i} \ldots \mathbf{v}_{k_i,i})$, $i = 0, 1, 2, \ldots, g$ where the edge $\mathbf{v}_{0,i}\mathbf{v}_{1,i}$ is coloured by the colour 2. The relator obtained by C_i is:

$$R_i = \prod_{j=0}^{(k_i-1)/2} \beta(\mathbf{v}_{2j,i})\beta(\mathbf{v}_{2j+1,i})^{-1}\ , \quad 0 \leq i \leq g.$$

Add the relator $x_0 = 1$ into the presentation of the group after obtaining the relators. Thus the group G can be considered as a g-generator finitely presented abstract group.

Theorem 3.13 *[14] The group constructed in Construction 3.12 is isomorphic to the fundamental group of a manifold $\mathcal{M}$ represented by $\Gamma(f)$.*

Corrolary 3.14 *Fundamental group of a 3-manifold of genus g is a finitely presented group of rank at most g.*

Now we are ready to derive the fundamental groups of 3-manifolds represented by crystallisations depicted on Fig. 3.9. This way we basically prove that these crystallisations represents $S^1 \times S^2$, as expected. Denote by Γ_a, Γ_b the crystallisation depicted on Fig. 3.9a, Fig. 3.9b, respectively. There are two 0-1 factors in the crystallisation Γ_a. Using Gagliardi's algorithm we derive the relator $ab^{-1}ba^{-1}bb^{-1} = 1$. One can easily see that this relator is trivial. Following the algorithm we exclude the generator b getting $\pi_1(\Gamma_a) \cong \mathbb{Z}$. The crystallisation Γ_b admits two generators given by 0-1 cycles, as well. Using Gagliardi's algorithm we derive relators $aa^{-1} = 1$ and $ab^{-1}aa^{-1}ba^{-1} = 1$, which are trivial, as well. Hence $\pi_1(\Gamma_b) \cong \mathbb{Z}$.

Agreement. Since all manifolds we deal with are coded by $2(g+1)$-tuples in $\mathcal{F}_g$, we will talk about the fundamental groups given by $2(g+1)$-tuple f, or shortly about the fundamental group of $2(g+1)$-tuple f. We will use the symbol $\pi_1(f)$ for the fundamental group of manifold $\mathcal{M}(f)$. Similarly we talk about the homology group of $2(g+1)$-tuple, $H_1(f)$.

In the following we focus mostly on the 3-manifolds of genus at most two. By Lemma 3.11 and Theorem 3.10, these manifolds are represented by 4-coloured graphs coded by 6-tuples in $\mathcal{F}_2$. It is easy to design an algorithm to verify the conditions (i) – (vi) for a given integer vector with six items. The most complicated seems to be to verify the condition (vi), but the complexity of this algorithm is polynomial, depending on complexity of given 6-tuple f. Therefore we can construct the set $\mathcal{F}_2$ up to a fixed complexity in an effective way.

3.4 Equivalences on $\mathcal{F}_2$

Now we introduce the equivalence relations on $\mathcal{F}_2$ defined in [6, 17, 21]. If $f = (h_0, h_1, h_2, q_0, q_1, q_2)$ is an admissible 6-tuple define the permutations ψ_1, ψ_2, ψ_3 acting on $\mathcal{F}_2$ as follows [6]:

$$\begin{aligned}
\psi_1(h_0, h_1, h_2, q_0, q_1, q_2) &= (h_1, h_2, h_0; q_1, q_2, q_0)\\
\psi_2(h_0, h_1, h_2, q_0, q_1, q_2) &= (h_2, h_1, h_0; q_0, q_2, q_1)\\
\psi_3(h_0, h_1, h_2, q_0, q_1, q_2) &= (h_0, h_1, h_2; 2l_0 - q_0, 2l_1 - q_1, 2l_2 - q_2)
\end{aligned}$$

The above described permutations represents some re-colourings of the graph $\Gamma_{\Delta_4}(f)$.

Definition 12 ($\mathcal{H}$-orbit) *Let $f, g \in \mathcal{F}_2$. Let us define the relation*

$$f \overset{\mathcal{H}}{\approx} g : \exists \eta \in \langle \psi_1, \psi_2, \psi_3 \rangle\,,\ \eta(f) = g$$

This relation is an equivalence and we will call it ***$\mathcal{H}$-equivalence on*** *$\mathcal{F}_2$. The equivalence classes will be called* ***$\mathcal{H}$-orbits****.*

Lemma 3.15 $\mathcal{H}$-equivalence preserves the admissibility of the 6-tuple.

Proof. See the Proposition 16 in [6]. □

Lemma 3.16 (Order of $\langle \psi_1, \psi_2, \psi_3 \rangle$) The group $\mathcal{H} = \langle \psi_1, \psi_2, \psi_3 \rangle$ is isomorphic to $\mathcal{D}_{12}$, where D_{12} is the group of symmetries of a regular hexagon. In particular, each $\mathcal{H}$-orbit has at most 12 elements.

Proof. It follows from the definition of ψ_1, ψ_2, ψ_3 that $\psi_1^3 = \psi_2^2 = \psi_3^2 = 1$. The group $\langle \psi_1, \psi_2 \rangle$ is isomorphic to the dihedral group D_6 because

$$\psi_2 \psi_1 \psi_2 = \psi_1^{-1}.$$

Also ψ_3 commutes with the members of $\langle \psi_1, \psi_2 \rangle$. Hence the group $\mathcal{H}$ satisfies the relations of dihedral group D_{12}. Thus $\mathcal{H}$ is an epimorphic image of D_{12}. To prove that the epimorphism is an isomorphism it is sufficient to find at least one admissible 6-tuple such that the respective $\mathcal{H}$-orbit has 12 different 6-tuples. The 6-tuple $(1, 3, 5, 2, 2, 2)$ is the such a 6-tuple. □

Following [17], let us define mapping $\sigma : \mathcal{F}_2 \to \mathcal{F}_2$:

$$\sigma(h_0, h_1, h_2, q_0, q_1, q_2) = \begin{cases} (h_0, h_1, h_2, q_0, q_1, q_2); & q_0 = 0 \\ (h'_0, h'_1, h'_2, q'_0, q'_1, q'_2); & q_0 \neq 0 \end{cases}$$

where $f = (h'_0, h'_1, h'_2, q'_0, q'_1, q'_2)$:

$$\begin{cases} h'_0 = h_0 + h_1 - q_0 \\ h'_1 = q_0 \\ h'_2 = h_2 + h_1 - q_0 \end{cases} \quad \begin{cases} q'_0 = h_0 + h_1 + h_2 - 2q_0 \\ q'_1 = q_0 + q_1 + h_1 \\ q'_2 = q_0 + q_2 + h_1 \end{cases} \quad ; 0 < q_0 < h_0, h_2$$

$$\begin{cases} h'_0 = q_0 + h_1 - h_2 \\ h'_1 = h_0 + h_2 - q_0 \\ h'_2 = q_0 + h_1 - h_0 \end{cases} \quad \begin{cases} q'_0 = h_1 \\ q'_1 = q_0 + q_1 - h_2 \\ q'_2 = q_0 + q_2 - h_0 \end{cases} \quad ; q_0 > h_0, h_2$$

$$\begin{cases} h'_0 = h_1 \\ h'_1 = h_0 \\ h'_2 = h_1 + h_2 - h_0 \end{cases} \quad \begin{cases} q'_0 = h_1 + h_2 - q_0 \\ q'_1 = q_1 \\ q'_2 = 2q_0 + q_2 + h_1 - h_0 \end{cases} \quad ; h_0 < q_0 < h_2$$

$$\begin{cases} h'_0 = h_1 + h_0 - h_2 \\ h'_1 = h_2 \\ h'_2 = h_1 \end{cases} \quad \begin{cases} q'_0 = h_1 + h_0 - q_0 \\ q'_1 = 2q_0 + q_1 + h_1 - h_2 \\ q'_2 = q_2 \end{cases} \quad ; h_2 < q_0 < h_0$$

The above described operation represents a sequence of dipole-moves such that applying it to the graph represented by an admissible 6-tuple we get the new graph, which can be represented by an admissible 6-tuple too.

Definition 13 ($\mathcal{G}$-equivalence) *Let $f, g \in \mathcal{F}_2$. We define a relation:*

$$f \overset{\mathcal{G}}{\approx} g : \exists \gamma \in \langle \psi_1, \psi_2, \psi_3, \sigma \rangle\,,\ \gamma(f) = g$$

*This relation will be called **$\mathcal{G}$-equivalence on** $\mathcal{F}_2$. The equivalence classes will be called **$\mathcal{G}$-orbits** and will be marked as usual $[f]_\mathcal{G}$.*

Agreement. Denote by $[f]_\mathcal{H}$ a $\mathcal{H}$-orbit containing f. Similar, denote by $[f]_\mathcal{G}$ a $\mathcal{G}$-orbit containing f.

Lemma 3.17 $\mathcal{G}$-equivalence preserves the admissibility of the given 6-tuple.

Proof. See Theorem 5.1 in [17] □

Obviously, any $\mathcal{G}$-orbit is a union of some $\mathcal{H}$-orbits.

Definition 14 (Derivation of f) *Let $\mathcal{H}, \mathcal{H}'$ be two different $\mathcal{H}$-orbits. Let $f \in \mathcal{H} \wedge g \in \mathcal{H}' : g = \sigma(f)$. Then we define a derivation of f as the difference $\delta(f) = z(g) - z(f)$ [17].*

Straightforward from Definitions 12 and 13 we get the following lemma.

Lemma 3.18 With the above notation

$$\delta(f) = \begin{cases} 0 & \text{iff } q_0 = 0 & \text{(a)} \\ h_1 - q_0 & \text{iff } 0 < q_0 < h_0, h_2 & \text{(b)} \\ h_1 - h_0 & \text{iff } h_0 < q_0 < h_2 & \text{(c)} \\ h_1 - h_2 & \text{iff } h_2 < q_0 < h_0 & \text{(d)} \\ q_0 + h_1 - h_0 - h_2 & \text{iff } q_0 > h_0, h_2 & \text{(e)} \end{cases}$$

Note. We denote by $f_i (i = 1, 2, \ldots, 6)$ the i-th item of the vector representing an admissible 6-tuple.

Definition 15 *Let $f, g \in \mathcal{F}_2$ be two 6-tuples. Let $I = \{1, 2, 3, 4, 5, 6\}$ be the set of indexes of components of these vectors. We define the lexical order $\prec$ as follows:*

$$f \prec g \Leftrightarrow \textit{for } j = \inf\{i | (i \in I) \wedge f(i) \neq g(i)\}, f(j) < g(j)$$

Definition 16 (Natural order on $\mathcal{F}_2$) *[21] Using the lexical order we derive an order on $\mathcal{F}_2$ in the following way*

$$f \ll g \Leftrightarrow (z(f) < z(g)) \vee ((z(f) = z(g)) \wedge (f \prec g))$$

We call this order the natural order on $\mathcal{F}_2$.

Finally, we define representatives of $\mathcal{G}$-orbits.

Definition 17 (Minimal representative of $\mathcal{G}$-orbit) *Let $F \subseteq \mathcal{F}_2$. The member f of F satisfying*

$$f \in F : \neg(\exists g \in F), g \ll f$$

is called a minimal representative of F.

Since $\mathcal{F}_2$ with respect to $\ll$ is a well-ordered set, for each F there exists a unique minimal representative which is in the same time the least representative of F.

Agreement. In the notation $[f]_{\mathcal{H}}$ denoting an orbit of $\mathcal{H}$-equivalence we shall always assume that 6-tuple f is minimal.

To create a $\mathcal{H}$-orbit from a given f is not difficult [6]. By Lemma 3.16 the members of $[f]_{\mathcal{H}}$ are

$$\begin{aligned}
f = \quad & (h_0, h_1, h_2, q_0, q_1, q_2) \\
\psi_1 f = \quad & (h_1, h_2, h_0, q_1, q_2, q_0) \\
\psi_2 f = \quad & (h_2, h_1, h_0, q_0, q_2, q_1) \\
\psi_3 f = \quad & (h_0, h_1, h_2, 2l_0 - q_0, 2l_1 - q_1, 2l_2 - q_2) \\
\psi_1^2 f = \quad & (h_2, h_0, h_1, q_2, q_0, q_1) \\
\psi_2\psi_1 f = \quad & (h_0, h_2, h_1, q_1, q_0, q_2) \\
\psi_3\psi_1 f = \quad & (h_1, h_2, h_0, 2l_1 - q_1, 2l_2 - q_2, 2l_0 - q_0) \\
\psi_2\psi_1^2 f = \quad & (h_1, h_0, h_2, q_2, q_1, q_0) \\
\psi_3\psi_1^2 f = \quad & (h_2, h_0, h_1, 2l_2 - q_2, 2l_0 - q_0, 2l_1 - q_1) \\
\psi_3\psi_2 f = \quad & (h_2, h_1, h_0, 2l_0 - q_0, 2l_2 - q_2, 2l_1 - q_1) \\
\psi_3\psi_2\psi_1 f = \quad & (h_0, h_2, h_1, 2l_1 - q_1, 2l_0 - q_0, 2l_2 - q_2) \\
\psi_3\psi_2\psi_1^2 f = \quad & (h_1, h_0, h_2, 2l_2 - q_2, 2l_1 - q_1, 2l_0 - q_0)
\end{aligned}$$

Similarly, it is not complicated to compute an image $\sigma(f)$ for any $f \in \mathcal{F}_2$. In contrast to $\mathcal{H}$-orbits a $\mathcal{G}$-orbit may be infinite. In what follows we give a simple method for deciding whether $g \approx_{\mathcal{G}} h$.

Example 3.19 The 6-tuple $(1, 3, k, 2, 2, k-1)$ $k \geq 3$ belongs to an infinite $\mathcal{G}$-orbit.
Since $\sigma((1, 3, k, 2, 2, k-1)) = (3, 1, k+2, k+1, 2, 2)$ and this 6-tuple is $\mathcal{H}$-equivalent to the 6-tuple $(1, 3, k+2, 2, 2, k+1)$. Hence $(1, 3, k+2, 2, 2, k+1) \approx_{\mathcal{G}} (1, 3, k, 2, 2, k-1)$, and $z(\sigma^{j+1}(f)) > z(\sigma^j(f))$ for every positive integer j. Thus $[(1, 3, k, 2, 2, k-1)]_{\mathcal{G}}$ is infinite.

Lemma 3.20 (Commutation rules in $\mathcal{G}$) Let $f \in \mathcal{F}_2$. Then the relations ψ_2, ψ_3 and σ satisfy

$$\begin{aligned}
\sigma^2 &= 1, \\
\psi_2\sigma &= \sigma\psi_2, \\
\psi_3\sigma &= \sigma\psi_3.
\end{aligned}$$

Proof. The proof of Lemma 3.20 is done by direct computation. Due to its technical complexity we refer the reader to our paper [21] for full reading. □

The application of σ is now easier. It follows that to calculate the action of σ it is sufficient to consider the images σf, $\sigma\psi_1 f$ and $\sigma\psi_1^2 f$ of the three members of an $\mathcal{H}$-orbit.

Definition 18 ($\mathcal{G}$-orbit graph) *Let $\mathcal{S} = \{V, E\}$ be a graph which vertices are $\mathcal{H}$-orbits and the adjacency relation is given by:*

$$[f]_{\mathcal{H}} \sim [g]_{\mathcal{H}} \Leftrightarrow \exists g' \in [g]_{\mathcal{H}} \wedge \exists f' \in [f]_{\mathcal{H}} : g' = \sigma f'.$$

Since $\sigma^2 = 1$, the graph $\mathcal{S}$ is undirected. Note that $\mathcal{S}$ contains loops.

The connectivity components of $\mathcal{S}$ are in one-to-one correspondence with $\mathcal{G}$-orbits. Therefore we call the connectivity components of $\mathcal{S}$, $\mathcal{G}$-orbits too. By the definition, a $\mathcal{G}$-orbit is a class of equivalence. We can describe its minimal representatives.

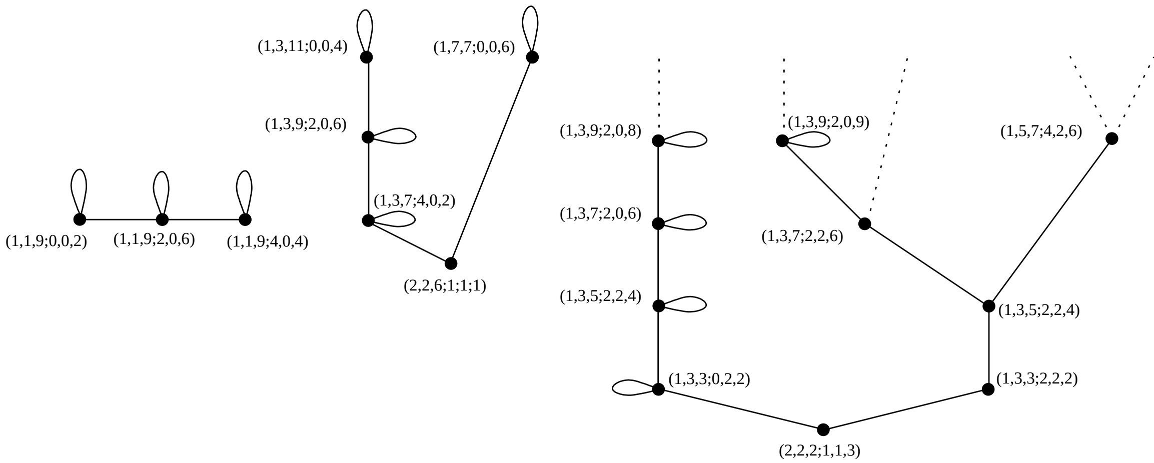

Figure 3.11: Some components of connectivity of $\mathcal{S}$

Agreement. Since the members of each $\mathcal{H}$-orbit have the same complexity, we define the complexity of a $\mathcal{H}$-orbit as the complexity of its members. Since each $\mathcal{H}$-orbit corresponds to a vertex in $\mathcal{S}$, we can speak about complexity of a vertex. Moreover, we say that $\mathbf{u} \ll \mathbf{v}$ for $\mathbf{u} = [f]_{\mathcal{H}}$ and $\mathbf{v} = [g]_{\mathcal{H}}$, if $f \ll g$.

Lemma 3.21 (Neighbourhood of vertex in $\mathcal{S}$) The set of neighbours of the vertex $\mathbf{u} = [f]_{\mathcal{H}}$ in the graph $\mathcal{S}$ is

$$N = \{[\sigma f]_{\mathcal{H}}, [\sigma\psi_1 f]_{\mathcal{H}}, [\sigma\psi_1^2 f]_{\mathcal{H}}\}.$$

In particular, a vertex in $\mathcal{S}$ has at most 3 neighbours.

Proof. Let $A = \langle \psi_2, \psi_3 \rangle$. Each $\mathcal{H}$-orbit decomposes into the orbits induced by the action of A. Since σ commutes with the elements of A (see Lemma 3.20), it follows that for $g = \phi f, \phi \in A$ we have $\sigma g = \sigma\phi f = \phi\sigma f$, hence $[\sigma g]_{\mathcal{H}} = [\sigma f]_{\mathcal{H}}$. Hence, the set of neighbours of vertex $\mathbf{u}$ is N. □

Theorem 3.22 (Main Theorem of [21]) *Let* $\mathbf{v}, \mathbf{u}, \mathbf{w}$ *be three pairwise distinct vertices in* $\mathcal{S}$. *Let* $\mathbf{u}$ *and* $\mathbf{w}$ *be neighbours of* $\mathbf{v}$. *Then*

$$\begin{aligned} z(\mathbf{u}) < z(\mathbf{v}) \quad &\Rightarrow z(\mathbf{w}) > z(\mathbf{v}), \\ z(\mathbf{u}) = z(\mathbf{v}) \quad &\Rightarrow z(\mathbf{w}) \geq z(\mathbf{v}). \end{aligned}$$

Proof. Let us analyse the derivation of complexity $\delta(f)$ for a vertex $\mathbf{v}, f \in [f]_{\mathcal{H}} = \mathbf{v}$. Recall that $f = (h_0, h_1, h_2, q_0, q_1, q_2)$ is the minimal representative of $[f]_{\mathcal{H}}$. It follows that $h_0 \leq h_1 \leq h_2$. By Lemma 3.21 $\mathbf{u}, \mathbf{w} \in \{[\sigma f]_{\mathcal{H}}, [\sigma\psi_1 f]_{\mathcal{H}}, [\sigma\psi_1^2 f]_{\mathcal{H}}\}$. Hence we need to analyse the three derivations: $\delta(f)$, $\delta(\psi_1 f)$ and $\delta(\psi_1^2 f)$. In the following discussion we refer to Lemma 3.18.

I. For $\delta(f)$ we get:

$$\begin{array}{ll}
(a) & q_0 = 0 \Rightarrow \delta(f) = 0, \\
(b) & 0 < q_0 < h_0, h_2 \Rightarrow \delta(f) > 0, \text{therefore } h_1 - q_0 \geq h_0 - q_0 > 0, \\
(c) & h_0 < q_0 < h_2, \text{we consider sub-cases:} \\
 & h_0 < q_0 < h_2 \Rightarrow \delta(f) > 0 \\
 & \text{or} \\
 & h_0 = h_1 < q_0 < h_2 \Rightarrow \delta(f) = 0, \\
(d) & h_2 < h_0 \text{ is in a contradiction with the minimality of } f, \\
(e) & q_0 > h_0, h_2 \Rightarrow \delta(f) > 0, \text{therefore } q_0 + h_1 - h_0 - h_2 > h_1 - h_0 \geq 0.
\end{array}$$

II. For $\delta(\psi_1 f)$ we get:

$$\begin{array}{ll}
(a) & q_1 = 0 \Rightarrow \delta(\psi_1 f) = 0, \\
(b) & 0 < q_1 < h_1, h_0 \Rightarrow \delta(\psi_1 f) > 0, \text{therefore } h_2 - q_1 \geq h_1 - q_1 > 0, \\
(c) & h_1 < h_0 \text{ is in a contradiction with the minimality of } f, \\
(d) & h_0 < q_1 < h_1 \Rightarrow \delta(\psi_1 f) > 0, \\
(e) & q_1 > h_1, h_0 \Rightarrow \delta(\psi_1 f) > 0, \text{therefore } q_1 + h_2 - h_1 - h_0 > h_2 - h_0 \geq 0.
\end{array}$$

III. For $\delta(\psi_1^2 f)$ we get:

$$\begin{array}{ll}
(a) & q_2 = 0 \Rightarrow \delta(\psi_1^2 f) = 0, \\
(b) & 0 < q_2 < h_1, h_2 \text{ we must consider following cases:} \\
 & q_2 < h_0 \leq h_1, h_2 \Rightarrow \delta(\psi_1^2 f) > 0 \\
 & \text{or} \\
 & h_0 < q_2 < h_1 \Rightarrow \delta(\psi_1^2 f) < 0, \\
(c) & h_2 < h_1, \text{is in a contradiction with the minimality of } f, \\
(d) & h_1 < q_2 < h_2 \text{ we consider sub-cases:} \\
 & h_0 < h_1 < q_2 < h_2 \Rightarrow \delta(\psi_1^2 f) < 0 \\
 & \text{or} \\
 & h_0 = h_1 < q_2 < h_2 \Rightarrow \delta(\psi_1^2 f) = 0, \\
(e) & q_2 > h_1, h_2 \text{ we consider sub-cases:} \\
 & h_0 \leq h_1 \leq h_2 < q_2 < h_1 + h_2 - h_0 \Rightarrow \delta(\psi_1^2 f) < 0 \\
 & \text{or} \\
 & h_0 \leq h_1 \leq h_2 < q_2 = h_1 + h_2 - h_0 \Rightarrow \delta(\psi_1^2 f) = 0 \\
 & \text{or} \\
 & h_1, h_2, h_1 + h_2 - h_0 \leq q_2 \Rightarrow \delta(\psi_1^2 f) > 0.
\end{array}$$

The previous discussion describes the set of neighbours for the vertex **v**. The sub-cases are pairwise eliminative and they cover all the possibilities.

It follows from the previous discussion that two edges incident to vertices $\mathbf{u}$, $\mathbf{w}$ with given complexity never enter the vertex $\mathbf{v}$ with higher complexity, because only one from the three possible neighbours of a given vertex $\mathbf{x} = [f]_{\mathcal{H}}$ can have a smaller complexity as $\mathbf{x}$. This neighbour is $\mathbf{y} = [\sigma\psi_1^2 f]_{\mathcal{H}}$ and $z(\mathbf{y}) < z(\mathbf{x})$ in some sub-cases of Case **III**. The complexity of a neighbour of $\mathbf{v}$ can be smaller only in Case **III**.

Now assume $z(\mathbf{u}) < z(\mathbf{v})$. We have already observed $z(\mathbf{w}) \geq z(\mathbf{v})$. Assume $z(\mathbf{w}) = z(\mathbf{v})$. Analysing Cases **I**, **II** and **III** we see that $\mathbf{u}$ satisfies one of the conditions **III-b**, **III-d**, **III-e**. Moreover, $\mathbf{w}$ satisfies the condition **I-c**. Combining **I-c** with **III-b**, or **III-d**, or **III-e** we derive the following contradictions:

$$\begin{aligned}
&h_0 = h_1 < q_0 < h_2 \wedge h_0 = q_2 < h_1, h_2 \Rightarrow h_1 < q_0 \leq q_2 < h_1 \\
&h_0 = h_1 < q_0 < h_2 \wedge h_0 \leq h_1 \leq h_2 < q_2 = h_1 + h_2 - h_0 \Rightarrow h_2 < q_2 < h_2 \\
&h_0 = h_1 < q_0 < h_2 \wedge h_0 < h_1 < q_2 < h_2.
\end{aligned}$$

Hence $z(\mathbf{w}) > z(\mathbf{v})$ and we are done. □

Definition 19 (Horizontal branch) *A vertex $\mathbf{v}$ is in an horizontal branch $\mathcal{B} = \mathcal{B}(\mathbf{v})$ of a $\mathcal{G}$-orbit if the following holds:*

$$\mathbf{v} \in \mathcal{B}(\mathbf{v}) \Leftrightarrow \forall \mathbf{u} \in N(\mathbf{v}) : z(\mathbf{u}) \geq z(\mathbf{v})$$

Lemma 3.23 In every $\mathcal{G}$-orbit there is precisely one horizontal branch $\mathcal{B}$ and $\mathcal{B}$ contains the minimum element $\mathbf{m}$ of the $\mathcal{G}$-orbit with respect to the order $\ll$. The complexity of all elements of $\mathcal{B}$ is equal to $z(\mathbf{m})$.

Proof. By the definition a horizontal branch $\mathcal{B}$ consists of the 6-tuples with a fixed complexity. A minimal representative $\mathbf{m}$ of a $\mathcal{G}$-orbit containing $\mathcal{B}$ belongs to $\mathcal{B}$ as well. Moreover, Theorem 3.22 implies that the complexity of the 6-tuples in $\mathcal{B}(\mathbf{m})$ is equal to $z(\mathbf{m})$. □

Notice that a horizontal branch may contain only one vertex of $\mathcal{S}$.

Theorem 3.24 (Algorithmical complexity of $f \approx_{\mathcal{G}} g$ problem) *There exists a polynomial-time algorithm in terms of the complexities to decide whether two 6-tuples in $\mathcal{F}_2$ are $\mathcal{G}$-equivalent.*

Proof. Let f and g be 6-tuples in $\mathcal{F}_2$ such that $z(f) \geq z(g)$. Using Theorem 3.22 we find $f_1 \in N(f)$ and $g_1 \in N(g)$ so that $z(f_1) \leq z(f)$ and $z(g_1) \leq z(g)$. Note that if the complexity of f_1 (g_1) is less than $z(f)$ ($z(g)$), f_1 (g_1) is uniquely determined. By proceeding at most $z(f) = n$ iterations we reach the horizontal branches of the respective $\mathcal{G}$-orbits containing f and g. If $z(f_n) \neq z(g_n)$, the 6-tuples are not $\mathcal{G}$-equivalent. The complexity of this procedure is $O(n)$. If $z(f_n) = z(g_n)$ the algorithm continues. By Lemma 3.23 we can choose the minimal representative of horizontal branches $\mathcal{B}_1(f_n)$ and $\mathcal{B}_2(g_n)$ containing f_n and g_n. If the minimal representatives are equal then $f \approx_{\mathcal{G}} g$. The complexity of this part of algorithm can be roughly estimated by $O(z_n(f)^3)$. The 6-tuples f and g are not $\mathcal{G}$-equivalent in the other case. □

3.5 Catalogues of minimal representatives

Previous results allows us to generate the catalogues of minimal representatives of $\mathcal{G}$-classes of 3-manifolds of genus at most two.

The first one is a reduction of the catalogue of frames introduced in [6]. We have applied the $\mathcal{G}$-equivalence on it. New catalogue contains 309 of 6-tuples with complexity $z \leq 21$ instead of 695 6-tuples of the original. Note that in this catalogue were included only frames by Definition 10 representing 3-manifolds of genus exactly two.

The second catalogue is formed by computing of all admissible 6-tuples with complexity $z \leq 21$. Following code describes the algorithm to test the minimality of 6-tuple according to the natural order.

```
def isGmin(x):
    g1=Hmin(sigma(x))
    g2=Hmin(sigma(psi1(x)))
    g3=Hmin(sigma(psi1(psi1(x))))

    if z(g1)<z(x) or z(g2)<z(x) or z(g3)<z(x):
         return 0
    elif z(g1)==z(x) or z(g2)==z(x) or z(g3)==z(x):
         hbranch=[]
         hbranch.append(x)
         if z(g1)==z(x) and listsearch(g1,hbranch)==0 :
             hbranch.append(g1)
         if z(g2)==z(x) and listsearch(g2,hbranch)==0:
             hbranch.append(g2)
         if z(g3)==z(x) and listsearch(g3,hbranch)==0:
             hbranch.append(g3)
         for t in hbranch:
             g1=Hmin(sigma(t))
             g2=Hmin(sigma(psi1(t)))
             g3=Hmin(sigma(psi1(psi1(t))))
             if z(g1)==z(t) and listsearch(g1,hbranch)==0:
                 hbranch.append(g1)
             if z(g2)==z(t) and listsearch(g2,hbranch)==0:
                 hbranch.append(g2)
             if z(g3)==z(t) and listsearch(g3,hbranch)==0:
                 hbranch.append(g3)
 t=hbranch[0]
 for q in hbranch:
      if ll(q,t)==1:
      t=q
      if ll(t,x)==1:
          return 0
      else:
          return 1
    else:
         return 1
```

Next, $\mathcal{G}$-equivalence is applied. The reduced catalogue contains 433 6-tuples. In the process the 6-tuples producing relations of the form $a^k = b$ or $b^l = a$ were excluded (see Chapter 6 for details). By Theorem 2.13 and Corollary 2.14, these 6-tuples code 3-manifolds of genera at most one. Note that after the reduction there are still some 6-tuples representing 3-manifolds of genera less than two in the catalogue. In contrast to the catalogue of Casali [6] some non-frames are included.

Both catalogues are listed in Appendix A.

Chapter 4

Analysis of isomorphism classes of fundamental groups

Investigation of the fundamental groups of represented 3-manifolds gives us a possibility to abstract from the topological properties of a 3-manifold. We can use well-known techniques of group theory to determine whether two 3-manifolds are not homeomorphic. Restriction to fundamental groups reduces information on the structure of a considered 3-manifold. For example, note that fundamental groups of lens-spaces are $\mathbb{Z}_p$ disregarding the homotopy type given by the second parameter also. Despite this fact we can "construct" the fundamental groups of given manifolds and solve the isomorphism problem for some classes of 3-manifolds. The full solution of the isomorphism problem requires to get geometric interpretation of prime 3-manifolds.

The following section deals with presentations of fundamental groups associated with 6-tuples to determine isomorphism classes the presented manifolds. We fist introduce some definitions, lemmas and techniques of group theory used in this section. Note that the fundamental groups are considered as abstract 2-generators groups. GAP – Groups, Algorithms and Programming system* [16] was used in most of computations with groups presentations.

Finite groups which appear in the following text are called `[n,k]` using the notation of GAP, meaning `SmallGroup(n,k)`. Here `n` means the order of group and `k` means the position in the GAP library of small groups.

4.1 From 6-tuples to fundamental groups

We can attach a group to a crystallisation of a 3-manifold $\mathcal{M}$. This group is obtained by walking through $\{i, j\}$-residues in (Γ, γ). It can be proved, that this group is isomorphic to the fundamental group of represented manifold $\mathcal{M}$ [14].

We now apply the Construction 3.12 to the case of admissible six-tuples. Derived groups are finitely presented of the form $\langle a, b \mid R_1, R_2, R_3 \rangle$. The presentations will be simplified by the GAP (function `TzGoGo()`) to get as simple relators as possible. One example follows.

*in the following we do not cite GAP

Example 4.1 Let $f = (3, 3, 9, 2, 0, 4)$. Define the mapping $\beta : V(\Gamma) \to \{a, b, c\}$ as follows: $\beta(v) = a, b, c$ depending on whether $\mathbf{v}$ belongs to 0-1 cycles W_0, W_1 or W_2, respectively. The $\{2, 3\}$-factor is a union of the connectivity components:

```
W0:
   ([0, 0][1, 5][0, 10][2, 1][0, 4][2, 7][1, 0][2, 11]
    [0, 6][2, 5][1, 2][2, 9][0, 8][2, 3][0, 2][1, 3])
W1:
   ([0, 1][1, 4][0, 11][2, 0][0, 5][2, 6][1, 1][2, 10]
    [0, 7][2, 4])
W2:
   ([0, 3][2, 8][0, 9][2, 2])
```

To construct the components of connectivity we use the following simple algorithm :

```
while len(listed)<2*z(s):     # while we do not cover all vertices
    ver = findnew(s,listed)   # find first uncovered vertex
    ccomp = []                # create new empty component
                              #  of connectivity
    ver1 = [-1,-1]
    listed.append(ver)        # append vertex to list of
                              #  covered vertices
    ccomp.append(ver)         # append vertex to the current
                              #  component of connectivity
    while ver1 != ccomp[0]:   # while the component is not a cycle
        ver1 = alpha2(ver,s)  # create alpha_2 of vertex
        listed.append(ver1)
        ccomp.append(ver1)
        ver1 = alpha3(ver1,s) # create alpha_3 of vertex
        listed.append(ver1)
        ccomp.append(ver1)
        ver = ver1            # we come back to cycle
    ccomp.pop()               # cut last vertex in component,
                              #  it is the same as the first
    ccomp.append(orbit)       # append component to the known
                              #  components of connectivity
```

A vertex in a component of connectivity is coded by a pair of integers; the first one codes the cycle, which belongs to, the second one codes the order in the cycle and can be 0 for now. We apply β on the above cycles W_0, W_1, W_2 to find the respective relator symbols. In this way we get the following the presentation of the group $\pi_1(f)$ by using the rules set in Construction 3.12:

$$\begin{aligned}\pi_1(f) = \langle a, b, c \mid c = 1, ab^{-1}ac^{-1}ac^{-1}bc^{-1}ac^{-1}bc^{-1}ac^{-1}ab^{-1} = 1,\\ ab^{-1}ac^{-1}ac^{-1}bc^{-1}ac^{-1} = 1, ac^{-1}ac^{-1} = 1\rangle.\end{aligned}$$

and write the final form of the 2-generator fundamental group:

$$\pi_1(f) = \langle a, b \mid ab^{-1}a^2baba^2b^{-1} = 1, ab^{-1}a^2ba = 1, a^2 = 1\rangle.$$

Setting $c = 1$ and using Tietze transformations we finally get the normal form of presentation of the group – $\pi_1(f) = \langle a, b \mid a^2 = 1\rangle$.

4.2 Homology groups of 6-tuples

Knowing the presentations of fundamental groups of manifolds associated with 6-tuples we begin with recognising group invariants to determine the isomorphism classes of fundamental groups $\pi_1(f)$, where f ranges through the list of minimal representatives of $\mathcal{G}$-classes.

The first considered invariant is the homology group $H_1(f)$. We obtain homology group from the fundamental group by factorising $\pi_1(f)$ by the derived subgroup. In our case, this can be done by adding the relator $[a, b]$ into the presentation of fundamental group.

Example 4.2 (Determining homology group H_1) Determine the homology group† for some 6-tuples. Choose i.e. $f_1 = (7, 7, 7, 2, 2, 2)$. By Construction 3.12 and applying Tietze transformations we get the presentation

$$\pi_1(f_1) = \langle a, b \mid b^2a^2b^2 = ab^{-1}a, a^2b^2a^2 = ba^{-1}b\rangle$$

$$H_1(f_1) = \langle a, b \mid b^2a^2b^2 = ab^{-1}a, a^2b^2a^2 = ba^{-1}b, ab = ba\rangle$$

in additive form:

$$H_1(f_1) = \langle a, b \mid 5a + 5b = 0, 5a = 0\rangle$$

One can easily determine the homology group from previous presentation as $H_1(f_1) = \mathbb{Z}_5 \times \mathbb{Z}_5$.

The other example gives an infinite homology group, we deal with the presentation in Example 4.1. The 6-tuple was $f_2 = (3, 3, 9, 2, 0, 4)$, the fundamental group is $\pi_1(f_2) = \langle a, b \mid a^2 = 1\rangle$. The respective homology group is $H_1(f_2) = \mathbb{Z}_2 \times \mathbb{Z}$.

Third example is based on 6-tuple $f_3 = (4, 4, 8, 1, 1, 1)$. The fundamental group is $\pi_1(f_3) = \langle a, b \mid b^3 = ab^{-1}a, a^4b^2 = 1\rangle$ and homology group is $H_1(f_3) = \langle a, b \mid -2a + 4b = 0, 4a + 2b = 0\rangle$. We demonstrate the problems with determining of homology groups. Let us use elementary row operations on matrices to derive equivalent diagonal matrix. We have

$$\begin{pmatrix} -2 & 4 \\ 4 & 2 \end{pmatrix} \sim \begin{pmatrix} -2 & 4 \\ 0 & 10 \end{pmatrix} \sim \begin{pmatrix} -10 & 20 \\ 0 & 10 \end{pmatrix} \sim \begin{pmatrix} -10 & 0 \\ 0 & 10 \end{pmatrix} \sim \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}.$$

The last matrix suggests that the group is $\mathbb{Z}_{10} \times \mathbb{Z}_{10}$. However, this is false. Using GAP we check that $H_1(f_3) \cong \mathbb{Z}_2 \times \mathbb{Z}_{10}$. An explanation consists in fact that the two equations $10a = 0$ and $10b = 0$ are not equivalent with original ones but they are only consequences of them. Hence we have derived a group $H \cong \mathbb{Z}_{10} \times \mathbb{Z}_{10}$ such that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{10}$ is a quotient of it. To make our calculations with matrices correct we have to use "$\Rightarrow$" instead of "$\sim$".

The erroneous result we derived for f_3 in Example 4.2 appears in the case of finite homology groups. The explanation consists in distinguishing between calculation over the ring and over the field. Generally, we solve the system of such equations over the field, but in the case of finitely presented Abelian groups we have to compute over the ring $\mathbb{Z}$ only. Particular result for the case of infinite homology groups is solved by Lemma 4.3.

Lemma 4.3 Let H be an infinite Abelian group $H = \langle x, y \mid ax + by = 0, cx + dy = 0\rangle$. Let $\bar{a} = (a, c) = au + bv$ for some $u, v \in \mathbb{Z}$ and let $\bar{b} = bu + dv$.Then the group $H \cong \langle x, y \mid \bar{a}x + \bar{b}y = 0\rangle$ and one of the following cases hold:

† homology groups presentations are always considered in additive form

a) $\mathbb{Z} \times \mathbb{Z}$, iff $\bar{a} = 0$ and $\bar{b} = 0$

b) $\mathbb{Z} \times \mathbb{Z}_{|\bar{a}|}$, iff $\bar{b} = 0$

c) $\mathbb{Z} \times \mathbb{Z}_{|\bar{b}|}$, iff $\bar{a} = 0$

d) $\mathbb{Z} \times \mathbb{Z}_{\frac{|\bar{a}||\bar{b}|}{(\bar{a},\bar{b})}}$, iff $\bar{a}\bar{b} \neq 0$

Proof. Since H is infinite, the equations following from the presentation of the group H are linearly dependent. Note that division is not allowed over $\mathbb{Z}$. From Euclidean algorithm we know that there are $u, v \in \mathbb{Z}$ such that $(a, c) = q = au + cv$. Note that $c = pq, p \in \mathbb{Z}$. Rewrite the system of equations in the matrix form. Using Gauss elimination method we get:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} au & bu \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} q & bu + dv \\ c & d \end{pmatrix} \Rightarrow$$
$$\Rightarrow \begin{pmatrix} q & bu + dv \\ 0 & d - p(bu + dv) \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{d} \end{pmatrix}$$

Since the new equations are still linearly dependent, we know that $\bar{a}\bar{d} = 0$, and consequently $\bar{a} = q = (a, c) \neq 0 \Rightarrow \bar{d} = 0$. Thus two equations transform into the following one:

$$\bar{a}x + \bar{b}y = 0.$$

To prove the equivalence of the above equation and the system we have begun with, do the reverse transformation. This can be done as follows:

$$\begin{pmatrix} \bar{a} & \bar{b} \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} q & bu + dv \\ 0 & d - p(bu + dv) \end{pmatrix} \Rightarrow \begin{pmatrix} q & bu + dv \\ c & d \end{pmatrix}$$

The first equation one gets by multiplying the first line in matrix by integer $k = a/q$. Hence $\bar{a}x + \bar{b}y = 0$ is equivalent with the original system of equations over $\mathbb{Z}$.

To prove the cases of Lemma 4.3; in the Case a) we have two generated Abelian free group, since the relator is $0x + 0y = 0$, thus the group is $\mathbb{Z} \times \mathbb{Z}$.

In Cases b) and c) we have the direct products of $\mathbb{Z} \times \mathbb{Z}_{\bar{a}}$, $\mathbb{Z} \times \mathbb{Z}_{\bar{b}}$, since we have the relators $\bar{a}x = 0$, $\bar{b}y = 0$, respectively.

The equation $\bar{a}x = -\bar{b}y$ appears in Case d). The set of solutions of this equation over the field $\mathbb{Q}$ is $\{[q\bar{b}, -q\bar{a}],\ q \in \mathbb{Q}\}$. However, the set of admissible results must be a subset of $\mathbb{Z} \times \mathbb{Z}$. Choosing $q = 1/(-\bar{a}, \bar{b})$ we get the solution with the smallest positive value. Thus, all other solutions over $\mathbb{Z}$ are multiplicands of the chosen one. Set $m = |-\bar{a}|/(\bar{a}, \bar{b})$ and $n = |\bar{b}|/(\bar{a}, \bar{b})$. The group $A = \langle [m, n] \rangle$ is a normal subgroup of $\mathbb{Z} \times \mathbb{Z}$ and $H \cong \mathbb{Z} \times \mathbb{Z}/A$. Let $H \trianglerighteq K = \langle [0, m] \oplus H \rangle$. Every coset has the representative in the set $\{[x, y] \in \mathbb{Z} \times \mathbb{Z};\ 0 \leq x < n, 0 \leq y < n\}$, the order of $|H/K| = mn$. It follows that H/K is isomorphic to the torsion subgroup T of H and $H \cong K \times T \cong \mathbb{Z} \times (\mathbb{Z}_m \times \mathbb{Z}_n)$ [31, Th. 4.2.10]. Since m and n are relatively prime $H/K = \mathbb{Z} \times \mathbb{Z}_{mn}$. □

Note that the reduction technique used in the Proof of Theorem 4.3 gives an algorithm to reduce a system consisting of finitely many linearly dependent equations in $\mathbb{Z} \times \mathbb{Z}$. In each step we reduce one of them and finally we get one equation equivalent to the system we begin with.

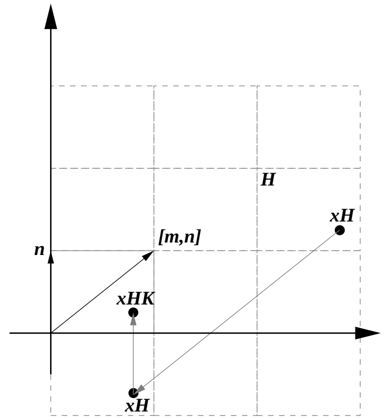

Figure 4.1: Illustration of cosets $\mathbb{Z}_{mn} \times \mathbb{Z}$

4.3 Using low-index subgroups

One of the group invariants which turned out to be useful is a list of subgroups of bounded index. This method is based on coset enumeration with upper bound, GAP can compute it in real time for small index n. The input are two finitely presented groups, possibly non isomorphic. Using GAP command `LowIndexSubgroupsFpGroup(G,n)` we obtain the list of representatives of conjugacy classes of subgroups of index lower or equal to `n` for each of the considered groups. Obviously, two groups with different lists of subgroups of index $\leq n$ are not isomorphic. In many considered cases the comparison of lengths of the lists is sufficient to distinguish non-isomorphic groups.

We use a particular improvement of the method selecting only normal subgroups from the list of low index subgroups. It is clear that we can factorise the group by each member of this restricted set to get all factors of bounded order. Since the considered normal subgroups have small indexes, it is easy to compare lists of factor groups and test isomorphisms of respective factors ”by hand”.

The small program in GAP script and one example follow.

```
LINormSubPrintGens := function (g,n)
% takes two args: the group and the maximal
% index of the subgroup
        local l,d;
        l:=LowIndexSubgroupsFpGroup(g,n);
        % create list of subgroup up to index n
        for d in l do
        % for all subgroups in the list
                if IsNormal(g,d)=true then
                % if the subgroup is normal
                        Print(IdSmallGroup(g/d));
                        % print identification of factor
                        Print(" ");
                        Print(GeneratorsOfGroup(d));
                        % print the subgroup generators
                        Print("\n");
                fi;od;end;;
```

One can use this example to create the functions which can be useful in situations which occurs in real computations.

Example 4.4 (Show non-isomorphism of two groups) Let us have two finitely presented groups

$$\begin{aligned} G_1 &= \langle a, b \mid a = b^2a^2b^2, b = a^3ba^3\rangle \\ G_2 &= \langle a, b \mid a^{-3} = ba^3b^{-1}, a^4 = b^2a^{-1}b^2\rangle \end{aligned}$$

We suppose that these groups are not isomorphic and we check it by using the function `LINormSubPrint`, which is a modification of above mentioned procedure.

```
gap> g:=F/ [a*b^-2*a^-2*b^-2,a^3*b*a^3*b^-1];
<fp group on the generators [ a, b ]>
gap> h:=F/[a*b^-1*a^3*b*a^2,a^4*b^-2*a*b^-2];
<fp group on the generators [ a, b ]>
gap> LINormSubPrint(g,12);
[ 1, 1 ] [ 2, 1 ] [ 4, 1 ] [ 8, 1 ] [ 3, 1 ] [ 6, 2 ] [ 12, 2 ]
[ 12, 3 ]
gap> LINormSubPrint(h,12);
[ 1, 1 ] [ 2, 1 ] [ 4, 1 ] [ 8, 1 ] [ 3, 1 ] [ 6, 2 ] [ 12, 2 ]
[ 10, 1 ] [ 12, 3 ]
```

From lengths of lists we deduce, that the group are not isomorphic. Moreover, the group G_2 contains the normal subgroup N such that $G/N \cong D_{10}$ which corresponds the vector `[10,1]`, but the group G_1 does not contain such a normal subgroup.

4.4 Some important groups

Oriented triangle groups. The oriented triangle group $\Delta^+(k, m, n)$ is the group with the presentation $\Delta^+(k, m, n) = \langle a, b \mid a^k = b^m = (ab)^n = 1\rangle;\ k \geq m \geq n$. The groups $\Delta^+(k, 2, 2)$ are the dihedral groups D_{2k}.

Oriented triangle groups can be regarded as the groups grooving up from regular tessellations of simply connected surfaces. The generators of these groups can be associated with some rotations around a vertex named 0 (a) and around the barycentre of face incident with 0 (b) (see Fig. 4.2). The form of the group is closely related to the geometry of the tessellated surface induced by the action of $\Delta^+(k, m, n)$. The geometry of the surface is:

- elliptic, iff $1/k + 1/m + 1/n > 1$
- Euclidean, iff $1/k + 1/m + 1/n = 1$
- hyperbolic, iff $1/k + 1/m + 1/n < 1$

Lemma 4.5 Oriented triangle groups $\Delta^+(k, m, n)$ have trivial centres, except $\Delta^+(k, 2, 2)$, where k is even.

Proof. ‡ Any central element $z \in \zeta(\Delta^+(k, m, n))$ must commute with x and y, hence fixes the unique fixed points of x and y in the surface onto which the group acts on; since z preserves orientation

‡thanks to Gareth Jones

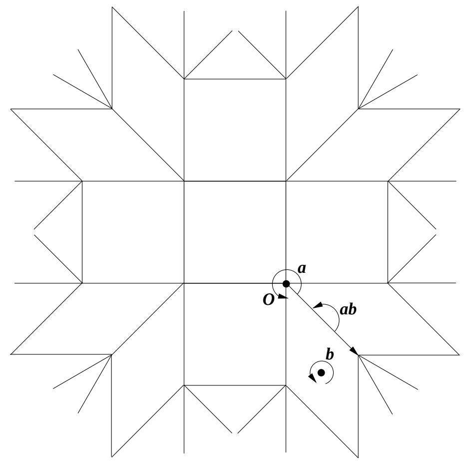

Figure 4.2: Action of the group $\Delta^+(5,4,2)$ on tessellation of type $\{5,4\}$.

and fixes two points, $z = 1$. This works for hyperbolic and Euclidean triangle groups, and with a little extra effort (since rotations of the sphere have 2 fixed points, permuted by z), it also tells that the centre of a spherical triangle group is trivial except for a dihedral group $D_{4n}, n \in \mathbb{N}$. $\square$

Lemma 4.6 Let Z' be a central subgroup $Z' \trianglelefteq G$. Let $G/Z' \cong \Delta^+(k,m,n)$, $k \geq m \geq n$, $m > 2$ if k even. Then the center $\zeta(G) = Z'$.

Proof. Clearly, $\zeta(G)/Z'$ is a central subgroup of G/Z'. By Lemma 4.5 $\zeta(G)/Z' = 1$. It follows $\zeta(G) = Z'$. $\square$

Extended triangle groups. The groups $\Delta(k,m,n) = \langle a, b \mid a^k = b^m = (ab)^n \rangle$ $k \geq m \geq n$ will be called extended triangle groups. Note that these groups has the center $\zeta\Delta(k,m,n) \cong \mathbb{Z}$, generated by the element a^k and the factor by the center is oriented triangle group $\Delta^+(k,m,n)$.

Generalised quaternion groups (Dicyclic groups). The generalised quaternion group Q_{4n}, $n \geq 2$ is the group with the presentation $Q_{4k} = \langle a, b \mid a^k = b^2 = (ab)^2 \rangle$[§].

Lemma 4.7 Let $G = \langle a, b \mid a^k = b^l = (a^r b^s)^m = 1 \rangle$ be a group and $(k,r) = 1$ and $(l,s) = 1$. Then $G \cong H = \langle x, y \mid x^k = y^l = (xy)^m = 1 \rangle$.

Proof. Set the map $\phi : a^r \mapsto x, b^s \mapsto y$. Since $(k,r) = 1$, $x \in \langle a \rangle$ and $|x| = |a|$. The situation is analogous for y. Thus $G \to H$ is a group isomorphism. $\square$

Lemma 4.8 Let $G = \langle a, b \mid a^k = (a^l b^m)^n, \ldots \rangle$ be a group. Then $G = \langle a, b \mid a^k = (b^m a^l)^n, \ldots \rangle$.

Proof. Immediate, since conjugation by an element a^l of the group is an inner automorphism of the group taking $a^k \mapsto a^k$, $a^l b^m \mapsto b^m a^l$. $\square$

[§]these groups are noted as $\langle 2, 2, k \rangle$ in [8]

4.5 Finite cyclic fundamental groups and lens spaces

By Corollary 2.14 a 3-manifold represented by an admissible 6-tuple f with a cyclic fundamental group is either a lens-space $\mathcal{L}(p,q)$ or $S^1 \times S^2$ (see Theorem 4.11). An obstacle to recognise these spaces from a presentation of the fundamental group $\pi_1(f)$ comes from the fact that no algorithm to decide whether $\pi_1(f)$ is cyclic is known to us. Even if we know that f determines a lens-space $\mathcal{L}(p,q)$ we know no procedure to determine the second parameter of the lens-space directly from f [13]. In what follows we give a combinatorial definition of lens-spaces by means of crystallisations and prove the structural results on the associated fundamental groups. On the other hand, if and admissible 6-tuple is a non-frame (which can be easily checked) we know that it represents a 3-manifold of genus at most one.

Recall the definition of the graph coded by an admissible $2(g+1)$-tuple and apply its definition for $g = 1$. A 4-tuple (h_0, h_1, q_0, q_1) codes the graph Λ_{Δ_4}. Since this graph is a crystallisation, by Lemma 3.11 codes a 3-manifold $\mathcal{L}(f)$ of genus less or equal one.

Definition 20 (Lens space $\mathcal{L}(f)$) *The 3-manifold $\mathcal{M}(f)$ coded by an admissible 4-tuple $f = (h_0, h_1, q_0, q_1)$ is called a lens space.*

Note that our definition of lens space includes also the Poincaré sphere S^3 coded by 4-tuples $(h_0, h_1, 0, 0)$. Generally, a lens space is given by two integers p and q, $(p,q) = 1$, see Chapter 2. The relation between an admissible 4-tuple (h_0, h_1, q_0, q_1) and integers p, q is apparent. Firstly, the admissibility forces $q_0 = q_1$. Secondly, observe that if $h'_0 + h'_1 = h_0 + h_1$ then the graphs that correspond to (h_0, h_1, q_0, q_0) and (h'_0, h'_1, q_0, q_0) are isomorphic. Hence, setting $p = (h_0 + h_1)/2$ and $q = q_0/2 = q_1/2$ we get the two parameters p, q describing the lens space $\mathcal{L}(p,q)$, $(p,q) = 1$ in the standard definition.

Let us deal with a concrete example of lens space $\mathcal{L}(5,2)$ represented by 4-tuple $f = (5,5,4,4)$. The simplicial complex represented by the graph $\mathcal{L}(f)$ (see Fig. 4.3) can be imagined as two $2p$-sided pyramids glued together in socles which triangular faces are glued with shift $2q$. Using the

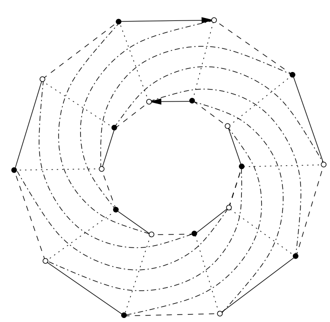

Figure 4.3: Lens space $\Lambda_{\Delta_4}(5,5,4,4)$

Construction 3.12 one can easily seen that the following lemma holds.

Lemma 4.9 Fundamental group of a lens space given by an admissible 4-tuple (h_0, h_1, q_0, q_1) is isomorphic to finite cyclic group $\mathbb{Z}_n$, where $n = p/(p,q)$.

To see the opposite implication is a more delicate problem. Its proof requires explicit or implicit use of Poincaré Conjecture or of Thurston's Geometrisation Conjecture. Using known results one can prove the following proposition paraphrasing Corollary 2.14.

Proposition 4.10 *Let $\mathcal{M}$ be a compact connected and orientable 3-manifold of genus at most two. Then $\pi_1(\mathcal{M}) \cong \mathbb{Z}_p$ implies that $\mathcal{M}$ is homeomorphic to a lens space $\mathcal{L}(p,q)$ for some $q \in \mathbb{N}$ coprime to p.*

Proof. If the genus of $\mathcal{M}$ is at most one then by Proposition 2.9 $\mathcal{M}$ is one of $\mathcal{L}(p,q)$, $(p,q)=1$, S^3 or $S^2 \times S_1$. Since the fundamental group of $S^2 \times S^1$ is known to be infinite cyclic the statement holds in this case. Assume the genus of $\mathcal{M}$ is two. By Theorem 2.13 $\mathcal{M}$ is a quotient of S^3 by the fundamental group $\pi_1(\mathcal{M}) \cong \mathbb{Z}_p$. Free actions of cyclic groups $\mathbb{Z}_p$ on S^3 are classified [33, pp. 250–251]. The quotient $S^3/\mathbb{Z}_p \cong \mathcal{L}(p,q)$ for some q coprime to p. □

In what follows we shall extensively use the following statement which is a consequence of Proposition 4.10.

Theorem 4.11 *Let f be an admissible 6-tuple. If $\pi_1(f)$ is cyclic, then $\mathcal{M}(f)$ is either a lens space or $S^1 \times S^2$.*

Browsing the six-tuples in our catalogue of representatives of $\mathcal{G}$-classes up to complexity 21 we have found 52 representatives yielding cyclic fundamental groups including $\mathbb{Z}$ and finite cyclic groups with orders in range from 2 to 29. See Appendix C for details.

Remark. It follows that admissible 4-tuples are in a correspondence with shifted rectangular grids which coincide with the set of simple crystallisations of genus one, see Section 3.2. As a consequence we get that the only 3-manifold of genus one which cannot be represented by an admissible 4-tuple is $S^1 \times S^2$. Proposition 3.8 suggests that $S^1 \times S^2$ can be defined alternatively using one of the crystallisations depicted on Fig. 3.9.

4.6 Connected sums and free products

Theorems 2.12 and 2.2 imply that a 3-manifold represented by an admissible 6-tuple is decomposable if and only if its fundamental groups is a free product of two cyclic groups. A problem is that given presentation of the fundamental group $\pi_1(f)$ we do not have an algorithm to recognise whether $\pi_1(f)$ has the above structure. In what follows we derive some sufficient conditions on an admissible 6-tuple f to represent a decomposable 3-manifold.

Let $f_1 \in \mathcal{F}_g, f_2 \in \mathcal{F}'_g$ for some integers g, g'. Let $\Gamma_1 = \Gamma(f_1)$ and $\Gamma_2 = \Gamma(f_2)$ be two crystallisations of 3-manifolds. We define a graph operation called *join* of Γ_1 and Γ_2, denoted $\Gamma_1 \# \Gamma_2$, in the following way [12].

Construction 4.12 (Join of Γ_1 and Γ_2) Choose vertex $\mathbf{u}$ in Γ_1 and $\mathbf{v}$ in Γ_2. Cut both vertices from the respective graphs and glue the hanging edges of the graphs preserving the colours. Denote the resulting graph $\Gamma = \Gamma_1 \# \Gamma_2$.

Note that the same operation is mentioned in the paper of Ferri and Gagliardi [12].

Lemma 4.13 Denote $\mathcal{M}$,$\mathcal{N}$ the 3-manifolds represented by crystallisations Γ_1,Γ_2, respectively. Then the graph $\Gamma = \Gamma_1 \# \Gamma_2$ is a crystallisation representing the connected sum $\mathcal{M} \# \mathcal{N}$.

Proof. By cutting only one vertex of Γ_1, Γ_2 we cut one simplex of the simplicial complex represented by Γ_1, Γ_2, respectively. Since a simplex is contractible, the gluing of the edges preserving the colours is equivalent to creating the connected sum of represented 3-manifolds. Thus Γ represents $\mathcal{M} \# \mathcal{N}$.

By Theorem 3.3, it remains to prove that Γ is contracted. Choose a colour $c \in \Delta_4$. By the assumptions $\Gamma_{\hat{c}}$ splits into two subgraphs $(\Gamma_1)_{\hat{c}} \setminus \mathbf{u}$ and $(\Gamma_2)_{\hat{c}} \setminus \mathbf{v}$ joined by a 3-edge-cut. Since Γ_1, Γ_2 are crystallisations, the subgraphs $(\Gamma_1)_{\hat{c}}$,$(\Gamma_2)_{\hat{c}}$ are connected. We claim that they are 2-connected. Assume, to the contrary that say, $(\Gamma_1)_{\hat{c}}$, has a cut-vertex. Then $(\Gamma_1)_{\hat{c}}$ has a bridge. However a cubic graph containing a bridge does not admit a 3-edge-colouring. It follows that both $(\Gamma_1)_{\hat{c}} \setminus \mathbf{u}$ and $(\Gamma_2)_{\hat{c}} \setminus \mathbf{v}$ are connected, and consequently, $\Gamma_{\hat{c}}$ is connected as well. □

In the particular case of joins of lens spaces we have the following lemma.

Lemma 4.14 Let $f_1 = (h_0, h_1, q, q)$ and $f_2 = (h'_0, h'_1, q', q')$ be admissible 4-tuples. The join $\Lambda_{\Delta_4}(f_1)\#\Lambda_{\Delta_4}(f_2)$ is a crystallisation of a connected sum of lens spaces $\mathcal{L}(p_1, q_1)\#\mathcal{L}(p_2, q_2)$ which is coded by the 6-tuple $f = (1, p_1 - 1, p_2 - 1, q_2, q_1, 0)$, where $p_1 = (h_0 + h_1)/2$, $p_2 = (h'_0 + h'_1)/2$, $q_1 = q$ and $q_2 = q'$.

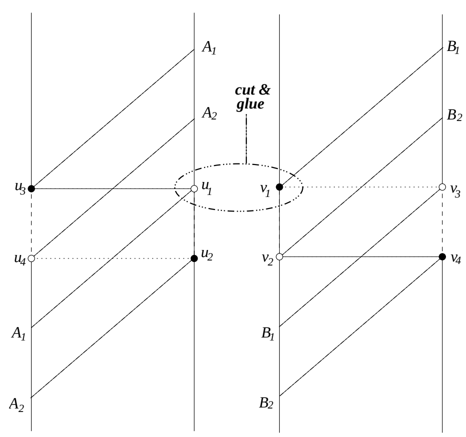

Figure 4.4: Join of lens-space graphs

Proof. Recall that the crystallisations of lens spaces are bipartite and assume that they are represented by 4-tuples $(2p_1 - 1, 1, q_1, q_1)$ and $(1, 2p_2 - 1, q_2, q_2)$. We will cut the vertices $\mathbf{u}_1 \in V(\Lambda_{\Delta_4}(f_1))$ and $\mathbf{v}_1 \in \Lambda_{\Delta_4}(f_2)$. The proof is done if the conditions of admissibility for 6-tuple will be observed. The conditions (i) – (v) are trivial to prove, since they come from the definition of crystallisations $\Lambda_{\Delta_4}(f)$ and definition of join.

Let $\mathbf{u}_2\mathbf{u}_1$, $\mathbf{u}_3\mathbf{u}_1$, $\mathbf{u}_3\mathbf{u}_4$, $\mathbf{u}_2\mathbf{u}_4$ be the edges in $\Gamma_1 = \Lambda_{\Delta_4}(f_1)$ coloured by 0, 2, 0, 2 respectively. It follows that the $\{2, 3\}$-factor in Γ_1 consists of two cycles of the form $(\mathbf{u}_3\mathbf{u}_1 A_1)$ and $(\mathbf{u}_4\mathbf{u}_2 A_2)$, where A_1, A_2 are $\{2, 3\}$-coloured paths. Similarly, the $\{2, 3\}$-factor splits into two $\{2, 3\}$-coloured cycles of the form $(\mathbf{v}_3\mathbf{v}_1 B_1)$ and $(\mathbf{v}_4\mathbf{v}_2 B_2)$ in $\Gamma_2 = \Lambda_{\Delta_4}(f_2)$ (see Fig. 4.4). By the definition, the join in vertices $\mathbf{u}_1$ and $\mathbf{v}_1$ gives rise to three $\{2, 3\}$-coloured cycles of the form $C_1 = (\mathbf{u}_3\mathbf{v}_3 B_1 A_1)$, $C_2 = (\mathbf{u}_4\mathbf{u}_2 A_2)$ and $C_3 = (\mathbf{v}_4\mathbf{v}_2 B_2)$. Hence the condition (vi) is satisfied and $\Gamma = \Gamma_1\#\Gamma_2$ is coded by an admissible 6-tuple f. By direct checking we derive f has the required form (see Fig. 4.5). □

The following statement gives an algebraic criterion to recognise admissible 6-tuples representing decomposable 3-manifolds of genus two.

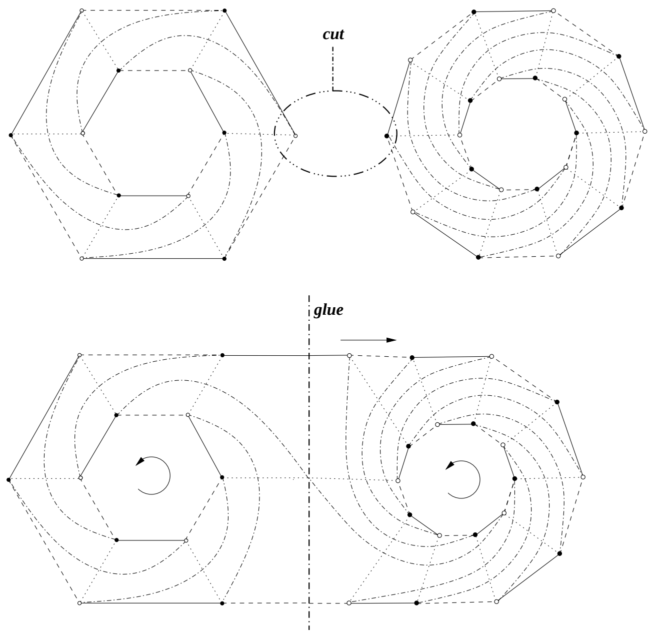

Figure 4.5: A crystallisation representing $\mathcal{L}(3,1)\#\mathcal{L}(5,2)$

Theorem 4.15 *Let f be an admissible 6-tuple. Let $\pi_1(f) = C_1 * C_2$ be a free product of cyclic groups. Then $\mathcal{M} = \mathcal{M}_1\#\mathcal{M}_2$ with $\pi_1(\mathcal{M}_1) = C_1$ and $\pi_1(\mathcal{M}_2) = C_2$. Moreover either $C_i = \mathbb{Z}_p$ and $\mathcal{M}_i \cong \mathcal{L}(p,q)$ for some q coprime to p, or $C_i = \mathbb{Z}$ and $\mathcal{M}_i \cong S_1 \times S_2$, for $i = 1, 2$.*

Proof. First part follows from Theorem 2.2. The second part follows from Theorem 2.12. □

It follows from Theorem 2.1 that fundamental group of an admissible 6-tuple $f = (1, 2p-1, 2r-1, 2s, 2q, 0)$ representing $\mathcal{L}(p,q)\#\mathcal{L}(r,s)$ is $\mathbb{Z}_p * \mathbb{Z}_r$. The parameters of the lens spaces into which the 3-manifold coded by f decomposes can be derived directly from f. Unfortunately, there are admissible 6-tuples with fundamental group isomorphic to a free product $\mathbb{Z}_p * \mathbb{Z}_r$, which are not in the above form. By Theorem 4.15 we deduce that the 3-manifold is a connected sum of lens spaces $\mathcal{L}(p,q)\#\mathcal{L}(r,s)$, for some integers q, s. This time we are not able to compute q and s from the 6-tuple. There are 137 minimal representatives of $\mathcal{G}$-classes up to complexity 21 which fundamental groups are free products of cyclic groups (see Appendix C).

4.7 Acyclic indecomposable fundamental groups

In what follows, we shall analyse the fundamental groups of 3-manifolds represented by admissible 6-tuples from the list obtained in Chapter 3. Using the methods described in Section 4.2 we firstly decompose the input list into the homology classes. Each homology class is analysed case-by-case. The analysis of a homology class begins with a table containing minimal representatives of $\mathcal{G}$-classes (see Chapter 3 and Appendix A) of complexity at most 21 belonging to the considered class as well as the respective fundamental groups obtained by using the algorithm of Gagliardi described in Section 4.1. Since our primary aim is to identify prime 3-manifolds of genus two included in our catalogue, in the first step we exclude from the consideration representatives which fundamental groups are either finite cyclic or free products of finite cyclic groups.

Each homology class is examined in a separate paragraph starting by a table of respective presentations of fundamental groups obtained from 6-tuples by using the Gagliardi's algorithm. Knowing these presentations GAP has been used to test the finiteness of fundamental groups employing Knuth-Bendix rewriting systems related to presented groups [10]. This was done by using a simple script described in Chapter 6. Knowing, that rewriting system is confluent, a finiteness of the group was deduced and it was localised (with some exceptions) in GAP library of small groups. Many isomorphisms between groups within homology classes were set in this way. The rest of groups were tested using low-index subgroups procedure to set possible non-isomorphisms. Finally, ad-hoc methods were used to set isomorphisms between groups with the same lists of low-index subgroups. These derivations are described in the text of respective paragraph describing a homology class. Every paragraph contains in the end the table of representatives of isomorphism classes obtained by using procedure described above. Using substitutions, the generators may change. Thus, the generators "a" and "b" of the presentations in the output table may be different from the original ones. The results are summarised in the last section of this chapter. Many extended triangle groups appeared in all the list of representatives.

In what follows in every homology class we provide a similar procedure to distinguish isomorphism classes of fundamental groups. At first we use low-index subgroups to determine non-isomorphisms in the list. Further, we try to find the isomorphisms between the groups with the same list of low-index subgroups. The following Example gives a detailed insight to the method.

Example 4.16 Let have three groups given by presentations:

$G = \langle a, b \mid ab^{-1}a^{-1}baba^{-1}b^{-1}, a^2b^{-1}a^{-1}ba^{-1}b^{-1}, ab^{-1}a^{-2}b^{-1}ab\rangle,$

$H = \langle a, b \mid ab^{-1}a^{-1}ba^{-1}b^{-1}, a^2b^{-1}a^{-4}b^{-1}, ab^{-1}a^{-3}b^{-1}ab\rangle,$

$K = \langle a, b \mid ab^{-1}a^{-2}b^{-1}ab^{-1}, ab^{-1}a^{-1}b^{-1}, ab^{-1}a^{-2}b^{-2}a^{-2}b^{-1}ab^{-1}abab^{-1}\rangle.$

All of them have the abelianisation isomorphic to $\mathbb{Z}$. Using the GAP script `LINormSubPrint` we checked that the lists of factor groups by low-index normal subgroups up to index 12 are the same for G and H and contain 15 members. The list of respective factors contains 12 members for the group K. It follows $G \ncong K$ and $H \ncong K$. We are to prove a possible isomorphism $G \cong H$. It is done by following way.

Rewrite the presentation of the group G in the generators a and $c = ab^{-1}$. The first relator is $a^2 = cac^{-2}ac$, the second relator is $cac = aca$. The third relator follows from previous two. Rewrite the relators to the form $a^3 = (ac)ac^{-2}ac$ and $(ac)^2 = a(ac)a$. Rewrite the presentation in the generators a and $d = ac$. The second relator is $d^2 = ada$ and the first one is $a^3 = dad^{-1}ad^{-1}ad$. Let us multiply the second relator from left and from right by a. The new form of this relator is $a^5 = adad^{-1}ad^{-1}ada$. Insert now the second relator into the first. The result is $a^5 = dad$. Then left multiply the first relator by a and the second relator from right by d. The final form of the presentation of this group is $G = \langle a, d \mid a^6 = d^3 = (ad)^2\rangle$.

In the case of the group H the rewriting of the presentation in generators a and $c = ab^{-1}$ is useful. The first relator transforms into $c^2 = aca$, the second relator transforms into $a^5 = cac$. The third relator can be derived by using previous two relators. Now right multiply the first relator by c and the second relator from left by a. Now it is easy to derive the presentation $H = \langle a, c \mid a^6 = c^3 = (ac)^2 \rangle$.

In both cases the rewriting process terminates with the same presentation. We claim that this means $G \cong H$. Our argument is based on von Dyck's Theorem [31, p. 51]. In each step the substitution of generators a, b by some words $c = \{a, b, a^{-1}, b^{-1}\}^+$ and $d = \{a, b, a^{-1}, b^{-1}\}^+$ determines an epimorphism ϵ

$$\epsilon : \langle a, b \mid R \rangle \to \langle c, d \mid Q \rangle$$

from $\langle a, b \rangle$ onto $\langle c, d \rangle \leq \langle a, b \rangle$. To see that ϵ is an automorphism it is enough to express the generators a and b in terms of c and d. In the above rewriting process this can be easily seen. The automorphism $G \to H$ can be obtained by composing of the sequence of the particular substitutions. In a few cases where the substitution are more complex (i.e. in homology class $\mathbb{Z}_5 \times \mathbb{Z}_5$) we define an inverse epimorphism as well proving that the considered are isomorphic. Note also that explicit epimorphisms are constructed by GAP in all cases of finite groups.

Now let us solve the group K. Rewrite the second relator in the presentation of the group K to the form $a = bab$ to get the equation $a = b$. One can easily see that $K \cong \mathbb{Z}$, since substituting a to other relators gives no new relation for a.

Finally we get two isomorphism classes – $\mathbb{Z}$ and $\langle a, b \mid a^6 = b^3 = (ab)^2 \rangle$.

4.7.1 Finite homology groups

Homology class $H_1 = 1$

Table 4.1: Original relators for $H_1 = 1$

No.	f	G
1	(3, 7,11, 4, 6, 0)	$\langle a, b \mid ab^{-3}a^3, ab^{-5}a^6, ab^{-2}a^2\rangle$
2	(3, 5, 9, 4, 4,12)	$\langle a, b \mid ab^{-2}ab, b^{-1}a^4b^{-1}a^{-1}\rangle$
3	(4, 6,10, 3, 5,15)	$\langle a, b \mid ab^{-2}ab, b^{-1}a^4b^{-1}a^{-1}\rangle$
4	(3, 5,13, 4, 4,16)	$\langle a, b \mid ab^{-2}ab, b^{-1}a^6b^{-1}a^{-1}\rangle$
5	(4, 4, 8, 3, 3, 7)	$\langle a, b \mid ab^{-1}a^{-1}ba^{-1}b^{-1}, b^3ab^{-2}a\rangle$
6	(4, 4,12, 3, 3,11)	$\langle a, b \mid b^{-1}aba^{-1}ba, bab^{-1}a^{-4}b^{-1}a\rangle$
7	(5, 5, 5, 4, 4, 4)	$\langle a, b \mid ab^{-1}a^{-1}b^{-1}ab, b^{-1}a^{-1}b^3a^{-1}b^{-1}\rangle$
8	(5, 7, 7, 6, 6, 4)	$\langle a, b \mid a^{-1}b^{-1}abab^{-1}, b^{-1}a^{-1}b^4a^{-1}b^{-1}a\rangle$
9	(5, 5, 9, 4, 4, 8)	$\langle a, b \mid ab^{-1}a^{-1}ba^{-1}b^{-1}a, a^2ba^{-1}b^{-1}a^{-1}ba\rangle$
10	(7, 7, 7, 6, 6, 6)	$\langle a, b \mid ab^{-1}a^{-1}baba^{-1}b^{-1}a, ab^{-1}a^{-1}b^2a^{-1}b^{-1}ab\rangle$

The group G_9 is the only trivial group in the list.

The orders of fundamental groups G_1, G_2, G_3 and G_7 are all 120. The mutual isomorphism of these groups can be checked by using GAP. This group was recognised as `[120,5]`. The structure of this group can be described as follows. The center of this group is isomorphic to $\mathbb{Z}_2$ and factor $G/\zeta(G)$ is isomorphic to the alternating group on five elements. Hence, the group is a central extension of $\mathbb{Z}_2$ by A_5. The canonical presentation of the group can be obtained i.e. from the presentation of G_3. Rewrite relators into the form $b^2 = aba$ and $a^4 = bab$. Then right multiply the first relator by b and left multiply the second relator by a. Thus the presentation of the group is $\langle a, b \mid a^5 = b^3 = (ab)^2\rangle$.

Let us examine the presentation of the group G_4. At first rewrite the presentation in generators $c = a$ and $d = a^2b^{-1}$. Note that c and d generates the group G_4. The first relator turns into $d^2c^{-1}d^{-1}c^{-1} = 1$ and further simplifies into $d^3 = (cd)^2$. Second relator changes into the form $c^2d^{-1}c^{-1}dc^{-1}d^{-1} = 1$. Using the first relator we derive $d^{-1}c^{-1} = cd^{-2}$. Substituting it into the second relator we get $c^6d^{-1}c^{-1}d^{-1} = 1$. Hence $c^7 = (cd)^2$. Finally, we rewrite everything in the letters a and b to get the presentation $G_4 = \langle a, b \mid a^7 = b^3 = (ab)^2\rangle$.

$\mathbf{G_5 \cong G_4}$. To recognise the type of G_5 we rewrite the presentation in the generators b and $c = ab$. From the first relation we get $c^{-1}b^{-1}c^2b = 1$, the second one transforms into the form $c^{-1}b^5c^{-1}b^{-1}cb^{-1} = 1$. Rewrite the first relator to the form $c^{-1}b^{-1} = bc^{-2}$ and substitute it twice into the second relator to get $c^{-1}b^7c^{-2} = 1$. Rewrite the fist relator to the form $c^2 = bcb$ and by right multiplication by c we obtain $c^3 = (bc)^2$. The assignment $\phi : b \mapsto a,\ c \mapsto b$ extends to an isomorphism $G_5 \cong G_4$, since the generators are mapped onto the generators and the relations are preserved.

$\mathbf{G_6 \cong G_4}$. To examine G_6 we first rewrite the relations in the generators a and $c = b^{-1}a$. The relations transform into $ac^{-2}ac = 1$ and $cac^{-1}aca^{-5} = 1$. Rewriting the first relation into the form $cac^{-1} = a^{-1}c$ and substituting it into the second one we get $caca^{-6} = 1$. It follows that $c^2 = aca$ and $a^6 = cac$. The assignment $\phi : a \mapsto a, c \mapsto b$ extends to a group isomorphism $G_6 \cong G_4$, since it maps the generators onto the generators and the relations are preserved.

$\mathbf{G_8 \cong G_4}$. An isomorphism of these groups can be easily checked by rewriting the relators of G_8 into the form $b^2 = aba$ for the first one, and $a^6 = bab$ for the second one. The isomorphism $G_8 \cong G_4$ can be easily set by taking $\phi : a \mapsto a, b \mapsto b$.

$\mathbf{G_{10} \cong G_4}$. The situation is more complicated in the case of the 6-tuple $(7, 7, 7, 6, 6, 6)$. It is not easy to define a group isomorphism proving $G_{10} \cong G_4$. However, it is noted in [6] that the 3-manifold represented by this 6-tuple is homeomorphic to the manifold represented by the 6-tuple $(4, 4, 12, 3, 3, 11)$: "The $\Gamma(7, 7, 7; 6, 6, 6)$ represents the 2-fold covering space S^3 branched over the torus link $\{3, 7\}$. The same 3-manifold is also represented by $\Gamma(4, 4, 12; 5, 3, 9)$, as the 2-fold covering space of S^3 branched over the knot K_1'" [6]. Since $(4, 4, 12, 5, 3, 9)$ is equivalent to $(4, 4, 12, 3, 3, 11)$ (one can see it applying the σ-operator defined in [17]), it follows that the fundamental groups $G_{10} \cong G_6 \cong G_4$.

It transpires that the homology class $H_1 = 1$ consists of the following three isomorphism classes of fundamental groups:

Table 4.2: Isomorphism classes in $H_1 = 1$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(3, 7,11, 6, 0,12)	1	1	1	1
(5, 5, 5, 4, 4, 4)	$\langle a, b \mid a^5 = b^3 = (ab)^2 \rangle$	$\mathbb{Z}_2$	A_5	4
(5, 5, 9, 4, 4, 8)	$\langle a, b \mid a^7 = b^3 = (ab)^2 \rangle$	$\mathbb{Z}$	$\Delta^+(7, 3, 2)$	5

Homology class $H_1 = \mathbb{Z}_2$

Table 4.3: Original relators for $H_1 = \mathbb{Z}_2$

No.	f	G
1	(4, 4, 6, 3, 3, 5)	$\langle a, b \mid b^2ab^{-2}a, bab^{-2}ab, ab^{-1}a^{-1}ba^{-1}b^{-1} \rangle$
2	(3, 5, 7, 4, 4,10)	$\langle a, b \mid ab^{-2}ab, b^{-1}a^3b^{-1}a^{-1} \rangle$
3	(4, 6, 8, 3, 5,13)	$\langle a, b \mid ab^{-2}ab, b^{-1}a^3b^{-1}a^{-1} \rangle$
4	(4, 6, 8, 5, 5,11)	$\langle a, b \mid ab^{-3}ab, ab^{-1}a^{-1}b^{-1}a^3 \rangle$
5	(3, 7, 9, 4, 4,14)	$\langle a, b \mid ab^{-3}ab, ab^{-1}a^{-1}b^{-1}a^3 \rangle$
6	(4, 6,10, 5, 3,15)	$\langle a, b \mid ab^{-2}ab, ab^{-1}a^{-1}b^{-1}a^2 \rangle$
7	(6, 6, 8, 3, 3, 9)	$\langle a, b \mid ab^{-1}ab^2, a^{-1}b^{-1}ab^{-1}a^{-2} \rangle$
8	(6, 6, 8, 3, 7, 7)	$\langle a, b \mid aba^{-2}ba, ab^{-1}a^{-1}bababa^{-1}b^{-1} \rangle$
9	(5, 5,11, 4, 4,10)	$\langle a, b \mid ab^{-1}a^{-1}ba^{-1}b^{-1}a, a^3ba^{-1}b^{-1}a^{-1}ba \rangle$
10	(5, 7, 9, 2, 2,12)	$\langle a, b \mid ab^{-1}ab^2, ab^{-1}a^{-3}b^{-1} \rangle$
11	(5, 7, 9, 6, 6,12)	$\langle a, b \mid ab^{-1}a^{-1}b^{-1}ab, a^2ba^{-6}b \rangle$

The groups G_1, G_2, G_3, G_6, G_7 and G_{10} are isomorphic to the group, which is an extension of $\mathbb{Z}_2$ by S_4. The result comes from GAP. The presentation of the group can be derived e.g. from the presentation of G_2. Rewrite the fist relator into the form $b^2 = aba$ and the second one into the form $a^3 = bab$. Let us now right multiplicate the first relation by b and left multiplicate the second relation by a to get the following presentation of the group $G_2 = \langle a, b \mid a^4 = b^3 = (ab)^2 \rangle \cong \Delta(4, 3, 2)$.

$\mathbf{G_5 \cong G_4}$. The isomorphism of the groups G_4 and G_5 is clear since the groups have the same presentation. The relators can be rewritten to a simpler form as follows. Rewrite the first relator into the form $b^3 = aba$ and the second one to the form $a^4 = bab$. Multiplicate the relators b from right or a form left respectively. The resulting group is $G_4 = \langle a, b \mid a^5 = b^4 = (ab)^2 \rangle \cong \Delta(5, 4, 2)$.

$\mathbf{G_8 \cong G_4}$. In the case of the group G_8 rewrite the presentation in generators a and $c = ab$. The first relator is $a^3 = cac$ after substitution, the second $a^2c^{-1}a^{-1}c^3a^{-1}c^{-1} = 1$. The first relator can be written in the following forms: $a^{-3}c = c^{-1}a^{-1}$ and $ca^{-3} = a^{-1}c^{-1}$. Let us substitute these forms of the first relator into the second relator at the appropriate places. We get $a^4 = c^5$ as the second relation. Let us do right multiplication of the first relation by a to get $c^5 = a^4 = (ca)^2$. The isomorphism $G_8 \cong G_4$ is given by setting $\phi : c \mapsto a, a \mapsto b$.

$\mathbf{G_{11} \cong G_9}$. Rewriting of presentation in generators a and $c = ab$ helps to analyse the presentation of the group G_{11}. After the substitution we get the relations $c^2 = aca$ and $a^7 = cac$. By right multiplication of the first relation by c and by the left multiplication of the second relation by a we get the presentation $G_{11} = \langle a, b \mid a^8 = b^3 = (ab)^2 \rangle \cong \Delta(8, 3, 2)$.

Let us rewrite the relators of G_9 at first. From the second relation $a^4ba^{-1}b^{-1}a^{-1}b = 1$ we get $ba^{-1}b^{-1} = a^{-4}b^{-1}a$. Inserting it into the first relation we have $a^2b^{-1}a^{-1}a^{-4}b^{-1}a = 1$. This relation reduces to $b^{-1}a^3b^{-1} = a^5$. Set $c = b^{-1}a^3$. Clearly $G_9 = \langle a, c \rangle$. From previous step we get $c^2 = a^8$. Rewriting the first relation in generators a, c we derive $a^{-1}ca^{-1}c^{-1}a^{-1}c = 1$. It follows $(a^{-1}c)^2a^{-1}c^{-1} = 1$. By right multiplication by c^2 we obtain $(a^{-1}c)^3 = c^2$. Hence the group G_9 has presentation

$$\Delta(8, 3, 2) \cong G_9 = \langle a, c \mid a^8 = (a^{-1}c)^3 = c^2 \rangle \cong G_{11}.$$

It follows that the homology class $\mathbb{Z}_2$ is a union of the following three isomorphism classes.

Table 4.4: Isomorphism classes in $H_1 = \mathbb{Z}_2$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 4, 4, 6, 3, 3, 5)`	$\langle a, b \mid a^4 = b^3 = (ab)^2 \rangle$	$\mathbb{Z}_2$	S_4	6
`( 4, 6, 8, 5, 5,11)`	$\langle a, b \mid a^5 = b^4 = (ab)^2 \rangle$	$\mathbb{Z}$	$\Delta^+(5, 4, 2)$	3
`( 5, 5,11, 4, 4,10)`	$\langle a, b \mid a^8 = b^3 = (ab)^2 \rangle$	$\mathbb{Z}$	$\Delta^+(8, 3, 2)$	2

Homology class $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$

Table 4.5: Original relators for $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$

No.	f	G
1	`( 3, 3, 3, 2, 2, 2)`	$\langle a, b \mid b^2a^2, ba^2b, ab^{-1}a^{-1}b^{-1} \rangle$
2	`( 3, 3, 7, 2, 2, 6)`	$\langle a, b \mid b^{-1}aba, b^{-1}a^{-4}b^{-1} \rangle$
3	`( 4, 4, 6, 3, 3, 3)`	$\langle a, b \mid b^2a^2, ab^{-1}a^{-1}b^{-1} \rangle$
4	`( 5, 5, 5, 2, 2, 6)`	$\langle a, b \mid ba^2b, ab^{-1}a^{-1}b^{-1} \rangle$
5	`( 3, 3,11, 2, 2,10)`	$\langle a, b \mid b^{-1}aba, b^{-1}a^{-6}b^{-1} \rangle$
6	`( 4, 4,10, 1, 5, 9)`	$\langle a, b \mid b^{-1}aba, b^{-1}a^{-4}b^{-1} \rangle$
7	`( 4, 6, 8, 5, 7,11)`	$\langle a, b \mid ab^{-1}ab, b^{-1}a^{-1}b^{-1}a^3 \rangle$
8	`( 5, 5, 9, 2, 2,10)`	$\langle a, b \mid ab^{-1}ab, b^{-1}a^{-4}b^{-1} \rangle$
9	`( 5, 5, 9, 4, 4, 4)`	$\langle a, b \mid b^2a^2, ab^{-1}a^{-1}b^{-1} \rangle$
10	`( 4, 6,10, 5, 5,13)`	$\langle a, b \mid ab^{-3}ab, aba^{-5}b \rangle$
11	`( 6, 6, 8, 3, 7, 3)`	$\langle a, b \mid b^2a^2, ab^{-1}a^{-1}b^{-1} \rangle$
12	`( 6, 6, 8, 5, 3, 5)`	$\langle a, b \mid b^2a^2, ab^{-1}a^{-1}b^{-1} \rangle$
13	`( 3, 3,15, 2, 2,14)`	$\langle a, b \mid b^{-1}aba, b^{-1}a^{-8}b^{-1} \rangle$
14	`( 3, 7,11, 4, 4,16)`	$\langle a, b \mid ab^{-3}ab, b^{-1}a^5b^{-1}a^{-1} \rangle$

The groups G_1, G_3, G_4, G_9, G_{11} and G_{12} were recognised by the GAP to be the same. The group to which they are isomorphic is `[8,4]`. This group has the center $\mathbb{Z}_2$ and the Abelian factor by the center is of size 4. There is only one such a group – the group of quaternions Q_8 [8, p. 19].

The fundamental groups G_2, G_6, G_7 and G_8 were recognised to be isomorphic to `[16,9]`. It has the center $\mathbb{Z}_2$. The presentation can be computed i.e. for the 6-tuple `( 3, 3, 7, 2, 2, 6)`. It has the fundamental group $G = \langle a, b \mid b^{-1}aba = 1, b^{-1}a^4b^{-1} = 1 \rangle$. From the second relation of presentation we have $a^{-4} = b^2$. From $b^{-1}aba = b^{-1}a^4b^{-1}$ we get $(ab)^2 = a^{-4}$. Providing the derivations

$$\begin{aligned} abab &= a^{-4} = b^2 \\ aba &= b \\ b^{-1}ab &= a^{-1} \end{aligned}$$

we get a tool to prove $a^4 = b^{-1}b^2b = b^{-1}a^4b = (b^{-1}ab)^4 = a^{-4}$. Finally, $G_2 = \langle a, b \mid a^4 = b^2 = (ab)^2\rangle$, which is the group Q_{16}[8, p. 19].

The analysis of the groups G_5 and G_{13} can be done in the similar way as it was shown in previous paragraph. We get the fundamental groups $G_5 = \langle a, b \mid a^6 = b^2 = (ab)^2\rangle \cong Q_{24}$ and $G_{13} = \langle a, b \mid a^8 = b^2 = (ab)^2\rangle \cong Q_{32}$ respectively.

$\mathbf{G_{14} \cong G_{10}}$. Since the presentations of groups G_{10} and G_{14} are the same, they can be dealt together. Rewrite the first relation to the form $b^3 = aba$ and the second relation to the form $a^5 = bab$. The second step is to right multiply the first relation by b and left multiply the second relation by a. We get the presentation $G_{10} = \langle a, b \mid a^6 = b^4 = (ab)^2\rangle \cong \Delta(6, 4, 2)$.

Summarising we have the following five isomorphism classes in the homotopy class $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Table 4.6: Isomorphism classes in $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 3, 3, 3, 2, 2, 2)`	$\langle a, b \mid a^2 = b^2 = (ab)^2\rangle$	$\mathbb{Z}_2$	D_4	6
`( 3, 3, 7, 2, 2, 6)`	$\langle a, b \mid a^4 = b^2 = (ab)^2\rangle$	$\mathbb{Z}_2$	D_8	4
`( 3, 3,11, 2, 2,10)`	$\langle a, b \mid a^6 = b^2 = (ab)^2\rangle$	$\mathbb{Z}_2$	D_{12}	1
`( 3, 3,15, 2, 2,14)`	$\langle a, b \mid a^8 = b^2 = (ab)^2\rangle$	$\mathbb{Z}_2$	D_{16}	1
`( 4, 6,10, 5, 5,13)`	$\langle a, b \mid a^6 = b^4 = (ab)^2\rangle$	$\mathbb{Z}$	$\Delta^+(6, 4, 2)$	2

Homology class $H_1 = \mathbb{Z}_3$

Table 4.7: Original relators for $H_1 = \mathbb{Z}_3$

No.	f	G
1	`( 4, 4, 4, 3, 3, 3)`	$\langle a, b \mid ab^{-1}a^{-1}b^{-1}a, ab^{-2}ab\rangle$
2	`( 3, 5, 5, 4, 4, 2)`	$\langle a, b \mid ab^{-1}a^{-1}b^{-1}a, ab^{-2}ab\rangle$
3	`( 4, 6, 6, 3, 5,11)`	$\langle a, b \mid b^{-1}a^2b^{-1}a^{-1}, ab^{-2}ab\rangle$
4	`( 6, 6, 6, 3, 3, 7)`	$\langle a, b \mid ab^{-1}a^{-2}b^{-1}, ab^{-1}ab^2\rangle$
5	`( 5, 7, 7, 2, 2,10)`	$\langle a, b \mid ab^{-1}a^{-2}b^{-1}, ab^{-1}ab^2\rangle$
6	`( 5, 7, 7, 4, 6,12)`	$\langle a, b \mid ab^{-3}a^3, ab^{-2}ab^{-2}ab\rangle$
7	`( 4, 6,10, 7, 5, 3)`	$\langle a, b \mid ab^{-2}ab, ab^{-1}a^{-1}b^{-1}a\rangle$
8	`( 6, 6, 8, 1, 5,11)`	$\langle a, b \mid ba^{-1}b^{-1}a^{-1}b, a^2b^{-1}a^{-1}b^{-1}\rangle$
9	`( 5, 5,11, 2, 6,10)`	$\langle a, b \mid ab^{-1}ab^{-1}a^{-3}b^{-1}, ab^{-2}ab^{-1}a^{-1}ba^{-1}b^{-1}\rangle$
10	`( 7, 7, 7, 2, 8, 8)`	$\langle a, b \mid ab^{-1}ab^{-1}a^{-2}b^{-1}, a^{-1}babababa^{-1}\rangle$
11	`( 7, 7, 7, 4, 6, 6)`	$\langle a, b \mid ba^{-1}b^{-1}a^{-1}b, ab^{-1}a^{-1}b^{-1}a\rangle$

Most of groups in the subclass, namely G_1, G_2, G_3, G_4, G_5, G_7, G_8 and G_{11} were identified by the GAP to be isomorphic to `[24,3]`. This group has the center $\mathbb{Z}_2$ and a factor by the center of size 12. This factor is isomorphic to the alternating group A_4. The presentation of this group can be easily derived from the presentation of G_1. Rewrite the relators into the form $a^2 = bab$ and $b^2 = aba$. Left multiply the first relator by a and right multiply the second relator by b. Thus $G_1 = \langle a, b \mid a^3 = b^3 = (ab)^2\rangle \cong \Delta(3, 3, 2)$.

The presentation of the group G_6 contains the relation $a^4 = b^3$. Thus we are about to rewrite the second relation. Right multiplication by b^{-3} helps. The presentation is $G_6 = \langle a, b \mid a^4 = b^3 = (ab^{-2})^{-3}\rangle$.

$\mathbf{G_9 \cong G_6}$. The presentation of the group G_9 with generators a and $c = ab^{-1}$ is more suitable to determine the fundamental group. First relation becomes $a^4 = c^3$. The second relation can be derived as follows. In the generators a and c we get $(c^2a^{-1})^2c^{-1}a^{-1} = 1$. It follows that

$$\begin{aligned}(c^2a^{-1})^2c^{-1}a^{-1} &= 1,\\ (ac^{-2})^{-2} &= ac,\\ (ac^{-2})^{-3} &= c^3.\end{aligned}$$

Thus the presentation of G_9 is $\langle a, b \mid a^4 = c^3 = (ac^{-2})^{-3}$. Set the isomorphism $G_9 \cong G_6$ to be $\phi : a \mapsto a, c \mapsto b$.

$\mathbf{G_{10} \cong G_6}$. In the case of the group G_{10} we rewrite the presentation in the generators a and $c = ab$. New relators are $a^3 = c^4$ and $(a^2c^{-1})^2 = ca(a^{-3}a^3) = (ca^{-2})a^3$. The desired isomorphism $G_{10} \cong G_6$ is set by the map $\phi : c \mapsto a, a \mapsto b$.

The homology class contains only two different isomorphism classes of fundamental groups.

Table 4.8: Isomorphism classes in $H_1 = \mathbb{Z}_3$

f	G	$\zeta(G)$	$G/\zeta(G)$	Id
`( 4, 4, 4, 3, 3, 3)`	$\langle a, b \mid a^3 = b^3 = (ab)^2\rangle$	$\mathbb{Z}_2$	A_4	8
`( 5, 7, 7, 4, 6,12)`	$\langle a, b \mid a^4 = b^3 = (ab^{-2})^{-3}\rangle$	$\mathbb{Z}$	$\Delta^+(4, 3, 3)$	3

Homology class $H_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$

Table 4.9: Original relators for $H_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$

No.	f	G
1	`( 4, 6, 6, 1, 1, 9)`	$\langle a, b \mid a^3b^3, ab^2ab^{-1}ab^{-1}\rangle$
2	`( 5, 5, 7, 2, 6, 2)`	$\langle a, b \mid ab^{-3}a^2, ab^{-1}ab^{-1}a^{-2}b^{-1}\rangle$
3	`( 4, 8, 8, 5, 7, 3)`	$\langle a, b \mid ab^{-3}a^2, a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^2\rangle$
4	`( 5, 7, 9, 4, 6, 2)`	$\langle a, b \mid ab^{-3}a^2, ab^{-1}ab^{-1}a^{-2}b^{-1}\rangle$

In the case of the group G_1 rewriting the relators into the form $a^3 = b^{-3}$ and $(ab^{-1})^2ab^2 = 1$ helps. Let us right multiply the second relator by b^{-3} to get $a^3 = b^{-3} = (ab^{-1})^3$. Now choose new generators a and $c = b^{-1}$ and rewrite the presentation into the form $G_1 = \langle a, c \mid a^3 = c^3 = (ac)^3\rangle \cong \Delta(3, 3, 3)$.

$\mathbf{G_3 \cong G_1}$. In the case of the group G_3 rewrite the presentation with the generators $c = a^{-1}$ and $d = b^{-1}$. We have $c^3 = d^3$ and $(cd)^2cd^{-2} = 1$. To get final form of the presentation, right multiply the second relator by d^3. The isomorphism $G_3 \cong G_1$ can be set by defining the mapping $\phi : c \mapsto a, d \mapsto c$.

$\mathbf{G_4 \cong G_2 \cong G_1}$. Groups G_2 and G_4 have the same presentation. The first relator is $a^3 = b^3$, the second relator can be rewritten into the form $a^{-2}b^{-1}(ab^{-1}) = 1$. Simply left multiplicate the second relator by b^3. Now let us rewrite the presentation of G_2 in the generators $c = ab^{-1}$ and b. The isomorphism $G_2 \cong G_1$ is obvious by setting $\phi : c \mapsto a, b \mapsto b$.

Table 4.10: Isomorphism classes in $H_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 4, 6, 6, 1, 1, 9)`	$\langle a, b \mid a^3 = b^3 = (ab)^3\rangle$	$\mathbb{Z}$	$\Delta^+(3, 3, 3)$	4

Homology class $H_1 = \mathbb{Z}_4$

Table 4.11: Original relators for $H_1 = \mathbb{Z}_4$

No.	f	G
1	`( 3, 3, 5, 2, 2, 4)`	$\langle a, b \mid ab^{-1}ab, b^{-1}a^{-3}b^{-1}\rangle$
2	`( 4, 4, 6, 1, 5, 5)`	$\langle a, b \mid b^{-1}aba, b^{-1}a^{-3}b^{-1}\rangle$
3	`( 3, 3, 9, 2, 2, 8)`	$\langle a, b \mid b^{-1}aba, b^{-1}a^{-5}b^{-1}\rangle$
4	`( 4, 6, 6, 3, 5, 3)`	$\langle a, b \mid ab^{-1}a^{-1}b^{-1}, b^3a^2\rangle$
5	`( 5, 5, 7, 2, 2, 8)`	$\langle a, b \mid ab^{-1}ab, b^{-1}a^{-3}b^{-1}\rangle$
6	`( 4, 4,10, 3, 3, 7)`	$\langle a, b \mid b^{-1}aba, b^{-1}a^{-3}b^{-1}\rangle$
7	`( 6, 6, 6, 1, 5, 9)`	$\langle a, b \mid a^2b^{-2}, a^{-1}ba^{-1}ba^{-1}b^{-1}\rangle$
8	`( 3, 3,13, 2, 2,12)`	$\langle a, b \mid b^{-1}aba, b^{-1}a^{-7}b^{-1}\rangle$
9	`( 4, 6,10, 5, 7,13)`	$\langle a, b \mid ab^{-1}ab, b^{-1}a^{-1}b^{-1}a^4\rangle$
10	`( 6, 6, 8, 3, 5, 7)`	$\langle a, b \mid b^{-1}aba, b^{-1}a^{-3}b^{-1}\rangle$
11	`( 5, 5,11, 2, 2,12)`	$\langle a, b \mid ab^{-1}ab, b^{-1}a^{-5}b^{-1}\rangle$
12	`( 5, 5,11, 2, 6, 8)`	$\langle a, b \mid b^{-1}aba, b^{-1}a^{-3}b^{-1}\rangle$
13	`( 5, 7, 9, 4, 6, 4)`	$\langle a, b \mid ab^{-1}a^{-1}b^{-1}, b^3a^2\rangle$
14	`( 5, 7, 9, 6, 4, 0)`	$\langle a, b \mid b^{-1}a^{-1}ba^{-1}, b^{-1}a^{-1}b^{-1}a^2\rangle$
15	`( 7, 7, 7, 2, 6,10)`	$\langle a, b \mid ab^{-2}ab^{-1}a^{-3}b^{-1}, ab^{-1}a^{-1}ba^2ba^{-1}b^{-1}\rangle$

The first subclass was determined by GAP to be the group `[12,1]`. The center is $\mathbb{Z}_2$ and the factor of group by its center is S_3, which can be easily checked in GAP. The fundamental group is some extension of $\mathbb{Z}_2$ by the group S_3. Actually it is easy to derive the presentation of the group $\Delta(3, 2, 2)$ in this case. Rewrite the presentation of G_1 in generators $c = a^{-1}$ and b. The relators are $b = cbc$ and $c^3 = b^2$. Right multiply the first relator by b. The groups belonging into this subclass are G_1, G_2, G_4, G_5, G_6, G_7, G_{10}, G_{12}, G_{13} and G_{14}.

The groups G_3, G_9 and G_{11} are isomorphic to `[20,1]`. Similarly, the center is $\mathbb{Z}_2$. The factor of this group by its center is dihedral group D_{10}. The presentation of the group $\Delta(5, 2, 2)$ was derived in the similar way as in previous isomorphism class.

The fundamental group G_8 can be determined in the similar way as extension of $\mathbb{Z}_2$ by D_{14}. The GAP recognised it as the group `[28,1]`. The presentation of the group $\Delta(7, 2, 2)$ was derived in the similar way as in previous isomorphism classes.

Rewrite the relators of G_{15} in the generators a and $c = ab^{-1}$. The first relator is in the form $ca^{-1}c^2a^{-4}c = 1$ and it is useful to rewrite it into the form $c^2a^{-1}c^2 = a^4$. The second relator is in the form $ca^{-1}c^{-1}a^3c^{-1}a^{-1}c = 1$ after the substitution. Rewrite this relator to the form $a^{-1}c^{-1}a^3c^{-1}a^{-1} = c^{-2}$. Set $d = c^2a^{-1}$ and extend the relator to the form $c^{-1}a^3c^{-3}c^2a^{-1} = ac^{-2}$. Substitute d into the last form: $c^{-1}a^3c^{-3} = d^{-2}$. Use the first relator to express a^3 in the second relator: $c^{-1}(a^{-1}c^2a^{-1}c^2)c^{-2} = d^{-2}$ and extend the relator into the form $c^{-3}c^2a^{-1}dc^{-1} = d^{-2}$. Derive the final form of the second relator $c^{-3}d^2c^{-1} = d^{-2}$. It is easy to derive $d^2 = a^3$ from the first relator. Now, get back the generator a into the second relator: $c^{-3}a^3c^{-1} = a^{-3}$. Rewrite the relator to the final form: $c^3 = a^3c^{-1}a^3$.

Mutual non-isomorphism of these four groups can be checked in GAP by using low index subgroups procedures. Proof can be done by comparing the numbers of conjugacy classes of subgroups up to index 12. We refer to Section 4.3 and Appendix C. Using the procedure `LISubNum` we get the following results. There is 6 such conjugacy classes in G_1, 5 conjugacy classes in G_3, 4 conjugacy classes in G_8 and 37 conjugacy classes in G_{15}.Thus we have four isomorphism classes of fundamental groups in the homology class $\mathbb{Z}_4$.

Table 4.12: Isomorphism classes in $H_1 = \mathbb{Z}_4$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 3, 3, 5, 2, 2, 4)`	$\langle a, b \mid a^3 = b^2 = (ab)^2 \rangle$	$\mathbb{Z}_2$	D_6	10
`( 3, 3, 9, 2, 2, 8)`	$\langle a, b \mid a^5 = b^2 = (ab)^2 \rangle$	$\mathbb{Z}_2$	D_{10}	3
`( 3, 3,13, 2, 2,12)`	$\langle a, b \mid a^7 = b^2 = (ab)^2 \rangle$	$\mathbb{Z}_2$	D_{14}	1
`( 7, 7, 7, 2, 6,10)`	$\langle a, b \mid a^4 = b^2a^{-1}b^2, b^3 = a^3b^{-1}a^3 \rangle$	?	?	1

Homology class $H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$

Table 4.13: Original relators for $H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$

No.	f	G
1	`( 5, 5, 5, 2, 2, 2)`	$\langle a, b \mid ab^{-2}a^{-1}b^{-2}, ab^{-1}a^2ba \rangle$
2	`( 5, 5, 7, 2, 4, 2)`	$\langle a, b \mid b^{-1}a^2ba^2, a^3b^{-2}ab^{-2} \rangle$
3	`( 5, 5, 9, 4, 8, 4)`	$\langle a, b \mid ab^{-1}a^2ba, b^{-1}a^{-3}b^{-2}a^{-1}b^{-1} \rangle$

Rewrite the relators of G_1 in the following way. The first relator rewrite to $a = b^2ab^2$ and left multiply to get $a^2 = (ab^2)^2$. The second relator rewrite to $b = a^2ba^2$ and right multiply to get $b^2 = (a^2b)^2$. Hence the presentation of the group G_1 is $\langle a, b \mid a^2 = (ab^2)^2, b^2 = (a^2b)^2 \rangle$.

$\mathbf{G_3 \cong G_2}$. Set the isomorphism $G_2 \cong G_3$ by map $\phi : a \mapsto a, b \mapsto b^{-1}$. It is easy to see that relations are mapped onto the relations. In fact, rewrite the first relator to get $b = a^2ba^2$ and right multiply by b to get $b^2 = (a^2b)^2$. The second relator derive to the form $a^3 = b^2a^{-1}b^2$ and left multiply by a^{-1} to get $a^2 = (a^{-1}b^2)^2$. Thus $G_2 = \langle a, b \mid a^2 = (a^{-1}b^2)^2, b^2 = (a^2b)^2 \rangle$

The groups G_1 and G_2 are not isomorphic. To prove it, let us browse the normal subgroups of low index of the respective groups. The group G_1 contains three normal subgroups of index 10, while the group G_2 has only two such subgroups.

Remark.. Note that the group G_1 is isomorphic to a subgroup of the group of isometries of E^3. Set $\alpha : [x, y, z] \mapsto [x + 1, -y, -z]$ and $\beta : [x, y, z] \mapsto [-x, y + 1, 1 - z]$. The isomorphism can be set by mapping $\psi : a \mapsto \alpha$, $b \mapsto \beta$. Using this identification one can prove that the respective 3-manifold is Euclidean. Similarly, the group $G_3 \cong G_2$ is isomorphic to a subgroup of the group of affine transformations of E^3. Let $\alpha : [x, y, z] \mapsto [x - 2y + 1, -y, -z]$ and $\beta : [x, y, z] \mapsto [-x, y + 1, 1 - z]$ be transformations of the Euclidean space E^3. The embedding of G_3 into the group of transformations can be set by mapping $\phi : a \mapsto \alpha$, $b \mapsto \beta$. However, one can prove that the 3-manifold represented by $(5, 5, 7, 2, 4, 2)$ is not Euclidean.

Table 4.14: Isomorphism classes in $H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 5, 5, 5, 2, 2, 2)`	$\langle a, b \mid a^2 = (ab^2)^2, b^2 = (a^2b)^2 \rangle$	?	?	1
`( 5, 5, 7, 2, 4, 2)`	$\langle a, b \mid a^2 = (a^{-1}b^2)^2, b^2 = (a^2b)^2 \rangle$	?	?	2

Homology class $H_1 = \mathbb{Z}_5$

Table 4.15: Original relators for $H_1 = \mathbb{Z}_5$

No.	f	G
1	(4, 8, 8, 5, 5,13)	$\langle a,b \mid ab^{-4}ab, ab^{-1}a^{-1}b^{-1}a^3\rangle$
2	(3, 9, 9, 4, 4,16)	$\langle a,b \mid ab^{-4}ab, ab^{-1}a^{-1}b^{-1}a^3\rangle$
3	(5, 5,11, 4, 8, 4)	$\langle a,b \mid ab^{-1}a^{-1}b^{-1}a^2ba, a^2ba^{-2}ba^2b^{-1}\rangle$
4	(7, 7, 7, 4, 4, 8)	$\langle a,b \mid b^{-1}a^{-1}baba^{-1}, b^{-1}a^{-1}b^2a^{-1}b^{-1}ab^{-1}a^{-2}b^{-1}\rangle$

$\mathbf{G_2} \cong \mathbf{G_1}$. Since the relators of G_1 and G_2 are the same, the resulting relators for both groups are easy to obtain e.g. from relators of G_1. Rewrite the relators into the form $a^4 = bab$ and $b^4 = aba$ and multiply the first relator by a from left and the second one by b from right.

Rewrite the presentation of G_3 into the form $G_3 = \langle a,b \mid bab = a^2ba^2, ba^{-2}b = a^{-2}ba^{-2}\rangle$ and rewrite it in generators a and $c = a^{-2}b$. The presentation turns into $G_3 = \langle a,c \mid ca^5c = a^2, a^2c^2 = ca^{-2}\rangle$. Now substitute $d = a^2c^{-1}$. We get a presentation $G_3 = \langle a,d \mid d^{-1}a^7d^{-1}a^2 = a^2, a^2d^{-1}a^2d^{-1}a^2 = d^{-1}\rangle$. The first relator gives rise to the relator $a^7 = d^2$. The second relator can be rewritten into the form $(da^{-2})^3 = d^2$. By using Lemma 4.8 the final form of presentation can be set.

In the case of the group G_4 the we first rewrite the relators in the generators a and $c = b^{-1}a^{-1}$. The first relator transforms into the form $c^2 = a^{-1}ca^{-1}$. Let us right multiply the first relator by c to get $c^3 = (a^{-1}c)^2$. The second relator can be derived by a couple exchanges of the substring $a^{-1}ca^{-1}$ inside it. We get the relator $c^3a^3c^3a^4 = 1$. Since $c^3 = (a^{-1}c)^2$ is a central element in G_4, we can simplify the second relator into $a^7 = c^{-6}$. We have got the presentation $G_4 = \langle a,c \mid a^7 = c^{-6}, c^3 = (a^{-1}c)^2\rangle$.

$\mathbf{G_3} \cong \mathbf{G_4}$. To prove $G_3 \cong G_4$ is a bit tricky. Rename the generators in G_4 to c and d to get $G_4 = \langle c,d \mid c^7 = d^{-6}, d^3 = (c^{-1}d)^2\rangle$ and set its images in $\phi : G_4 \to G_3 = \langle a,b \mid a^7 = b^2 = (a^{-2}b)^3\rangle$ to be $\phi(c) = a^{-2}$ and $\phi(d) = a^{-2}b$. This mapping is a homomorphism, because ϕ maps relations of G_4 onto the relations of G_3. Since a^{-2} and $a^{-2}b$ generates the group G_3, the homomorphism is surjective. To see it write a as $a = a^8a^{-7} = (a^2)^4b^{-2}$ in G_3. On the other hand, the homomorphism $\psi : G_3 \to G_4$ defined by mapping $\psi(a) = c^{-4}(cd^{-1})^2$ and $\psi(b) = dc^{-1}$ is surjective. Since $\phi = \psi^{-1}$, $G_3 \cong G_4$.

Table 4.16: Isomorphism classes in $H_1 = \mathbb{Z}_5$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(4, 8, 8, 5, 5,13)	$\langle a,b \mid a^5 = b^5 = (ab)^2\rangle$	$\mathbb{Z}$	$\Delta^+(5,5,2)$	2
(5, 5,11, 4, 8, 4)	$\langle a,b \mid a^7 = b^2 = (a^{-2}b)^3\rangle$	$\mathbb{Z}$	$\Delta^+(7,3,2)$	2

Homology class $H_1 = \mathbb{Z}_5 \times \mathbb{Z}_5$

Table 4.17: Original relators for $H_1 = \mathbb{Z}_5 \times \mathbb{Z}_5$

No.	f	G
1	(7, 7, 7, 2, 2, 2)	$\langle a,b \mid ab^{-2}a^{-2}b^{-2}ab^{-1}, ab^{-1}a^2b^2a^2b^{-1}\rangle$
2	(7, 7, 7, 4, 4, 4)	$\langle a,b \mid ab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}ab^{-2}, ab^{-1}ababab^{-1}a\rangle$

Rewrite the first relator of G_1 into the form $ab^{-1}a = b^2a^2b^2$ and left multiply it by a^2 to get $a^3b^{-1}a = (a^2b^2)^2$. The second relator of G_1 can be rewritten into the form $ba^{-1}b = a^2b^2a^2$ and right multiplied by b^2 to get $ba^{-1}b^3 = (a^2b^2)^2$.

In the presentation of G_2 simply rewrite the first relator into the form $ab^{-2}a = b(ab)^2$ and left multiply by a. Similarly, rewrite the second relator into the form $ba^{-2}b = (ab)^2a$ and right multiply it by it b.

$\mathbf{G_1 \cong G_2}$[¶]. It was not easy to found the isomorphism. Rewrite the presentations of the groups as follows. The presentation of G_1 is left in generators a and b, $G_1 = \langle a, b \mid a^3b^{-1}a = ba^{-1}b^3 = (a^2b^2)^2\rangle$. We rewrite the presentation of the group G_2 in generators $c = a$ and $d = b$ getting $G_2 = \langle c, d \mid c^2d^{-2}c = dc^{-2}d^2 = (cd)^3\rangle$. Set the homomorphism $\phi : a \mapsto c^2d^{-1}, b \mapsto cd$. This homomorphism is an epimorphism. Set the reverse homomorphism to be $\psi : c \mapsto a^2b^2, b \mapsto a^2b^{-1}a$. It can be checked that ψ is a homomorphism inverse to ϕ. Thus $\phi : G_1 \to G_2$ is an isomorphism.

Table 4.18: Isomorphism classes in $H_1 = \mathbb{Z}_5 \times \mathbb{Z}_5$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(7, 7, 7, 2, 2, 2)	$\langle a, b \mid a^3b^{-1}a = ba^{-1}b^3 = (a^2b^2)^2\rangle$	?	?	2

Homology class $H_1 = \mathbb{Z}_6$

Table 4.19: Original relators for $H_1 = \mathbb{Z}_6$

No.	f	G
1	(4, 6,10, 3, 5, 3)	$\langle a, b \mid ab^{-1}a^{-2}b^{-1}a^2, b^2a^{-2}b^{-1}a^{-2}b\rangle$
2	(6, 6, 8, 5, 3, 7)	$\langle a, b \mid ab^{-1}a^{-2}b^{-1}a, a^{-1}baba^{-1}b^{-1}a^{-1}ba^{-1}b^{-1}\rangle$
3	(3, 7,11, 4, 6, 2)	$\langle a, b \mid ab^{-1}a^{-2}b^{-1}a^2, ab^{-3}a^2ba\rangle$
4	(5, 7, 9, 4, 6,14)	$\langle a, b \mid ab^{-2}abab^{-2}, ab^{-3}a^4\rangle$
5	(5, 7, 9, 6, 8,12)	$\langle a, b \mid ab^{-1}a^{-1}b^{-1}a^2, ab^{-1}abab^{-2}ab\rangle$

Rewrite the presentation of the group G_4. The first relator can be rewritten into the form $(ab^{-2})^2ab = 1$. Let us right multiply this relator by b^{-3} and rewrite it into the form $(ab^{-2})^{-3} = b^{-3}$. Thus the presentation is $G_4 = \langle a, b \mid a^5 = b^3 = (ab^{-2})^{-3}\rangle$.

In the case of the group G_1 rewrite the first relator to the form $a^3 = ba^2b$ and left multiply it by a^2. The second relator can be examined similarly by rewriting it into the form $b^3 = a^2ba^2$ and right multiplying by b. The case of the group G_3 can be processed in a similar way. The derivations of relators are straightforward.

$\mathbf{G_3 \cong G_1}$. The isomorphism $G_3 \cong G_1$ is set by mapping $\phi : a \mapsto a, b \mapsto b$. The factor group by the center can be derived by using Lemma 4.7.

Rewrite the presentation of the group G_2 in the generators a and $c = a^{-1}b$. The first relator can be rewritten into the form $ac = a^{-2}c^{-1}a^{-1}a^2$ and then into the form $ca^3ca^2 = a^3$ (by right multiplying the relator by a^4). The second relator transforms into the form $(ca^2)^2ca^{-2}ca^2ca^{-2} = 1$. Rewrite now the group presentation in the generators a and $d = ca^2$ and left multiply the first relator by a. We obtain $G_2 = \langle a, d \mid a^4 = (ad)^2 = d^3a^{-4}d^2\rangle$. Since $\langle a^4\rangle$ is central, the element a^{-4} commutes with any other element of the group. Thus, the relators of G_2 can be rewritten into the form $G_2 = \langle a, d \mid a^4 = (ad)^2, a^8 = d^5\rangle$.

Rewrite the presentation of the group G_5 in the generators a and $c = ab$. The final form of the presentation is $G_5 = \langle a, c \mid a^4 = c^2, ac^{-1}ac(ac^{-1})^2ac = 1\rangle$. It follows from $a^4 = c^2$, that $c^{-1} = ca^{-4}$ holds. Substituting it into the second relator and taking into the account that a^{-4} is central we get $G_5 = \langle a, c | \, a^4 = c^2, a^{12} = (ac)^5\rangle$.

The groups $G_1 = \langle d, e \mid d^5 = e^4 = (d^2e)^2\rangle$, $G_2 = \langle a, b \mid a(ab)^4 = 1, a^8 = b^5\rangle$ and $G_5 = \langle a, c \mid a^4 = c^3, ac^{-1}ac(ac^{-1})^2ac = 1\rangle$ are isomorphic.

[¶]thanks to George Havas

$\mathbf{G_2} \cong \mathbf{G_1}$ Since $(d^2)^5 = e^8$ and $(d^2e)^2 = e^4$, the elements of the group G_2 satisfy the relations of the group G_1. Thus there exists homomorphism $\phi : G_2 \to G_1$, taking $a \mapsto e$ and $b \mapsto d^2$. Since $d = d^5d^{-4} = e^4(d^2)^{-2}$, d^2 and e generate G_1. Thus ϕ is an epimorphism. A mapping $\psi : G_1 \to G_2$ can be constructed in a similar way. The central element a^4 commutes with ab and with elements b and b^3, as well. Thus $(b^3a^{-4})^5 = b^6a^{-8} = b^6b^{-5} = b$. The elements a and b^3a^{-4} satisfies the relations of the group G_1. Thus there exists a homomorphism $\psi : G_1 \to G_2$, taking $e \mapsto a$ and $d \mapsto b^3a^{-4}$. The mapping ψ is an epimorphism, as well. The following equalities hold: $\phi(\psi(d)) = \phi(b^3a^{-4}) = d^6e^{-4} = d$ and $\phi(\psi(e)) = \psi(a) = e$. It follows that $\forall x \in G_1 : \phi(\psi(x)) = x$. Since ψ is injective, it is an isomorphism as well.

$\mathbf{G_5} \cong \mathbf{G_2}$ Rewrite the presentation of G_2 in the generators a and $c = ab$. The new presentation is $G_2 = \langle a, c \mid a^4 = c^2, a^8 = (a^{-1}c)^5\rangle$. Using the first relation we may replace the element c by a^4c^{-1}. We get $a^8 = (a^{-1}a^4c^{-1})^5 = a^{20}(a^{-1}c^{-1})^5$. This means, that $a^{-12} = (a^{-1}c^{-1})^5$ and $a^{12} = (ca)^5 = (ac)^5$, according to Lemma 4.8.

Table 4.20: Isomorphism classes in $H_1 = \mathbb{Z}_6$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 4, 6,10, 3, 5, 3)`	$\langle a, b \mid a^5 = b^4 = (a^2b)^2\rangle$	$\mathbb{Z}$	$\Delta^+(5, 4, 2)$	4
`( 5, 7, 9, 4, 6,14)`	$\langle a, b \mid a^5 = b^3 = (ab^{-2})^{-3}\rangle$	$\mathbb{Z}$	$\Delta^+(5, 3, 3)$	1

Homology class $H_1 = \mathbb{Z}_6 \times \mathbb{Z}_2$

Table 4.21: Original relators for $H_1 = \mathbb{Z}_6 \times \mathbb{Z}_2$

No.	f	G
1	`( 4, 4, 4, 1, 1, 1)`	$\langle a, b \mid a^2b^2, a^3b^{-1}ab^{-1}\rangle$
2	`( 4, 4, 6, 1, 5, 1)`	$\langle a, b \mid b^2a^{-4}, ba^{-1}ba^3, ab^{-1}ab^{-3}\rangle$
3	`( 3, 3, 9, 2, 0, 2)`	$\langle a, b \mid ab^2a, ab^{-1}ab^{-3}\rangle$
4	`( 3, 3,11, 2, 2, 2)`	$\langle a, b \mid ab^{-3}ab^{-1}, ab^{-1}a^{-3}b^{-1}\rangle$
5	`( 5, 5, 7, 2, 0, 2)`	$\langle a, b \mid ab^2a, b^{-2}ab^{-1}ab^{-1}\rangle$
6	`( 5, 5, 7, 2, 0, 8)`	$\langle a, b \mid a^2b^2, ab^{-1}ab^{-1}a^{-1}b^{-1}ab^{-1}\rangle$
7	`( 5, 7, 7, 2, 8, 0)`	$\langle a, b \mid ba^{-1}ba^3, ab^{-1}ab^{-3}\rangle$
8	`( 4, 4,12, 3, 7, 3)`	$\langle a, b \mid ab^2a, ab^{-1}ab^{-3}\rangle$

The groups G_1, G_3, G_5 and G_8 are isomorphic. This fact was checked by the GAP. The group was identified by the GAP as the group `[24,11]`. This group is a central extension of $\mathbb{Z}_6$ by the Klein group. The presentation can be derived from the presentation of G_1. Rewrite the second relator into the form $a^{-3} = b^{-1}ab^{-1}$ and left multiply it by a. Then rewrite the presentation of the group in generators $c = a^{-1}$ and b. Using Lemma 4.8 we get the presentation $G_1 = \langle c, b \mid c^2 = b^2 = (cb)^{-2}\rangle$.

The remaining groups are isomorphic to the group `[48,27]`. This is a central extension of $\mathbb{Z}_6$ by the dihedral group D_8. The presentation of the group can be derived e.g. from presentation of G_7. Rewrite the relators into the form $a^3 = b^{-1}ab^{-1}$ and $b^3 = ab^{-1}a$. Multiply the relators by a or b^{-1} from left and right, respectively.

Table 4.22: Isomorphism classes in $H_1 = \mathbb{Z}_6 \times \mathbb{Z}_2$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 4, 4, 4, 1, 1, 1)`	$\langle a, b \mid a^2 = b^2 = (ab)^{-2}\rangle$	$\mathbb{Z}_6$	D_4	4
`( 4, 4, 6, 1, 5, 1)`	$\langle a, b \mid a^4 = b^2 = (ab^{-1})^2\rangle$	$\mathbb{Z}_6$	D_8	4

Homology class $H_1 = \mathbb{Z}_6 \times \mathbb{Z}_3$

Table 4.23: Original relators for $H_1 = \mathbb{Z}_6 \times \mathbb{Z}_3$

No.	f	G
1	`( 4, 6,10, 5, 9, 3)`	$\langle a, b \mid ab^3a^2, ab^{-1}a^2b^{-1}a^{-1}b^{-1}a \rangle$
2	`( 4, 8, 8, 1, 7, 3)`	$\langle a, b \mid ab^{-3}a^2, ab^{-1}a^{-2}b^{-1}a^{-2}b^{-1} \rangle$
3	`( 5, 5,11, 4, 2, 8)`	$\langle a, b \mid ab^{-1}a^{-2}b^{-1}a^{-2}b^{-1}, ab^{-1}a^{-1}b^{-1}a^2b^{-1}a \rangle$
4	`( 5, 7, 9, 4, 2, 2)`	$\langle a, b \mid ab^{-3}a^2, a^{-2}ba^{-2}ba^{-2}b^{-2} \rangle$

Rewrite the relators of G_1 into the form $a^3 = b^{-3}$ and $a^{-1}b^{-1}(a^2b^{-1})^2 = 1$ and left multiply the second one by a^3. Rewrite the presentation of G_1 in the generators a and $c = b^{-1}$.

Relators of the group G_2 can be rewritten in the following way. Rewrite the first relator into the form $a^{-3} = b^{-3}$ and the second relator into the form $ab^{-1}(a^{-2}b^{-1})^2 = 1$. Left multiply the second relator by a^{-3}. Rewrite the presentation of G_2 in the generators $c = a^{-1}$ and $d = b^{-1}$.

$\mathbf{G_2 \cong G_1}$. The isomorphism of the groups G_1 and G_2 can be set by mapping $\phi : a \mapsto c, b \mapsto d$.

Rewrite the relators of G_3 in following way. First relator can be rewritten into the form $(ba^2)^2ba^{-1} = 1$. Next right multiply it by a^3 and use Lemma 4.8 to get the new form. Left multiply second relator by a^3 to get it into the form $a^3 = (a^2b^{-1})^3$. We have derived the presentation $G_3 = \langle a, b \mid a^3 = (ba^2)^3 = (a^2b^{-1})^3 \rangle$.

$\mathbf{G_3 \cong G_1}$. Rewrite the presentation of the group G_3 in the generators a and $c = ba^2$. We get $G_3 = \langle a, c \mid a^3 = c^3 = (a^4c^{-1})^3 \rangle$. It follows that $a^3 = (a^4c^{-1})^3 = (ac^3c^{-1})^3 = (ac^2)^3$. The isomorphism $G_1 \cong G_3$ can be set by mapping $\phi : a \mapsto a, b \mapsto c$.

The presentation of G_4 can be simplified as follows. Rewrite the second relator into the form $b^2 = (a^{-2}b)^2a^{-2} = 1$ and right multiply it by b. The derivation of the first relator follows. Finally, rewrite the presentation of G_4 in the generators a and $c = a^{-2}b$.

$\mathbf{G_4 \cong G_1}$. The groups G_1 and G_4 are isomorphic by setting $\phi : a \mapsto a, b \mapsto c$.

Table 4.24: Isomorphism classes in $H_1 = \mathbb{Z}_6 \times \mathbb{Z}_3$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 4, 6,10, 5, 9, 3)`	$\langle a, b \mid a^3 = b^3 = (a^2b)^3 \rangle$	$\mathbb{Z}$	$\Delta^+(3,3,3)$	4

Homology class $H_1 = \mathbb{Z}_7$

Table 4.25: Original relators for $H_1 = \mathbb{Z}_7$

No.	f	G
1	`( 4, 4,10, 3, 3, 3)`	$\langle a, b \mid ab^{-1}a^{-2}b^{-1}a^2, ab^{-2}a^2ba \rangle$
2	`( 6, 6, 6, 3, 5, 5)`	$\langle a, b \mid ab^{-2}ab, ab^{-1}a^{-1}b^{-1}a^{-1}bab, a^2b^{-1}a^{-1}b^{-1}ab^{-1}a^{-1}b^{-1} \rangle$
3	`( 3, 5,11, 4, 4, 2)`	$\langle a, b \mid ab^{-1}a^{-2}b^{-1}a^2, ab^{-2}a^2ba \rangle$
4	`( 5, 7, 7, 4, 6, 4)`	$\langle a, b \mid ab^{-2}ab, ab^{-1}a^{-1}b^{-1}a^{-1}bab, ab^{-1}a^{-1}b^{-1}ab^{-1}a^{-1}b^{-1}a \rangle$
5	`( 5, 5,11, 4, 2, 4)`	$\langle a, b \mid a^3b^{-1}a^{-2}b^{-1}, ab^{-2}a^2ba \rangle$
6	`( 5, 7, 9, 6, 6,14)`	$\langle a, b \mid ab^{-1}a^{-1}b^{-1}a, ab^{-1}abab^{-2}ab \rangle$

All fundamental groups are isomorphic to a group of size 840. The center of this group is $\mathbb{Z}_{14}$. The factor of this group by the center is the alternating group A_5. The result was obtained by the GAP . The group was recognised as the group `[840,13]`. The presentation of this group is $G_1 = \langle a, b \mid a^5 = b^3 = (a^2b)^2 \rangle$. This presentation was obtained from the presentation of G_1 by using techniques used in many previous examples.

Table 4.26: Isomorphism classes in $H_1 = \mathbb{Z}_7$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 4, 4,10, 3, 3, 3)`	$\langle a, b \mid a^5 = b^3 = (a^2b)^2$	$\mathbb{Z}_{14}$	A_5	6

Homology class $H_1 = \mathbb{Z}_8$

Table 4.27: Original relators for $H_1 = \mathbb{Z}_8$

No.	f	G
1	`( 4, 4, 4, 1, 1, 5)`	$\langle a, b \mid ab^2a, abab^{-1}ab^{-1}\rangle$
2	`( 3, 3, 7, 2, 2, 2)`	$\langle a, b \mid ab^{-1}a^{-2}b^{-1}, ab^{-2}a^2\rangle$
3	`( 4, 4, 6, 3, 1, 5)`	$\langle a, b \mid aba^{-1}b, ba^2bab^{-1}a\rangle$
4	`( 3, 3,11, 2, 2, 4)`	$\langle a, b \mid ab^{-1}a^2ba, a^3b^{-1}a^{-2}b^{-1}\rangle$
5	`( 5, 5, 7, 0, 2, 8)`	$\langle a, b \mid a^2b^2, abab^{-1}ab^{-1}\rangle$
6	`( 5, 5, 7, 2, 4, 6)`	$\langle a, b \mid ab^{-1}ab, b^{-1}a^{-3}b^{-2}a^{-2}b^{-1}\rangle$
7	`( 5, 5, 7, 4, 4, 4)`	$\langle a, b \mid b^{-1}aba, b^{-1}a^{-2}b^{-2}a^{-3}b^{-1}\rangle$
8	`( 5, 5, 9, 2, 6, 2)`	$\langle a, b \mid ab^{-1}a^{-2}b^{-1}, ab^{-2}a^2\rangle$
9	`( 5, 7, 7, 2, 6, 4)`	$\langle a, b \mid ab^{-2}a, a^{-1}ba^{-1}b^{-1}a^{-1}b^{-1}\rangle$
10	`( 4, 4,12, 1, 5, 5)`	$\langle a, b \mid b^{-1}a^2ba^2, a^3b^{-1}a^{-2}b^{-1}\rangle$
11	`( 4, 4,12, 3, 3, 3)`	$\langle a, b \mid ab^{-1}a^{-2}b^{-1}, ab^{-2}a^2\rangle$
12	`( 4, 6,10, 3, 3, 3)`	$\langle a, b \mid aba^{-1}ba, ab^{-2}a^2\rangle$
13	`( 4, 8, 8, 1, 1,13)`	$\langle a, b \mid a^3b^4a, ab^{-1}a^{-3}b^{-1}ab^{-1}\rangle$
14	`( 3, 3,15, 2, 2, 6)`	$\langle a, b \mid b^{-1}a^2ba^2, a^3b^{-1}a^{-4}b^{-1}\rangle$
15	`( 5, 5,11, 2, 4,10)`	$\langle a, b \mid ab^{-1}ab, b^{-1}a^{-4}b^{-2}a^{-3}b^{-1}\rangle$
16	`( 5, 5,11, 4, 0,10)`	$\langle a, b \mid a^2b^2, ab^{-1}a^{-1}b^{-1}ab^{-1}\rangle$
17	`( 5, 5,11, 4, 4, 8)`	$\langle a, b \mid b^{-1}aba, b^{-1}a^{-3}b^{-2}a^{-4}b^{-1}\rangle$
18	`( 7, 7, 7, 2, 2, 8)`	$\langle a, b \mid ab^2a, abab^{-1}ab^{-1}\rangle$
19	`( 7, 7, 7, 2, 2,10)`	$\langle a, b \mid a^3b^4a, ab^{-1}a^{-3}b^{-1}ab^{-1}\rangle$

The first isomorphism class has the fundamental group of size 24, which was recognized by the GAP as `[24,1]`. The center of this group is the cyclic group $\mathbb{Z}_4$ and the respective factor was recognised by the GAP as the group D_6. The members of this class are G_1, G_2, G_3, G_5, G_8, G_9, G_{11}, G_{12}, G_{16} and G_{18}. The relators of the group can be obtained e.g. from relators of G_2. Rewrite the first relator into the form $a = ba^2b$ and left multiply by a^2 to get $a^3 = (a^2b)^2$. The derivation of second relator is obvious.

The second isomorphism class is represented by the group `[40,1]` . The relators come from the relators of the group G_4. Rewrite the fist relator to the form $b = a^2ba^2$ an right multiply it by b to get $b^2 = (a^2b)^2$. The second relator can be rewritten to the form $a^3 = ba^2b$ and left multiplied by a^2 to get the form $a^5 = (a^2b)^2$. The members of this isomorphism class are G_4, G_6, G_7 and G_{10}.

The last finite fundamental group in the homology class is the group `[56,1]`. The relators of the group can be derived e.g from the relators of G_{14}. Rewrite the first relator into the form $b = a^2ba^2$ and right multiply it by b to get $b^2 = (a^2b)^2$. The second relator can be rewritten into the form $a^3 = ba^4b$. By left multiplying it by a^4 we get $a^7 = (a^4b)^2$. The isomorphism class contains the groups G_{14}, G_{15} and G_{17}

$\mathbf{G_{19} \cong G_{13}}$. The groups G_{13} and G_{19} have the same presentation. Let us derive a presentation of G_{13}. Rewrite the second relator into the form $a^3 = b^{-1}(ab^{-1})^2$ and left multiply it by a. The derivation of first relator is obvious. Rewrite the group presentation in the generators a and $c = b^{-1}$.

Table 4.28: Isomorphism classes in $H_1 = \mathbb{Z}_8$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(4, 4, 4, 1, 1, 5)	$\langle a, b \mid a^3 = b^2 = (a^2b)^2\rangle$	$\mathbb{Z}_4$	D_6	10
(3, 3,11, 2, 2, 4)	$\langle a, b \mid a^5 = b^2 = (a^2b)^2\rangle$	$\mathbb{Z}_4$	D_{10}	4
(3, 3,15, 2, 2, 6)	$\langle a, b \mid a^7 = (a^4b)^2, b^2 = (a^2b)^2\rangle$	$\mathbb{Z}_4$	D_{14}	3
(4, 8, 8, 1, 1,13)	$\langle a, b \mid a^4 = b^4 = (ab)^3\rangle$	$\mathbb{Z}$	$\Delta^+(4, 4, 3)$	2

Homology class $H_1 = \mathbb{Z}_8 \times \mathbb{Z}_2$

Table 4.29: Original relators for $H_1 = \mathbb{Z}_8 \times \mathbb{Z}_2$

No.	f	G
1	(4, 6, 8, 3, 9,13)	$\langle a, b \mid b^{-3}a^{-1}ba^{-1}, b^{-3}a^4b^{-1}\rangle$
2	(3, 7,11, 2, 6, 2)	$\langle a, b \mid ab^{-1}a^{-3}b^{-1}, ab^{-5}ab^{-1}\rangle$
3	(5, 5,11, 2, 0,12)	$\langle a, b \mid a^4b^2, ab^{-1}ab^{-1}abab^{-1}\rangle$

Rewrite the presentation of G_1 as follows. The first relator rewrite into the form $b^3 = a^{-1}ba^{-1}$ and right multiply it by b. The derivation of second relator is easy.

$\mathbf{G_2} \cong \mathbf{G_1}$. To analyse the group G_2, rewrite the first relator into the form $a^3 = b^{-1}ab^{-1}$ and right multiply it by a. The second relator can be rewritten into the form $b^5 = ab^{-1}a$ and left multiplied by b^{-1} to get the desired form. The isomorphism $G_2 \cong G_1$ is defined by setting $\phi : a \mapsto b, b \mapsto a$.

In the case of the group G_3 rewrite the second relator into the form $(ab^{-1})ab = 1$. Multiply the relator by b^{-2} from right to get $(ab^{-1})^3 = b^{-2}$. Then rewrite the relators of the group in the new generators a and $c = ab^{-1}$. We obtain the relators $a^4 = c^4 = (a^{-1}c)^2$.

$\mathbf{G_3} \cong \mathbf{G_1}$. The isomorphism $G_3 \cong G_1$ can be set by taking $\phi : a \mapsto a, c \mapsto b$.

Table 4.30: Isomorphism classes in $H_1 = \mathbb{Z}_8 \times \mathbb{Z}_2$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(4, 6, 8, 3, 9,13)	$\langle a, b \mid a^4 = b^4 = (a^{-1}b)^2\rangle$	$\mathbb{Z}$	$\Delta^+(4, 4, 2)$	3

Homology class $H_1 = \mathbb{Z}_9$

Table 4.31: Original relators for $H_1 = \mathbb{Z}_9$

No.	f	G
1	(4, 4, 6, 1, 1, 7)	$\langle a, b \mid a^3b^2, abab^{-1}ab^{-1}\rangle$
2	(3, 5, 7, 2, 4, 2)	$\langle a, b \mid ab^{-1}a^{-2}b^{-1}, ab^{-3}a^2\rangle$
3	(4, 4, 8, 3, 1, 7)	$\langle a, b \mid ab^{-1}a^{-2}b^{-1}, ab^{-1}a^2b^{-1}ab\rangle$
4	(4, 4, 8, 3, 7, 3)	$\langle a, b \mid ab^{-1}a^{-1}b^{-1}a, a^2b^3a\rangle$
5	(4, 6, 8, 1, 1,11)	$\langle a, b \mid ab^{-1}ab^2ab^{-1}, a^4b^3\rangle$
6	(4, 6, 8, 5, 5, 3)	$\langle a, b \mid ab^{-2}a^2, b^{-1}a^{-1}b^{-1}a^{-1}ba^{-1}\rangle$
7	(5, 7, 7, 0, 2,10)	$\langle a, b \mid a^2b^3, a^{-1}b^{-1}ab^{-1}ab^{-1}\rangle$
8	(5, 7, 7, 2, 8, 2)	$\langle a, b \mid ab^{-1}ab^{-1}a^{-2}b^{-1}, ab^{-4}a^2\rangle$
9	(4, 6,10, 1, 1,13)	$\langle a, b \mid ab^{-1}ab^2ab^{-1}, b^{-1}a^{-5}b^{-2}\rangle$
10	(4, 6,10, 5, 5, 1)	$\langle a, b \mid aba^{-1}ba, ab^{-3}a^2\rangle$
11	(5, 5,11, 0, 2,12)	$\langle a, b \mid a^3b^2, abab^{-1}ab^{-1}\rangle$
12	(5, 7, 9, 2, 8, 2)	$\langle a, b \mid ab^{-1}a^{-2}b^{-1}, b^3a^{-3}\rangle$
13	(5, 7, 9, 4,10,14)	$\langle a, b \mid ab^{-1}ab^2ab^{-1}, a^{-4}b^3a^{-1}\rangle$
14	(5, 7, 9, 6,10,12)	$\langle a, b \mid ab^{-2}a^2, abab^{-1}ab\rangle$

The groups G_1, G_2, G_3, G_4, G_6, G_7, G_{10}, G_{11}, G_{12} and G_{14} were recognised by GAP to be isomorphic to the group `[72,3]`. The center of this group is $\mathbb{Z}_6$. The factor of this group by its center is the alternating group A_4. The presentation can be derived from the relators of the group G_2. Rewrite the first relator to the form $a^2 = b^{-1}ab^{-1}$ and left multiply it by a. The derivation of the second relator can be easily done.

The presentation of the group G_5 can be derived as follows. Rewrite the first relator into the form $(ab^{-1})^2a = b^{-2}$ an right multiply the relator by b^{-1}. The derivation of second relator is straightforward. Now rewrite presentation of the group G_5 in the generators a and $c = b^{-1}$.

Write $G_8 = \langle a, b \mid a^3 = b^4, a^{-2}b^{-1}(ab^{-1})^2 = 1\rangle$. Then left multiply the second relator by a^3. The final form of the presentation is $G_8 = \langle a, b \mid b^4 = a^3 = (b^{-1}a)^3\rangle$ (using Lemma 4.8).

$\mathbf{G_8 \cong G_5}$. Finally, rewrite the presentation in the generators b and $d = b^{-1}a$. One can easily see that the assignment $b \mapsto a$, $d \mapsto c$ defines an isomorphism $G_8 \to G_5$.

Rewrite the relators of G_9 into the form $a^5 = b^{-3}$ and $(ab^{-1})ab^2 = 1$. Right multiply the second relator by b^{-3} and rewrite the presentation of G_9 in the generators a and $c = b^{-1}$.

In the case of the group G_{13} rewrite the presentation in the generators $d = a^{-1}$ and $e = b^{-1}$. The relators are $d^4 = e^3$ and $(d^{-1}e)^2d^{-1}e^{-2} = 1$. Now right multiply the second relator by d^3 and rewrite the presentation of the group in the generators d and $f = d^{-1}e$.

$\mathbf{G_{13} \cong G_9}$. Set the isomorphism $G_{13} \cong G_9$ by taking $\phi : d \mapsto a, f \mapsto b$.

Table 4.32: Isomorphism classes in $H_1 = \mathbb{Z}_9$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 4, 4, 6, 1, 1, 7)`	$\langle a, b \mid a^3 = b^3 = (a^{-1}b)^2\rangle$	$\mathbb{Z}_6$	A_4	10
`( 4, 6, 8, 1, 1,11)`	$\langle a, b \mid a^4 = b^3 = (ab)^3\rangle$	$\mathbb{Z}$	$\Delta^+(4,3,3)$	2
`( 5, 7, 9, 4,10,14)`	$\langle a, b \mid a^5 = b^3 = (ab)^3\rangle$	$\mathbb{Z}$	$\Delta^+(5,3,3)$	2

Homology class $H_1 = \mathbb{Z}_9 \times \mathbb{Z}_3$

Table 4.33: Original relators for $H_1 = \mathbb{Z}_9 \times \mathbb{Z}_3$

No.	f	G
1	`( 6, 6, 6, 1, 1, 1)`	$\langle a, b \mid a^3b^3, a^4b^{-1}ab^{-1}ab^{-1}\rangle$

There is only one 6-tuple in the class. It is easy to derive the presentation of the group G_1 by rewriting the second relator into the form $a^{-4} = b^{-1}(ab^{-1})^2$ and left multiplying it by a. Rewrite the presentation in the generators $c = a^{-1}$ and b. By using Lemma 4.8 we get the presentation $G_1 = \langle c, b \mid c^3 = b^3 = (cb)^{-3}\rangle$.

Table 4.34: Isomorphism classes in $H_1 = \mathbb{Z}_9 \times \mathbb{Z}_3$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 6, 6, 6, 1, 1, 1)`	$\langle a, b \mid a^3 = b^3 = (ab)^{-3}\rangle$	$\mathbb{Z}$	$\Delta^+(3,3,3)$	1

Homology class $H_1 = \mathbb{Z}_{10}$

Table 4.35: Original relators for $H_1 = \mathbb{Z}_{10}$

No.	f	G
1	(4, 4, 8, 1, 1, 9)	$\langle a, b \mid ab^{-1}abab^{-1}, a^4b^2 \rangle$
2	(3, 7, 7, 2, 6, 2)	$\langle a, b \mid ab^{-1}a^{-2}b^{-1}, ab^{-4}a^2 \rangle$
3	(4, 4,10, 3, 1, 9)	$\langle a, b \mid ab^{-1}a^{-3}b^{-1}, ab^{-1}abab^{-1}a \rangle$
4	(4, 6, 8, 3, 9, 3)	$\langle a, b \mid ab^{-1}a^{-1}b^{-1}a, a^2b^4a \rangle$
5	(5, 7, 9, 2, 0,12)	$\langle a, b \mid ab^{-1}abab^{-1}, a^4b^2 \rangle$

Only one type of the group appears in the homology class. It has be recognised by GAP as the group [240,102]. The center is the cyclic group $\mathbb{Z}_{10}$. The factor $G/\zeta(G) \cong S_4$. Derive the relators of this group from the relators of G_1. Rewrite the relators in generators a and $c = b^{-1}$. Rewrite the first relator into the form $(ac)^2ac^{-1} = 1$ and right multiply it by c^2. We get the presentation $G_1 = \langle a, c \mid a^4 = c^2 = (ac)^3 \rangle$.

Table 4.36: Isomorphism classes in $H_1 = \mathbb{Z}_{10}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(4, 4, 8, 1, 1, 9)	$a^4 = b^2 = (ab)^3$	$\mathbb{Z}_{10}$	S_4	5

Homology class $H_1 = \mathbb{Z}_{10} \times \mathbb{Z}_2$

Table 4.37: Original relators for $H_1 = \mathbb{Z}_{10} \times \mathbb{Z}_2$

No.	f	G
1	(4, 4, 8, 1, 1, 1)	$\langle a, b \mid ab^{-3}ab^{-1}, ab^2a^3 \rangle$
2	(4, 4,10, 1, 5, 1)	$\langle a, b \mid ab^{-3}ab^{-1}, ba^{-6}b \rangle$
3	(3, 7, 9, 2, 0, 2)	$\langle a, b \mid ab^{-1}a^3b^{-1}, ab^4a \rangle$
4	(4, 4,12, 1, 1, 5)	$\langle a, b \mid ab^{-1}a^3b^{-1}, b^{-1}a^{-2}b^{-2}ab^{-1}a^{-1} \rangle$
5	(4, 8, 8, 3, 3, 1)	$\langle a, b \mid a^2b^2, a^3b^{-1}a^3b^{-3} \rangle$
6	(5, 7, 9, 0, 2, 2)	$\langle a, b \mid ab^{-3}ab^{-1}, ab^2a^3 \rangle$
7	(5, 7, 9, 0, 4,14)	$\langle a, b \mid ab^{-3}ab^{-1}, b^2a^{-6} \rangle$

The groups G_4 and G_5 are isomorphic to [40,11] . The factor $G/\zeta(G) = \mathbb{Z}_2 \times \mathbb{Z}_2$. The relators of this group are easy to derive e.g. from relators of G_5. Rewrite the second relator to the form $a^3b^{-3}a^3b^{-1} = 1$ and right multiply it by b^{-2}. The derivation of first relator is obvious: $a^2 = b^{-2}$. Rewrite the presentation in the generators a and $c = b^{-1}$.

There are three isomorphic groups in the second homotopy class – G_1, G_3 and G_6. The GAP recognises this group as [80,27] . This group has the center $\mathbb{Z}_{10}$ and factor by the center is the dihedral group D_8. The relators of this group follows up from the relators of G_1. Rewrite the first relator to the form $b^3 = ab^{-1}a$ and right multiply it by b^{-1}. The second relator can be rewritten into the form $a^{-4} = b^2$. Rewrite this group in the generators $c = a^{-1}$ and b and use Lemma 4.8.

The groups G_2 and G_7 are isomorphic to the group [120,21] . Form the relators e.g. from the presentation of G_2. Rewrite the second relator to the form $b^3 = ab^{-1}a$ and right multiply it by b^{-1}.

Table 4.38: Isomorphism classes in $H_1 = \mathbb{Z}_{10} \times \mathbb{Z}_2$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(4, 4, 8, 1, 1, 1)	$\langle a, b \mid a^4 = b^2 = (ab)^{-2} \rangle$	$\mathbb{Z}_{10}$	D_8	3
(4, 4,10, 1, 5, 1)	$\langle a, b \mid a^6 = b^2 = (ab^{-1})^2 \rangle$	$\mathbb{Z}_{10}$	D_{12}	2
(4, 8, 8, 3, 3, 1)	$\langle a, b \mid a^2 = b^2 = (a^3b^3)^2 \rangle$	$\mathbb{Z}_{10}$	D_4	2

Homology class $H_1 = \mathbb{Z}_{11}$

Table 4.39: Original relators for $H_1 = \mathbb{Z}_{11}$

No.	f	G
1	(4, 4,10, 1, 1,11)	$\langle a, b \mid ab^{-1}abab^{-1}, a^5b^2 \rangle$
2	(3, 7, 9, 4, 8,14)	$\langle a, b \mid ab^{-1}ab^2, a^2b^{-3}a^3 \rangle$
3	(4, 4,12, 3, 1,11)	$\langle a, b \mid ab^{-1}a^{-4}b^{-1}, ab^{-1}abab^{-1}a \rangle$
4	(4, 8, 8, 1, 9,13)	$\langle a, b \mid ab^{-2}ab, a^{-4}b^{-3}a^{-1} \rangle$
5	(6, 6, 8, 3,11, 3)	$\langle a, b \mid ab^{-1}a^{-1}b^{-1}a, a^2b^5a \rangle$

Only one fundamental group appears in this homology class. The GAP identified it as the group [1320,14]. The center of this group is the cyclic group $\mathbb{Z}_{22}$. The factor of the group by its center is the alternating group A_5. To obtain the relators for this group is suitable to use the relators of G_1. Rewrite the second relator into the form $(ab^{-1})^2ab = 1$ and right multiplicate it by b^{-2}.

Table 4.40: Isomorphism classes in $H_1 = \mathbb{Z}_{11}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(4, 4,10, 1, 1,11)	$\langle a, b \mid a^5 = b^{-2} = (ab^{-1})^3 \rangle$	$\mathbb{Z}_{22}$	A_5	5

Homology class $H_1 = \mathbb{Z}_{12}$

Table 4.41: Original relators for $H_1 = \mathbb{Z}_{12}$

No.	f	G
1	(4, 4,10, 1, 5, 7)	$\langle a, b \mid b^{-1}a^{-1}b^{-2}ab^{-1}, aba^{-4}b \rangle$
2	(6, 6, 6, 1, 3, 7)	$\langle a, b \mid ab^2a, abab^{-1}ab^{-1}a^{-1}b^{-1}ab^{-1} \rangle$
3	(3, 3,13, 2, 2, 8)	$\langle a, b \mid b^{-1}a^{-1}b^{-2}ab^{-1}, aba^{-4}b \rangle$
4	(4, 4,12, 1, 1,13)	$\langle a, b \mid ab^{-1}abab^{-1}, b^{-1}a^{-6}b^{-1} \rangle$
5	(6, 6, 8, 5, 1, 7)	$\langle a, b \mid aba^{-1}b, ba^2bab^{-1}abab^{-1}a \rangle$
6	(3, 7,11, 4, 8,16)	$\langle a, b \mid ab^{-1}ab^2, a^2b^{-3}a^4 \rangle$
7	(5, 7, 9, 6, 8, 2)	$\langle a, b \mid a^4b^{-2}a, ab^{-1}a^{-3}b^{-1}a \rangle$

The groups G_1, G_2, G_3, G_5 and G_7 are isomorphic to the group [60,2] . The center of this group is the cyclic group $\mathbb{Z}_6$ and the factor of the group by its center is the dihedral group D_{10}. The relators of this group can be derived e.g. from the presentation of the group G_7. Rewrite the second relator to the form $a^3 = b^{-1}a^2b^{-1}$ and left multiply this relator by a^2. The derivation of the first relator is straightforward.

Rewrite presentation of the group G_4 as follows: The second relator rewrite into the form $(ab^{-1})^2ab = 1$ and right multiply it by b^{-2}. Rewrite the first relator into the form $a^6 = b^{-2}$. Then rewrite the presentation in the generators a and $c = b^{-1}$. We get $G_4 = \langle a, b \mid a^6 = c^2 = (ac)^3 \rangle$. Finally, rewrite the presentation in generators a and $d = ac$ getting $G_4 = \langle a, d \mid a^6 = d^3 = (a^{-1}d)^2 \rangle$.

$\mathbf{G_6} \cong \mathbf{G_4}$. In the case of the group G_6 we rewrite the relations as follows. Rewrite the first relator into the form $b^2 = a^{-1}ba^{-1}$ and right multiply it by b. The derivation of the first relator is simple. The isomorphism $G_4 \cong G_6$ can be set by mapping $\phi : a \mapsto a, d \mapsto b$.

Table 4.42: Isomorphism classes in $H_1 = \mathbb{Z}_{12}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 4, 4,10, 1, 5, 7)`	$\langle a, b \mid a^5 = b^2 = (a^2b^{-1})^2$	$\mathbb{Z}_6$	D_{10}	5
`( 4, 4,12, 1, 1,13)`	$\langle a, b \mid a^6 = b^3 = (a^{-1}b)^2\rangle$	$\mathbb{Z}$	$\Delta^+(6, 3, 2)$	2

Homology class $H_1 = \mathbb{Z}_{13}$

Table 4.43: Original relators for $H_1 = \mathbb{Z}_{13}$

No.	f	G
1	`( 4, 6,10, 7, 3,15)`	$\langle a, b \mid ab^{-1}a^{-3}b^{-1}a, ab^{-3}a^4\rangle$
2	`( 6, 6, 8, 3, 1, 9)`	$\langle a, b \mid a^3b^2, abab^{-1}ab^{-1}abab^{-1}\rangle$
3	`( 3, 5,13, 8, 4, 2)`	$\langle a, b \mid ab^{-1}a^{-3}b^{-1}a, ab^{-3}a^4\rangle$

All groups in the homology class were determined to be isomorphic to the group, which GAP knows as `[1560,13]` . The center of this group is the cyclic group $\mathbb{Z}_{26}$ and the factor of the group by its center is $G/\zeta(G) \cong A_5$. The relators of the group were obtained from the presentation of G_2. Rewrite the first relator into the form $a^3 = b^{-1}a^2b^{-1}$and left multiply it by a^2. The derivation of the second relator is easy.

Table 4.44: Isomorphism classes in $H_1 = \mathbb{Z}_{13}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 4, 6,10, 7, 3,15)`	$\langle a, b \mid a^5 = b^3 = (a^2b^{-1})^2\rangle$	$\mathbb{Z}_{26}$	A_5	3

Homology class $H_1 = \mathbb{Z}_{14}$

Table 4.45: Original relators for $H_1 = \mathbb{Z}_{14}$

No.	f	G
1	`( 4, 6, 6, 1, 7, 1)`	$\langle a, b \mid ab^{-1}a^{-3}b^{-1}, a^4b^{-3}\rangle$
2	`( 3, 5,11, 2, 4, 2)`	$\langle a, b \mid ab^{-1}a^{-3}b^{-1}, ab^{-4}ab^{-1}\rangle$
3	`( 5, 5, 9, 2, 0,10)`	$\langle a, b \mid a^3b^2, ab^{-1}ab^{-1}abab^{-1}\rangle$
4	`( 5, 7, 9, 0, 2,12)`	$\langle a, b \mid a^3b^2, abab^{-1}ab^{-1}ab^{-1}\rangle$
5	`( 7, 7, 7, 0, 2,10)`	$\langle a, b \mid a^2b^3, a^{-1}b^{-1}ab^{-1}ab^{-1}ab^{-1}\rangle$

Only the group `[336,115]` appears in this homology class. The center of the group is the cyclic group $\mathbb{Z}_{14}$, the factor by its center is S_4. The relators of this group can be derived e.g. from G_1. Rewrite the first relator into the form $a^3 = b^{-1}ab^{-1}$ and left multiply this relator by a. The first relator can be easily transformed to the required form.

Table 4.46: Isomorphism classes in $H_1 = \mathbb{Z}_{14}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 4, 6, 6, 1, 7, 1)`	$\langle a, b \mid a^4 = b^3 = (ab^{-1})^2\rangle$	$\mathbb{Z}_{14}$	S_4	5

Homology class $H_1 = \mathbb{Z}_{14} \times \mathbb{Z}_2$

Table 4.47: Original relators for $H_1 = \mathbb{Z}_{14} \times \mathbb{Z}_2$

No.	f	G
1	`( 4, 4,12, 1, 1, 1)`	$\langle a, b \mid ab^{-3}ab^{-1}, a^6b^2\rangle$

This group was identified by the GAP as the group `[168,29]`. The center of this group is the cyclic group $\mathbb{Z}_{14}$. The factor of the group by its center is isomorphic to the group D_{12}. The relators are easy to derive: rewrite the first relator into the form $b^3 = ab^{-1}a$ and right multiply it by b^{-1}. The second relator can be easily derived. Rewrite the presentation in the new generators $c = a^{-1}$ and b and use Lemma 4.8. We get $G_1 = \langle c, b \mid c^6 = b^2 = (cb)^{-2}\rangle$.

Table 4.48: Isomorphism classes in $H_1 = \mathbb{Z}_{14} \times \mathbb{Z}_2$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 4, 4,12, 1, 1, 1)`	$\langle a, b \mid a^6 = b^2 = (ab)^{-2}\rangle$	$\mathbb{Z}_{14}$	D_{12}	1

Homology class $H_1 = \mathbb{Z}_{15}$

Table 4.49: Original relators for $H_1 = \mathbb{Z}_{15}$

No.	f	G
1	`( 3, 7, 7, 2, 2, 2)`	$\langle a, b \mid ab^{-3}a^2, a^{-2}ba^{-2}b^{-2}\rangle$
2	`( 5, 5, 7, 4, 2, 4)`	$\langle a, b \mid ab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}, ab^{-2}a^2b^{-1}a\rangle$
3	`( 4, 4,10, 1, 1, 5)`	$\langle a, b \mid ab^{-1}a^{-2}b^{-1}ab^{-1}, a^3baba\rangle$
4	`( 4, 6, 8, 1, 5, 3)`	$\langle a, b \mid ab^{-2}a^2, ab^{-1}a^{-2}b^{-1}a^{-2}b^{-1}\rangle$
5	`( 6, 6, 6, 1, 1, 9)`	$\langle a, b \mid ab^3a^2, ab^2ab^{-1}ab^{-1}ab^{-1}\rangle$
6	`( 6, 6, 6, 3, 3, 5)`	$\langle a, b \mid ab^{-1}a^{-1}b^{-1}ab^{-2}, ab^{-1}abab^{-1}a\rangle$
7	`( 5, 7, 7, 4, 4, 2)`	$\langle a, b \mid b^{-2}a^{-3}b^{-1}, ab^{-2}a^{-1}b^{-2}a\rangle$
8	`( 5, 5,11, 2, 6, 2)`	$\langle a, b \mid ab^{-3}a^3, ab^{-1}ab^{-1}a^{-3}b^{-1}\rangle$

The group `[120,15]` is the group to which the groups G_1, G_2, G_3, G_4, G_6 and G_7 are isomorphic. The center of this group is the cyclic group $\mathbb{Z}_{10}$. The factor of the group by the center is the alternating group A_4. The relators of this group were derived from the group G_1 in the following way: rewrite the second relator into the form $b^2 = a^{-2}ba^{-2}$ and right multiply it by b. The derivation of the second relator is clear.

In the case of the group G_5 rewrite the second relator into the form $b^{-2} = (ab^{-1})^3a$ and right multiply it by b. Then rewrite the presentation in the generators a and $c = ab^{-1}$. Then use Lemma 4.8 to get the presentation $G_5 = \langle a, c \mid c^4 = a^3 = (ca^{-1})^3\rangle$.

For the group G_8 rewrite the relators following way: derive the second relator to the form $a^3 = b^{-1}(ab^{-1})^2$ and left multiply it by a. The derivation of the first relator is easy to do.

$\mathbf{G_8} \cong \mathbf{G_5}$. The isomorphism $G_8 \cong G_5$ is set by $\phi : a \mapsto c, b \mapsto a$.

Table 4.50: Isomorphism classes in $H_1 = \mathbb{Z}_{15}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 3, 7, 7, 2, 2, 2)`	$\langle a, b \mid a^3 = b^3 = (a^{-2}b)^2\rangle$	$\mathbb{Z}_{10}$	A_4	6
`( 5, 5,11, 2, 6, 2)`	$\langle a, b \mid a^4 = b^3 = (ab^{-1})^3\rangle$	$\mathbb{Z}$	$\Delta^+(4,3,3)$	2

Homology class $H_1 = \mathbb{Z}_{16}$

Table 4.51: Original relators for $H_1 = \mathbb{Z}_{16}$

No.	f	G
1	(4, 4, 6, 1, 1, 1)	$\langle a, b \mid ab^2a^2, ab^{-3}ab^{-1}\rangle$
2	(4, 4, 8, 1, 5, 1)	$\langle a, b \mid ab^{-1}ab^{-3}, b^2a^{-5}\rangle$
3	(3, 5, 9, 2, 0, 2)	$\langle a, b \mid ab^3a, ab^{-1}ab^{-1}a^2\rangle$
4	(3, 3,13, 2, 0, 2)	$\langle a, b \mid ab^2a^2, ab^{-3}ab^{-1}\rangle$
5	(5, 7, 7, 0, 2, 2)	$\langle a, b \mid ab^2a^2, b^{-2}ab^{-1}ab^{-1}\rangle$
6	(5, 7, 7, 0, 4,12)	$\langle a, b \mid ab^{-3}ab^{-1}, ba^{-5}b\rangle$
7	(6, 6, 8, 1, 1,11)	$\langle a, b \mid a^4b^3, ab^2ab^{-1}ab^{-1}ab^{-1}\rangle$
8	(3, 3,15, 2, 2, 2)	$\langle a, b \mid ab^{-3}ab^{-1}, ba^{-5}b\rangle$
9	(5, 5,11, 2, 0, 2)	$\langle a, b \mid ab^2a^2, ab^{-3}ab^{-1}\rangle$
10	(5, 7, 9, 2, 8, 0)	$\langle a, b \mid ab^{-1}ab^{-3}, ba^{-1}ba^4\rangle$

The group `[48,1]` is isomorphic to groups G_1, G_3, G_4, G_5 and G_9. This group has the center $\mathbb{Z}_8$ and the factor by the center is the group S_3. The relators of this group can be derived from the presentation of the group G_1. Rewrite the second relator into the form $b^3 = ab^{-1}a$ and right multiply this relator by b^{-1}. The derivation of the first relator is simple. Rewrite the presentation in the generators $c = a^{-1}$ and b and use Lemma 4.8 getting the presentation $G_1 = \langle a, b \mid a^3 = b^2 = (ab)^{-2}\rangle$.

The groups G_2, G_6, G_8 and G_{10} are isomorphic to the group `[80,1]`. The center of this group is $\mathbb{Z}_8$. The factor by the center is the dihedral group D_{10}. The relators of this group can be derived from the presentation of the group G_2. Rewrite the first relator into the form $b^3 = ab^{-1}a$ and right multiply this relator by b^{-1}.

In the case of the group G_7 the relators can be derived in the following way: rewrite the second relator into the form $b^{-2} = (ab^{-1})^3a$ and right multiply it by b^{-1}. Then rewrite the presentation in the generators a and $c = ab^{-1}$.

Table 4.52: Isomorphism classes in $H_1 = \mathbb{Z}_{16}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(4, 4, 6, 1, 1, 1)	$\langle a, b \mid a^3 = b^2 = (ab)^{-2}\rangle$	$\mathbb{Z}_8$	D_6	5
(4, 4, 8, 1, 5, 1)	$\langle a, b \mid a^5 = b^2 = (ab^{-1})^2\rangle$	$\mathbb{Z}_8$	D_{10}	4
(6, 6, 8, 1, 1,11)	$\langle a, b \mid a^4 = b^4 = (a^{-1}b)^3\rangle$	$\mathbb{Z}$	$\Delta^+(4, 4, 3)$	1

Homology class $H_1 = \mathbb{Z}_{16} \times \mathbb{Z}_2$

Table 4.53: Original relators for $H_1 = \mathbb{Z}_{16} \times \mathbb{Z}_2$

No.	f	G
1	(4, 8, 8, 1, 1, 1)	$\langle a, b \mid b^2a^4b^2, ab^{-5}ab^{-1}\rangle$

The derivations to obtain the relators of the fundamental group follow. Rewrite the second relator into the form $b^5 = ab^{-1}a$ and right multiply this relator by b^{-1}. First relator gives $a^{-4} = b^4$.

Table 4.54: Isomorphism classes in $H_1 = \mathbb{Z}_{16} \times \mathbb{Z}_2$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(4, 8, 8, 1, 1, 1)	$\langle a, b \mid a^4 = b^4 = (ab)^{-2}\rangle$	$\mathbb{Z}$	$\Delta^+(4, 4, 2)$	1

Homology class $H_1 = \mathbb{Z}_{17}$

Table 4.55: Original relators for $H_1 = \mathbb{Z}_{17}$

No.	f	G
1	(3, 7,11, 4, 2, 2)	$\langle a, b \mid ab^{-1}a^2b^2a, a^3b^{-3}a^2\rangle$
2	(5, 7, 9, 6, 4, 4)	$\langle a, b \mid ab^{-2}a^{-1}b^2, ab^{-1}abab, a^2b^{-2}a^2b^{-2}a\rangle$

Both fundamental groups have order 2040 . They cannot be identified in GAP as `SmallGroup(n,k)` because their size go over the upper bound of the size of groups included in the GAP library. The center of both groups is isomorphic to the cyclic group of order 34. The factors of each group by its center are isomorphic to the alternating group A_5.

Since there exists a unique involution in both groups, generated by 17th power of generator of the center, the groups can be factorised by this central subgroup of order 2, which means to add $b^{51} = 1$ or $b^{34} = 1$ to the presentations respectively. Factorisation by this involution gives the group $H = Z_{17} \times A_5$ in both cases. Thus the groups are some central extensions of $\mathbb{Z}_2$ by the group H.

On the other hand, since the groups are finite, the associated manifolds are factors of S^3 by fundamental groups. There exist unique central involution in S^3 [34], which identifies antipodal points in $S^3 - S^3/\langle\tau\rangle = PP^3$. Since the group A_5 is a simple group, the involution τ coincides with the unique central involution. Thus the extension of the group H is unique and $G_1 \cong G_2$.

We derive the presentation of the group from the presentation of G_1. We rewrite the first relator into the form $a^2b^2a^2b^{-1} = 1$ and right multiply it by b^3. The derivation of second relator is straightforward.

Table 4.56: Isomorphism classes in $H_1 = \mathbb{Z}_{17}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(3, 7,11, 4, 2, 2)	$\langle a, b \mid a^5 = b^3 = (a^2b^2)^2\rangle$	$\mathbb{Z}_{34}$	A_5	2

Homology class $H_1 = \mathbb{Z}_{18}$

Table 4.57: Original relators for $H_1 = \mathbb{Z}_{18}$

No.	f	G
1	(4, 6,10, 3, 9,15)	$\langle a, b \mid b^{-3}a^{-1}ba^{-1}, b^{-3}a^5b^{-1}\rangle$

The relators of this fundamental group are easy to obtain rewriting the first relator to the form $b^3 = a^{-1}ba^{-1}$ and right multiplying it by b. The derivation of the first relator is clear.

Table 4.58: Isomorphism classes in $H_1 = \mathbb{Z}_{18}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(4, 6,10, 3, 9,15)	$\langle a, b \mid a^5 = b^4 = (a^{-1}b)^2\rangle$	$\mathbb{Z}$	$\Delta^+(5, 4, 2)$	1

Homology class $H_1 = \mathbb{Z}_{19}$

Table 4.59: Original relators for $H_1 = \mathbb{Z}_{19}$

No.	f	G
1	(4, 6, 8, 1, 7, 1)	$\langle a, b \mid ab^{-1}ab^{-4}, ab^{-1}a^{-4}b^{-1}\rangle$
2	(5, 7, 9, 4, 0,14)	$\langle a, b \mid ab^{-1}a^{4}b^{-1}, ab^{-1}ab^{4}\rangle$

Both groups are of the order 2280. Unfortunately, GAP cannot deal with the groups of such big orders. However, the groups are isomorphic.

As concerns the group G_1, rewrite the first relator into the form $b^4 = ab^{-1}a$ and right multiply it by b. The second relator can be derived in a similar way rewriting it into the form $a^4 = b^{-1}ab^{-2}$ and left multiplying it by a.

To determine the relators of the group G_2 rewrite the first relator of the fundamental group into the form $a^4 = ba^{-1}b$ and right multiply it by a^{-1}. The second relator can be rewritten into the form $b^4 = a^{-1}ba^{-1}$ and left multiplied by b.

$\mathbf{G_2} \cong \mathbf{G_1}$. The isomorphism $G_2 \cong G_1$ can be constructed by setting $\phi : a \mapsto b, b \mapsto a$.

Table 4.60: Isomorphism classes in $H_1 = \mathbb{Z}_{19}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(4, 6, 8, 1, 7, 1)	$\langle a, b \mid a^5 = b^3 = (ab^{-1})^2\rangle$	$\mathbb{Z}_{38}$	A_5	2

Homology class $H_1 = \mathbb{Z}_{20}$

Table 4.61: Original relators for $H_1 = \mathbb{Z}_{20}$

No.	f	G
1	(4, 4,10, 1, 1, 7)	$\langle a, b \mid bab^2a^{-1}b, aba^4b\rangle$
2	(4, 6, 8, 1, 1, 3)	$\langle a, b \mid ab^{-1}a^3b^{-1}, aba^2b^2a\rangle$
3	(3, 7, 9, 2, 4, 2)	$\langle a, b \mid ab^3a, ab^{-2}ab^{-2}a^2\rangle$
4	(4, 4,12, 1, 1, 9)	$\langle a, b \mid ab^{-1}a^{-1}b^{-1}ab^{-1}, a^3ba^2ba\rangle$
5	(5, 7, 9, 2, 4, 4)	$\langle a, b \mid ab^{-1}a^{-2}b^{-1}ab^{-2}, ab^{-1}a^2baba\rangle$
6	(5, 7, 9, 4,10, 4)	$\langle a, b \mid ab^{-1}a^2b^2a, b^{-1}a^{-1}b^{-1}a^2b^{-1}a^{-1}b^{-2}\rangle$
7	(5, 7, 9, 6,10,14)	$\langle a, b \mid ab^{-2}a^{-1}b^{-2}a, ab^{-1}ab^2ab^{-1}a^2\rangle$

The groups G_1, G_2, G_3 and G_4 are isomorphic to the group `[60,1]` . The relators of this group can be derived e.g from the presentation of G_3. Rewrite the second relator to the form $a^{-2} = (ab^{-2})^2$ and the first relator to the form $b^3 = a^{-2}$. Rewrite the group presentation in the generators b and $c = a^{-1}$.

The groups G_5, G_6 and G_7 are isomorphic. Set the automorphism α of the group G_6 to be $\alpha : a \mapsto b, b^{-1} \mapsto a$. Rewrite the second relator into the form $a^{-3} = b^{-1}ab^2ab^{-1} = b^{-1}b^2ab^2b^{-3}$ and substitute the first relator into the second. The new presentation is $G_6 = \langle a, b \mid a^2 = b^2ab^2, a^3 = ba^{-3}b^3\rangle$.

Rewrite the presentation of the group G_7 as follows. Rewrite the first relator into the form $a^2 = b^2ab^2$. Substitute the first relator into the second as above. Thus $G_7 = \langle a, b \mid a^2 = b^2ab^2, a^3 = ba^{-3}b^3\rangle$.

$\mathbf{G_7} \cong \mathbf{G_6}$. The isomorphism $G_7 \cong G_6$ can be defined by setting $\phi : a \mapsto a, b \mapsto b$.

In the case of the group G_5 the rewriting the presentation in the generators a and $c = ba$ is used. The new relators are $a^2c^{-1}a^2 = c^{-2}$ and $a^2c^{-1}a^{-1}c^{-1}a^2c^{-1}ac^{-1} = 1$. Rewrite the presentation in the generators a and $d = c^{-1}$. Now substitute the first relator into the second in a similar way as in the case of G_7 to get $a^2d^{-1}d^3a^{-1}d = 1$. Rewrite the relator into the form $d^{-3} = a^{-1}da^2da^{-1} = a^{-1}da^2da^2a^{-3} = a^{-1}d^3a^{-3}$. We end with the presentation $G_5 = \langle d, a \mid d^2 = a^2da^2, d^3 = ad^{-3}a^3\rangle$.

$\mathbf{G_5 \cong G_6}$. The isomorphism $G_5 \cong G_6$ can be set by mapping $\phi : d \mapsto a, a \mapsto b$.

The groups G_1 and G_5 are not isomorphic. We prove it by checking the list of low index normal subgroup up to index 12. The factors by these subgroups follow:

```
g1: [ 1, 1 ] [ 2, 1 ] [ 4, 1 ] [ 5, 1 ] [ 10, 2 ] [ 6, 1 ] [ 12, 1 ],
g5: [ 1, 1 ] [ 2, 1 ] [ 4, 1 ] [ 5, 1 ] [ 10, 2 ].
```

Table 4.62: Isomorphism classes in $H_1 = \mathbb{Z}_{20}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(4, 4,10, 1, 1, 7)	$\langle a, b \mid a^3 = b^2 = (a^2b)^{-2}\rangle$	$\mathbb{Z}_{10}$	D_6	4
(5, 7, 9, 2, 4, 4)	$\langle a, b \mid a^2 = b^2ab^2, a^3 = ba^{-3}b^3\rangle$	?	?	3

Homology class $H_1 = \mathbb{Z}_{21}$

Table 4.63: Original relators for $H_1 = \mathbb{Z}_{21}$

No.	f	G
1	(4, 6, 6, 1, 1, 1)	$\langle a, b \mid ab^3a^2, a^4b^{-1}ab^{-1}\rangle$
2	(5, 5, 9, 2, 0, 2)	$\langle a, b \mid ab^3a, ab^{-1}ab^{-1}a^3b^{-1}\rangle$
3	(6, 6, 8, 1, 9, 1)	$\langle a, b \mid ab^{-3}a^4, ab^{-1}ab^{-1}a^{-4}b^{-1}\rangle$
4	(3, 5,13, 2, 0, 2)	$\langle a, b \mid ab^3a^2, ab^{-4}ab^{-1}\rangle$
5	(7, 7, 7, 0, 2, 2)	$\langle a, b \mid ab^2a^2, b^{-2}ab^{-1}ab^{-1}ab^{-1}\rangle$

All groups except G_3 are isomorphic to the group `[168,22]`. This group has the center $\mathbb{Z}_{14}$ and the factor by the center A_4. The relators of the group are easy to derive from the presentation of the group G_1: rewrite the second relator into the form $a^4 = ba^{-1}b$ and left multiply it by a^{-1}. The first relator can be rewritten into the form $a^3 = b^{-3}$. Rewrite the presentation in the generators a and $c = b^{-1}$ and use Lemma 4.8 to derive the final form of the presentation.

The case of the group G_3 can be solved as follows. Rewrite the second relator to the form $a^4 = b^{-1}(ab^{-1})^2$. Left multiply this relator by a. The derivation of the first relator is obvious.

Table 4.64: Isomorphism classes in $H_1 = \mathbb{Z}_{21}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(4, 6, 6, 1, 1, 1)	$\langle a, b \mid a^3 = b^3 = (ab)^{-2}\rangle$	$\mathbb{Z}_{14}$	A_4	4
(6, 6, 8, 1, 9, 1)	$\langle a, b \mid a^5 = b^3 = (ab^{-1})^3\rangle$	$\mathbb{Z}$	$\Delta^+(5, 3, 3)$	1

Homology class $H_1 = \mathbb{Z}_{22}$

Table 4.65: Original relators for $H_1 = \mathbb{Z}_{22}$

No.	f	G
1	(4, 6,10, 5, 1, 1)	$\langle a, b \mid ab^{-4}a^2, ab^{-1}a^2b^3a\rangle$
2	(4, 8, 8, 1, 9, 1)	$\langle a, b \mid ab^{-1}a^{-4}b^{-1}, ab^{-5}ab^{-1}\rangle$
3	(6, 6, 8, 1, 1, 9)	$\langle a, b \mid a^2b^2a^2b, ab^{-1}a^{-1}b^{-1}ab^{-1}ab^{-1}\rangle$
4	(3, 7,11, 2, 2, 2)	$\langle a, b \mid ab^{-3}a^3, ab^2a^{-1}b^2a^2\rangle$

The groups G_1, G_3 and G_4 are isomorphic to `[528,87]`. The center of this group is the cyclic group $\mathbb{Z}_{22}$. The factor by the center is the symmetric group S_4. The relators of this group can be derived from the relators of the G_1. Rewrite the second relator into the form $b^2 = a^{-2}ba^{-2}$ and left multiply this relator by b. The derivation of the first relator is clear.

The relators of the group G_2 can be derived as follows. Rewrite the first relator into the form $a^4 = b^{-1}ab^{-1}$ and left multiply this new relator by a. The second relator can be derived in a similar way. Rewrite it into the form $b^5 = ab^{-1}a$ and right multiply this relator by b^{-1}.

Table 4.66: Isomorphism classes in $H_1 = \mathbb{Z}_{22}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 4, 6, 10, 5, 1, 1)`	$\langle a, b \mid a^4 = b^3 = (ab^{-2})^2 \rangle$	$\mathbb{Z}_{22}$	S_4	3
`( 4, 8, 8, 1, 9, 1)`	$\langle a, b \mid a^5 = b^4 = (ab^{-1})^2 \rangle$	$\mathbb{Z}$	$\Delta^+(5, 4, 2)$	1

Homology class $H_1 = \mathbb{Z}_{24}$

Table 4.67: Original relators for $H_1 = \mathbb{Z}_{24}$

No.	f	G
1	`( 4, 4,10, 1, 1, 1)`	$\langle a, b \mid ab^{-3}ab^{-1}, ab^2a^4 \rangle$
2	`( 5, 5, 9, 2, 2, 2)`	$\langle a, b \mid ab^{-2}a^{-2}b^{-2}, a^3ba^3b^{-1} \rangle$
3	`( 5, 7, 7, 2, 2, 4)`	$\langle a, b \mid a^{-2}b^{-1}a^{-1}b^{-1}a^{-2}b, b^2a^{-1}b^2aba, ab^{-2}ab^{-1}a^2b^{-1} \rangle$
4	`( 4, 4,12, 1, 5, 1)`	$\langle a, b \mid ab^{-3}ab^{-1}, ba^{-7}b \rangle$
5	`( 4, 6,10, 1, 7, 1)`	$\langle a, b \mid ab^{-1}ab^{-4}, a^5ba^{-1}b \rangle$
6	`( 3, 9, 9, 0, 2, 2)`	$\langle a, b \mid ab^{-3}ab^{-1}, ab^2a^4 \rangle$
7	`( 5, 5,11, 2, 2, 8)`	$\langle a, b \mid ab^{-1}a^3ba^2, ab^{-1}a^{-1}b^{-1}a^{-2}b^{-1}a^{-1}b^{-1} \rangle$
8	`( 5, 5,11, 2, 4, 2)`	$\langle a, b \mid ab^{-1}a^3ba^2, a^4b^{-2}ab^{-2} \rangle$
9	`( 5, 7, 9, 2, 2, 4)`	$\langle a, b \mid ab^{-2}a^{-1}b^{-2}, a^2ba^2ba^2b^{-1} \rangle$
10	`( 7, 7, 7, 2, 2, 6)`	$\langle a, b \mid ab^{-2}ab^{-1}a^2b^{-1}, a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}ba^{-1}b \rangle$

The groups G_1 and G_6 are isomorphic to the group `[120,2]` . The center of this group is the cyclic group $\mathbb{Z}_{12}$ and the factor group by the center is the dihedral group D_{10}. The presentation of this group can be derived from the presentation of the group G_1. Rewrite the presentation in the generators a and $c = b^{-1}$. Rewrite the first relator to the form $c^3 = a^{-1}c^{-1}a^{-1}$ and left multiply it by c^{-1}. The derivation of the second relator is simple. In this way we get the presentation $G_1 = \langle a, c \mid a^5 = c^2 = (ac)^{-2} \rangle$.

The group G_4 is known as `[168,4]` in GAP. The center is the cyclic group $\mathbb{Z}_{12}$ and the factor group by the center is D_{14}. The presentation can be obtained by rewriting the first relator into the form $b^3 = ab^{-1}a$ and right multiplying it by b^{-1}. The second relator can be rewritten into the form $a^7 = b^2$.

In the case of the group G_5 rewrite the relators into the form $b^4 = ab^{-1}a$ and $a^5 = b^{-1}ab^{-1}$. Then right multiply the first relator by b^{-1} and left multiply the second relator by a. The center of this group is $\mathbb{Z}$ and the factor by the center is the group $\Delta^+(6, 3, 2)$.

Rewrite the the presentation of the group G_2 into the final form $G_2 = \langle a, b \mid a = b^2a^2b^2, b = a^3ba^3 \rangle$.

Now we shall deal with the group G_9. Rewrite the second relator into the form $(a^2b)^3 = b^2$ by right multiplying it by b^2. Rewrite the first relator into the form $a = b^2ab^2$ and substitute the second relator instead of b^2. In this way the first relator transforms into $a = (a^2b)^3a(a^2b)^3$. Rewrite the presentation of the group G_9 in the generators a and $c = a^2b$. The second relator is now $c^3 = a^{-2}ca^{-2}c$ and can be rewritten into the form $c = a^2c^2a^2$. We get $G_9 = \langle a, c \mid a = c^3ac^3, c = a^2c^2a^2 \rangle$.

$\mathbf{G_9 \cong G_2}$. The isomorphism $G_9 \to G_2$ can be set by mapping $\phi : c \mapsto a, a \mapsto b$.

The presentation of the group G_3 can be derived in the following way. Rewrite the first relator into the form $bab = a^{-2}ba^{-2}$ and left multiply this relator by a to get $(ab)^2 = a^{-1}ba^{-2}$. Rewrite the second relator into the form $aba = b^{-2}ab^{-2}$ and right multiply it by b to get $(ab)^2 = b^{-2}ab^{-1}$. Third relator can be derived from previous two. Rewrite the presentation of the group in the generators a and $c = ab$. The first relator is $c^2 = a^{-2}ca^{-2}$. The final form is $c = a^2c^2a^2$. The second relator is $c^2 = c^{-1}ac^{-1}a^2c^{-1}a$. It follows from the first relator that we can replace the last appearance of c^{-1} by $c^{-1} = a^{-2}c^{-2}a^{-2}$ in the second relator. In this way we get $c^2 = c^{-1}ac^{-1}a^2c^{-1}a = c^{-1}ac^{-1}a^2(a^{-2}c^{-2}a^{-2})a = c^{-1}ac^{-3}a^{-1}$, hence we have $a = c^3ac^3$. The group $G_3 = \langle a, c \mid c = a^2c^2a^2, a = c^3ac^3\rangle$.

$\mathbf{G_3} \cong \mathbf{G_2}$. The isomorphism $G_3 \to G_2$ can be set by mapping $\phi : c \mapsto a, a \mapsto b$.

Rewrite the presentation of the group G_7 in the generators a and $c = ba^{-1}$. The first relator turns into $c = a^3ca^3$. The second relator is $a^{-3} = ca^2c^2a^2c$. Replacing the left-hand side of the second relator by $a^{-3} = ca^3c^{-1}$ we get the final form $a = c^2a^2c^2$. Thus we get the presentation $G_7 = \langle a, c \mid a = c^2a^2c^2, c = a^3ca^3\rangle$.

$\mathbf{G_7} \cong \mathbf{G_2}$. The isomorphism $G_7 \to G_2$ can be set by mapping $\phi : a \mapsto a, c \mapsto b$.

Rewrite the presentation of the group G_8 into the form $G_8 = \langle a, b \mid a^{-3} = ba^3b^{-1}, a^4 = b^2a^{-1}b^2\rangle$.

$\mathbf{G_{10}} \cong \mathbf{G_8}$. The groups G_8 and G_{10} are isomorphic. To prove this fact, derive the relators of G_{10} at first. Rewrite the presentation of the group G_{10} in the generators a and $c = ba$. The second relator is $c^{-3} = ac^{-1}a^2c^{-1}a$. The first relator is $a^2c^{-1}ac^{-1}a^2c^{-1}a^3c^{-1} = 1$. Let us substitute the second relator into the first: $a^2c^{-1}ac^{-1}a^2c^{-1}a^3c^{-1} = a^2c^{-1}(ac^{-1}a^2c^{-1}a)a^2c^{-1} = a^2c^{-1}c^{-3}a^2c^{-1} = a^2c^{-4}a^2c^{-1} = 1$. The final form of the first relator is $c^4 = a^2c^{-1}a^2$. Use this form in the derivation of the second relator: $c^{-3} = ac^{-1}a^2c^{-1}a = ac^{-1}(a^2c^{-1}a^2)a^{-1} = ac^{-1}c^4a^{-1} = ac^3a^{-1}$. Hence we have a presentation $G_{10} = \langle a, c \mid c^4 = a^2c^{-1}a^2, c^{-3} = ac^3a^{-1}\rangle$. The isomorphism $G_{10} \to G_8$ can be set by mapping $\phi : c \mapsto a, a \mapsto b$.

The groups G_1, G_2, G_4, G_5 and G_8 are mutually non-isomorphic which can be checked by using GAP. The result follows from the examination of low index normal subgroups of these groups up to index 12. The factors by these normal subgroups follow.

```
g1 : [ 1, 1 ] [ 2, 1 ] [ 3, 1 ] [ 4, 1 ] [ 8, 1 ] [ 6, 2 ]
     [ 12, 2 ] [ 10, 1 ],
g4 : [ 1, 1 ] [ 2, 1 ] [ 4, 1 ] [ 3, 1 ] [ 6, 2 ] [ 12, 2 ] [ 8, 1 ],
g5 : [ 1, 1 ] [ 2, 1 ] [ 10, 1 ] [ 4, 1 ] [ 8, 1 ],
g9 : [ 1, 1 ] [ 2, 1 ] [ 4, 1 ] [ 8, 1 ] [ 3, 1 ] [ 6, 2 ] [ 12, 2 ]
     [ 12, 3 ],
g10: [ 1, 1 ] [ 2, 1 ] [ 4, 1 ] [ 8, 1 ] [ 3, 1 ] [ 6, 2 ] [ 12, 2 ]
     [ 10, 1 ] [ 12, 3 ].
```

Table 4.68: Isomorphism classes in $H_1 = \mathbb{Z}_{24}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(4, 4,10, 1, 1, 1)	$\langle a, b \mid a^5 = b^2 = (ab)^{-2}\rangle$	$\mathbb{Z}_{12}$	D_{10}	2
(4, 4,12, 1, 5, 1)	$\langle a, b \mid a^7 = b^2 = (ab^{-1})^2\rangle$	$\mathbb{Z}_{12}$	D_{14}	1
(4, 6,10, 1, 7, 1)	$\langle a, b \mid a^6 = b^3 = (ab^{-1})^2\rangle$	$\mathbb{Z}$	$\Delta^+(6,3,2)$	1
(5, 5, 9, 2, 2, 2)	$\langle a, b \mid a = b^2a^2b^2, b = a^3ba^3\rangle$	?	?	4
(5, 5,11, 2, 4, 2)	$\langle a, b \mid b = a^3ba^3, a^4 = b^2a^{-1}b^2\rangle$	?	?	2

Homology class $H_1 = \mathbb{Z}_{26}$

Table 4.69: Original relators for $H_1 = \mathbb{Z}_{26}$

No.	f	G
1	`( 4, 6, 8, 1, 1, 1)`	$\langle a, b \mid ab^{-4}ab^{-1}, ab^3a^3 \rangle$
2	`( 5, 7, 9, 2, 0, 2)`	$\langle a, b \mid ab^4a, ab^{-1}ab^{-1}a^3b^{-1} \rangle$

Both fundamental groups in the homology class were identified by the GAP as the group `[624,131]`. The center of this group is the cyclic group $\mathbb{Z}_{26}$ and the factor group by the center is the symmetric group S_4. The relators of the fundamental group can be derived e.g. from relators of the group G_1. Rewrite relators to the form $a^{-4} = b^3$ and $b^4 = ab^{-1}a$. Right multiply the second relator by b^{-1}. Then rewrite the presentation in the generators $c = a^{-1}$ and b and use Lemma 4.8 to get the final form of presentation.

Table 4.70: Isomorphism classes in $H_1 = \mathbb{Z}_{26}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 4, 6, 8, 1, 1, 1)`	$\langle a, b \mid a^4 = b^3 = (ab)^{-2} \rangle$	$\mathbb{Z}_{26}$	S_4	2

Homology class $H_1 = \mathbb{Z}_{27}$

Table 4.71: Original relators for $H_1 = \mathbb{Z}_{27}$

No.	f	G
1	`( 4, 6,10, 7, 1, 1)`	$\langle a, b \mid a^3b^3, a^4b^{-1}a^2b^{-1}a \rangle$
2	`( 6, 6, 8, 1, 1, 3)`	$\langle a, b \mid aba^2b^2a, a^3b^{-1}ab^{-1}ab^{-1} \rangle$

The fundamental groups are isomorphic to the group `[216,3]` . The center of this group is the cyclic group $\mathbb{Z}_{18}$, the factor of this group by its center is the alternating group A_4. The relators can be derived from the relators of G_1. Rewrite the relators to the form $a^3 = b^{-3}$ and $a^5 = ba^{-2}b$. Right multiply the second relator by a^{-2}. Then rewrite the presentation in the generators a and $c = b^{-1}$.

Table 4.72: Isomorphism classes in $H_1 = \mathbb{Z}_{27}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 4, 6,10, 7, 1, 1)`	$\langle a, b \mid a^3 = b^3 = (a^2b)^{-2} \rangle$	$\mathbb{Z}_{18}$	A_4	2

Homology class $H_1 = \mathbb{Z}_{31}$

Table 4.73: Original relators for $H_1 = \mathbb{Z}_{31}$

No.	f	G
1	`( 4, 6,10, 1, 1, 1)`	$\langle a, b \mid ab^{-4}ab^{-1}, a^5b^3 \rangle$

Since this group is of order 3720, it is not present in the GAP catalogue. Derive the relators of the fundamental group by rewriting the presentation in generators $c = a^{-1}$ and b. Then rewrite the second relator to the form $b^4 = c^{-1}b^{-1}c^{-1}$ and left multiply the new relator by b^{-1} to get $b^3 = (cb)^{-2}$. The derivation of the first relator is easy to do.

Table 4.74: Isomorphism classes in $H_1 = \mathbb{Z}_{31}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 4, 6,10, 1, 1, 1)`	$\langle a, b \mid a^5 = b^3 = (ab)^{-2} \rangle$	$\mathbb{Z}_{62}$	A_5	1

Homology class $H_1 = \mathbb{Z}_{33}$

Table 4.75: Original relators for $H_1 = \mathbb{Z}_{33}$

No.	f	G
1	(6, 6, 8, 1, 1, 1)	$\langle a, b \mid ab^3a^3, ab^{-4}ab^{-1}ab^{-1}\rangle$

Rewrite the presentation in the generators $c = a^{-1}$ and b. Rewriting the second relator to the form $b^4 = (c^{-1}b^{-1})^2c^{-1}$ and left multiplying by b^{-1} we get the final form of the relator: $b^3 = (cb)^{-3}$. The derivation of the first relator is obvious.

Table 4.76: Isomorphism classes in $H_1 = \mathbb{Z}_{33}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(6, 6, 8, 1, 1, 1)	$\langle a, b \mid a^4 = b^3 = (ab)^{-3}\rangle$	$\mathbb{Z}$	$\Delta^+(3,3,3)$	1

4.7.2 Infinite homology groups

Homology class $H_1 = \mathbb{Z}$

Table 4.77: Original relators for homology class $H_1 = \mathbb{Z}$

No.	f	G
1	(3, 7, 7, 4, 6, 0)	$\langle a, b \mid a^3b^{-3}, a^5b^{-5}, a^2b^{-2}\rangle$
2	(5, 5, 7, 4, 4, 6)	$\langle a, b \mid ab^{-1}a^{-1}baba^{-1}b^{-1}, a^2b^{-1}a^{-1}ba^{-1}b^{-1}, ab^{-1}a^{-2}b^{-1}ab\rangle$
3	(4, 4,10, 3, 3, 9)	$\langle a, b \mid ab^{-1}a^{-1}ba^{-1}b^{-1}, a^2b^{-1}a^{-4}b^{-1}, ab^{-1}a^{-3}b^{-1}ab\rangle$
4	(4, 6, 8, 5, 7,13)	$\langle a, b \mid a^3b^{-1}a^3b^{-2}a^{-1}b^{-2}, a^3b^{-3}, a^2b^{-2}\rangle$
5	(6, 6, 6, 5, 5, 5)	$\langle a, b \mid ab^{-1}a^{-1}baba^{-1}b^{-1}, a^2b^{-1}a^{-1}b^2a^{-1}b^{-1}, ab^{-1}a^{-1}ba^{-1}b^{-1}ab\rangle$
6	(3, 5,11, 4, 4,14)	$\langle a, b \mid ab^{-2}ab, a^6b^{-3}, a^5b^{-1}a^{-1}b^{-1}\rangle$
7	(6, 6, 8, 1, 3, 9)	$\langle a, b \mid ab^{-1}a^{-2}b^{-1}ab^{-1}, ab^{-1}a^{-1}b^{-1}, ab^{-1}a^{-2}b^{-2}a^{-2}b^{-1}ab^{-1}abab^{-1}\rangle$
8	(6, 6, 8, 5, 5, 7)	$\langle a, b \mid ab^{-1}a^{-1}ba^2ba^{-1}b^{-1}, a^2b^{-1}a^{-1}baba^{-1}b^{-1}, ab^{-1}a^{-1}ba^{-1}b^{-1}ab\rangle$

The group G_1 is isomorphic to $\mathbb{Z}$. The generator a is free and $a = b$. The same method can be used in the case of the group G_4. Rewrite the second relator in the presentation of the group G_7 to the form $a = bab$ to get the equation $a = b$. Thus $G_7 \cong \mathbb{Z}$, since substituting a to other relators gives no new relation for a.

Rewrite the presentation of the group G_2 in the generators a and $c = ab^{-1}$. The first relator is $a^2 = cac^{-2}ac$, the second relator is $cac = aca$. The third relator follows from previous two. Rewrite the relators to the form $a^3 = (ac)ac^{-2}ac$ and $(ac)^2 = a(ac)a$. Rewrite the presentation in the generators a and $d = ac$. The second relator is $d^2 = ada$ and the first one is $a^3 = dad^{-1}ad^{-1}ad$. Let us multiply the second relator from left and from right by a. The new form of this relator is $a^5 = adad^{-1}ad^{-1}ada$. Insert now the second relator into the first. The result is $a^5 = dad$. Then left multiply the first relator by a and the second relator from right by d. The final form of the presentation of this group is $G_2 = \langle a, d \mid a^6 = d^3 = (ad)^2\rangle$.

In the case of the group G_3 the rewriting of the presentation in generators a and $c = ab^{-1}$ is useful. The first relator transforms into $c^2 = aca$, the second relator transforms into $a^5 = cac$. The third relator can be derived by using previous two relators. Now right multiply the first relator by c and the second relator from left by a.

$\mathbf{G_3} \cong \mathbf{G_2}$ The isomorphism of the groups $G_3 \to G_2$ can be set by mapping $\phi : a \mapsto a, c \mapsto d$.

To derive the presentation of G_6 rewrite the first relator into the form $b^2 = aba$ and the third one into the form $a^5 = bab$. Now right multiply the first relator by b and left multiply the third relator by a. The second relator follows from the first and from the third relator. Since $G_2 = \langle a, d \mid a^6 = d^3 = (ad)^2\rangle$ and $G_6 = \langle a, b \mid a^6 = b^3 = (ab)^2\rangle$, the isomorphism of the groups $\phi : G_6 \to G_2$ can be defined by setting $\phi : a \mapsto a, b \mapsto d$.

In the case of group G_5 rewrite the second relator into the form $ab^{-2}a = b^{-1}a^2b^{-1}$. Combining the second and the third relator we get the relator $b^{-1}ab^2 = a^2ba^{-1}$.

The relators of G_8 rewrite as follows. Substituting the third relator into the second one we get $a^3 = b^{-1}ab^2ab^{-1}$. The first relator depends on the following two. The third relator can be rewritten into the form $ab^{-1}ab = b^{-1}a^{-1}ba^{-1}$.

It remains to show that G_1, G_2, G_5 and G_8 are mutually non isomorphic. The method to prove is based on the comparison of lengths of lists of normal subgroups of index up to 12. The group G_1 contains 12 such subgroups, the group G_2 contains 15 such subgroups, the group G_5 contains 131 such subgroups and the group G_8 contains 14 such subgroups. It follows that these groups cannot isomorphic.

Table 4.78: Isomorphism classes in $H_1 = \mathbb{Z}$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(3, 7, 7, 4, 6, 0)	$\mathbb{Z}$	$\mathbb{Z}$	1	3
(5, 5, 7, 4, 4, 6)	$\langle a, b \mid a^6 = b^3 = (ab)^2\rangle$	$\mathbb{Z}$	$\Delta^+(6, 3, 2)$	3
(6, 6, 6, 5, 5, 5)	$\langle a, b \mid a^2ba^{-1} = b^{-1}ab^2, b^{-1}a^2b^{-1} = ab^{-2}a\rangle$	?	?	1
(6, 6, 8, 5, 5, 7)	$\langle a, b \mid a^3 = b^{-1}ab^2ab^{-1}, ab^{-1}ab = b^{-1}a^{-1}ba^{-1}\rangle$	?	?	1

Homology class $H_1 = \mathbb{Z}_2 \times \mathbb{Z}$

Table 4.79: Original relators for homology class $H_1 = \mathbb{Z}_2 \times \mathbb{Z}$

No.	f	G
1	(1, 1, 7, 2, 0, 2)	$\langle a, b \mid a^2b^{-1}a^2b, a^2\rangle$
2	(2, 2,10, 3, 1, 3)	$\langle a, b \mid a^2b^{-1}a^{-2}ba^{-2}b^{-1}a^2b, a^2\rangle$
3	(3, 3, 9, 2, 0, 4)	$\langle a, b \mid ab^{-1}a^2baba^2b^{-1}, a^2b^{-1}a^2b, a^2\rangle$
4	(4, 6, 6, 5, 5, 3)	$\langle a, b \mid a^2b^{-1}a^{-1}b^2a^{-1}b^{-1}, a^3b^{-1}a^{-1}b^{-1}, ab^{-3}ab\rangle$
5	(3, 7, 7, 4, 4,12)	$\langle a, b \mid ab^{-3}ab, a^4b^{-4}, a^3b^{-1}a^{-1}b^{-1}\rangle$
6	(3, 3,13, 4, 2, 4)	$\langle a, b \mid a^2b^{-1}a^{-2}ba^2b^{-1}a^2ba^{-2}b^{-1}a^2b, a^2\rangle$
7	(5, 5,11, 2, 0, 6)	$\langle a, b \mid ab^{-1}a^2bababa^2b^{-1}ab^{-1}, ab^{-1}a^2baba^2b^{-1}, a^2\rangle$

The fact $G_1 \cong G_2 \cong G_3 \cong G_6 \cong G_7 \cong \mathbb{Z}_2 * \mathbb{Z}$ is obvious. After some substitutions the fact that b is free and $a^2 = 1$ can be easily derived.

Groups G_4 and G_5 are isomorphic. To prove it, rewrite the second relator of G_4 into the form $a^3 = bab$ and the third relator into the form $b^3 = aba$. Now left multiply the second relator by a and right multiply the third one by b. The first relator depends on the previous two. In the case of G_5 rewrite the first relator into the form $b^3 = aba$ and right multiply it by b. The derivation of the second relator is obvious. The third relator follows from previous two. Now set the isomorphism $G_5 \to G_4$ by mapping $\phi : a \mapsto a, b \mapsto b$.

Table 4.80: Isomorphism classes in $H_1 = \mathbb{Z} \times \mathbb{Z}_2$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(4, 6, 6, 5, 5, 3)	$\langle a, b \mid a^4 = b^4 = (ab)^2\rangle$	$\mathbb{Z}$	$\Delta^+(4, 4, 2)$	2
(2, 2,10, 3, 1, 3)	$\mathbb{Z}_2 * \mathbb{Z}$	1	$\mathbb{Z}_2 * \mathbb{Z}$	5

Homology class $H_1 = \mathbb{Z}_3 \times \mathbb{Z}$

Table 4.81: Original relators for homology class $H_1 = \mathbb{Z}_3 \times \mathbb{Z}$

No.	f	G
1	(1, 1,11, 2, 0, 2)	$\langle a, b \mid a^3b^{-1}a^3b, a^3 \rangle$
2	(5, 5, 7, 2, 6, 6)	$\langle a, b \mid ab^{-1}ab^{-1}a^{-2}b^{-1}, ab^{-1}a^{-1}baba^{-1}b^{-1},$ $a^2b^{-1}a^{-1}b^{-1}a^{-1}b^{-1} \rangle$
3	(2, 2,16, 3, 1, 3)	$\langle a, b \mid a^3b^{-1}a^{-3}ba^{-3}b^{-1}a^3b, a^3 \rangle$
4	(2, 2,16, 3, 1, 9)	$\langle a, b \mid a^3b^{-1}a^{-3}ba^{-3}b^{-1}a^3b, a^3 \rangle$
5	(6, 6, 8, 1, 9, 7)	$\langle a, b \mid ab^{-1}ab^{-1}ab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1},$ $ab^{-1}ab^{-1}a^{-2}b^{-1}, a^2b^{-1}a^{-1}b^{-1}a^{-1}b^{-1} \rangle$
6	(3, 3,15, 2, 0, 4)	$\langle a, b \mid ab^{-1}a^3ba^2ba^3b^{-1}, a^3b^{-1}a^3b, a^3 \rangle$

The isomorphisms $G_1 \cong G_3 \cong G_4 \cong G_6 \cong \mathbb{Z}_3 * \mathbb{Z}$ can be easily checked from the presentations.

Rewrite the relators of the group G_2 into the form $a^2 = b^{-1}ab^{-1}ab^{-1}$ and $a^2 = babab$. The second relator can be derived from the first relator. Left multiply both relators by a. Then rewrite the presentation of the group G_2 in the generators b and $c = ab^{-1}$ to get $G_2 = \langle c, b \mid c^3 = (cb)^3 = (cb^2)^3 \rangle$.

Rewrite the relators of G_5 to the form $a^2 = b^{-1}(ab^{-1})^2$ (the second) and $a^2 = b(ab)^2$ (the third). The first relator can be derived by using the second and the third relator. Left multiply both relators by a and rewrite the presentation in the generators $d = ab^{-1}$ and b. We get $G_5 = \langle b, d \mid b^3 = (db)^3 = (db^2)^3 \rangle$.

$\mathbf{G_2} \cong \mathbf{G_5}$ The isomorphism $G_2 \to G_5$ is done by setting $\phi : c \mapsto d, b \mapsto b$.

The group G_2 is not isomorphic to $\mathbb{Z}_3 * \mathbb{Z}$ since the list of factors by low-index subgroups up to index six contains the symmetric group S_3 but $\mathbb{Z}_3 * \mathbb{Z}$ admits no such a factor by low-index normal subgroup.

Remark. Referring Matveev [25] and the classification results in [3], 3-manifold with the fundamental group G_2 is Euclidean.

Table 4.82: Isomorphism classes in $H_1 = \mathbb{Z} \times \mathbb{Z}_3$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(5, 5, 7, 2, 6, 6)	$\langle a, b \mid a^3 = (ab)^3 = (ab^2)^3 \rangle$	?	?	2
(1, 1,11, 2, 0, 2)	$\mathbb{Z}_3 * \mathbb{Z}$	1	$\mathbb{Z}_3 * \mathbb{Z}$	4

Homology class $H_1 = \mathbb{Z}_4 \times \mathbb{Z}$

Table 4.83: Original relators for homology class $H_1 = \mathbb{Z}_4 \times \mathbb{Z}$

No.	f	G
1	(1, 1,15, 2, 0, 2)	$\langle a, b \mid a^4b^{-1}a^4b, a^4 \rangle$
2	(6, 6, 6, 1, 7, 7)	$\langle a, b \mid ab^{-1}ab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}, ab^{-2}ab^{-1}a^{-2}b^{-1},$ $ab^{-1}a^{-1}baba^{-1}b^{-1} \rangle$

The fact, that $G_1 = \mathbb{Z}_4 * \mathbb{Z}$ can be obtained from the presentation of G_1 in obvious way.

Rewrite the presentation of the group G_2 in the generators a and $c = ab^{-1}$. The presentation in these generators is $G_2 = \langle a, c \mid c^3 = a^2c^{-1}a, a^3 = c^2a^{-1}c^2 \rangle$. Third relator can be derived from the previous two.

The non-isomorphism of the groups G_2 and G_1 can be proved by browsing the list of subgroups of index three‖. In the case of G_2 there are seven subgroups of index three. The group G_1 contains only six conjugacy classes of subgroups of index three. Thus $G_2 \ncong G_1$.

Table 4.84: Isomorphism classes in $H_1 = \mathbb{Z} \times \mathbb{Z}_4$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 6, 6, 6, 1, 7, 7)`	$\langle a, b \mid a^3 = b^2a^{-1}b^2, b^3 = a^2b^{-1}a^2\rangle$	?	?	1
`( 1, 1,15, 2, 0, 2)`	$\mathbb{Z}_4 * \mathbb{Z}$	1	$\mathbb{Z}_4 * \mathbb{Z}$	1

Homology class $H_1 = \mathbb{Z}_5 \times \mathbb{Z}$

Table 4.85: Original relators for homology class $H_1 = \mathbb{Z}_5 \times \mathbb{Z}$

No.	f	G
1	`( 6, 6, 8, 5,11, 7)`	$\langle a, b \mid ab^{-1}aba^2bab^{-1}, a^2b^{-1}ababab^{-1}, ab^{-1}aba^{-1}b^{-1}a^{-1}b\rangle$
2	`( 1, 1,19, 2, 0, 2)`	$\langle a, b \mid a^5b^{-1}a^5b, a^5\rangle$
3	`( 1, 1,19, 2, 0, 6)`	$\langle a, b \mid a^5b^{-1}a^5b, a^5\rangle$

The groups G_2 and G_3 are isomorphic to the group $\mathbb{Z}_5 * \mathbb{Z}$, which is easy to prove straight from the presentations.

Rewrite the presentation of the group G_1 in the generators a and $c = ab$. The first relator is $cb^{-2}c^2 = b^2c^{-2}b$. Third relator can be transformed into the form $bc^{-1}b = b^{-1}c^2b^{-1}$. The second relator can be derived by the substitution of the third relator into the first.

The groups G_1 and G_2 are not isomorphic. The proof is based on browsing the classes of conjugacy of subgroups of index five. The list of conjugacy classes of index five contains 5 classes for the group G_1. The group G_2 contains only four (conjugated) subgroups of index five.

Table 4.86: Isomorphism classes in $H_1 = \mathbb{Z} \times \mathbb{Z}_5$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
`( 6, 6, 8, 5,11, 7)`	$\langle a, b \mid ab^{-2}a^2 = b^2a^{-2}b, ba^{-2}b = a^{-1}b^2a^{-1}\rangle$	?	?	1
`( 1, 1,19, 2, 0, 2)`	$\mathbb{Z}_5 * \mathbb{Z}$	1	$\mathbb{Z}_5 * \mathbb{Z}$	2

‖these subgroups are the representatives of conjugacy classes

4.8 Summary

Let us summarize our analysis of fundamental groups of 3-manifolds given by admissible 6-tuples. Our input catalogue contains **433** admissible 6-tuples of complexity at most 21. Let us recall that the catalogue was already reduced by excluding the six-tuples which were recognised as those coding lens spaces and their connected sums. We have found **78** isomorphism classes of fundamental groups of prime 3-manifolds of genus two comming from our catalogue. The remaining six-tuples in our catalogue give rise either to cyclic groups or to free products of cyclic groups. These six-tuples by Theorems 2.12 and 2.14 code either 3-manifolds of genus at most one or decomposable 3-manifolds of genus two. More detailed description of fundamental groups follows.

Trivial group – 3-manifolds of genus 0 There is only **one** admissible 6-tuple, namely $f =$`(3, 7,11, 6, 0,12)`, such that $\pi_1(f) \cong 1$.

Cyclic groups – 3-manifolds of genus 1 The 1019 admissible 6-tuples up to complexity 21 give rise to cyclic groups $\mathbb{Z}_n, 2 \leq n \leq 29$ and infinite cyclic group. Appendix C contains representatives of cyclic isomorphism classes derived from the reduced catalogue. The cases where we were able to determine the lens space $\mathcal{L}(p, q)$ coded by the 6-tuple are indicated by specifying the two paramaters p, q.

Free products of cyclic groups – decomposable 3-manifolds of genus 2 There are **137** non-prime 3-manifolds of genus 2 coded by admissible 6-tuples up to complexity 21. They are connected sums of lens spaces and $S^1 \times S^2$, respectively. Free products are of the form $\mathbb{Z}_m * \mathbb{Z}_n$ and $\mathbb{Z} * \mathbb{Z}_n$ for some parameters m, n, see Appendices C and B for details.

Acyclic groups – prime 3-manifolds of genus 2 There are **78** isomorphism classes of acyclic fundamental groups represented by admissible 6-tuples up to complexity 21. These 6-tuples represent prime 3-manifolds of genus 2. Among them, there are **71** 6-tuples admitting finite homology group and **7** ones admitting infinite homology group. Further, we have identified **39** classes with finite fundamental group and **39** classes with infinite fundamental group. Finite fundamental groups, we found, are all mentioned by Milnor [25, p. 405]. Moreover, we have found also **4** 6-tuples representing compact, connected Euclidean 3-manifolds. See Appendix C and B for details.

The main result of our analysis follow.

Theorem 4.17 *Prime 3-manifolds of genus at most two, represented by admissible 6-tuples of complexity ≤ 21, have fundamental groups of one of the following types:*

- *trivial group,*
- *cyclic groups $\mathbb{Z}_n, 2 \leq n \leq 29$,*
- *infinite cyclic group $\mathbb{Z}$,*
- *acyclic groups.*

*The list includes **78** isomorphism classes of non-trivial acyclic fundamental groups of prime 3-manifolds of genus two. Among them, there are **39** elliptic manifolds with finite groups, **4** Euclidean manifolds and **35** other manifolds with infinite fundamental groups.*

As concerns, Euclidean 3-manifolds of genus two we have the following statement.

Theorem 4.18 *There are exactly four isomorphism classes of fundamental groups of orientable Euclidean 3-manifolds of genus 2. The list of representing 6-tuples follows:*

1. `(5,5,5,2,2,2)` *with homology* $H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$
2. `(5,5,7,4,4,6)` *with homology* $H_1 = \mathbb{Z}$
3. `(4,6,6,5,5,3)` *with homology* $H_1 = \mathbb{Z} \times \mathbb{Z}_2$
4. `(5,5,7,2,6,6)` *with homology* $H_1 = \mathbb{Z} \times \mathbb{Z}_3$

Previous list covers the complete list of orientable Eucleidean 3-manifolds of genus 2 [22].

In the end of our analysis let us summarise our knowledge about the structure of fundamental groups appearing in the catalogue. The structure is well-known in the case of 39 finite groups we have found. Eleven of extended triangle groups was found in the class of infinite groups. These groups can be described geometrically and some related particular results are claimed in Section 4.4. In the case of 16 further groups we found out that their presentations are very similar to the extended triangle groups. Generally, this presentation can be paramatrised as

$$T = \langle a, b \mid a^k = b^l = (a^r b^s)^m \rangle,\ \ k, l, m, n, r, s \in \mathbb{Z}.$$

Note that the exponents r and s in the presentation of T are relatively small (at most 3 in absolute value). One can easily check that the center of the above groups is infinite cyclic. We do not know almost nothing about the structure of the following 12 groups:

1. $\langle a, b \mid a^4 = b^2 a^{-1} b^2, b^3 = a^3 b^{-1} a^3 \rangle$ with homology $\mathbb{Z}_4$,
2. $\langle a, b \mid a^2 = (ab^2)^2, b^2 = (a^2 b)^2 \rangle$ with homology $\mathbb{Z}_4 \times \mathbb{Z}_4$,
3. $\langle a, b \mid a^2 = (a^{-1} b^2)^2, b^2 = (a^2 b)^2 \rangle$ with homology $\mathbb{Z}_4 \times \mathbb{Z}_4$,
4. $\langle a, b \mid a^3 b^{-1} a = b a^{-1} b^3 = (a^2 b^2)^2 \rangle$ with homology $\mathbb{Z}_5 \times \mathbb{Z}_5$,
5. $\langle a, b \mid a^2 = b^2 a b^2, a^3 = b a^{-3} b^3 \rangle$ with homology $\mathbb{Z}_{20}$,
6. $\langle a, b \mid a = b^2 a^2 b^2, b = a^3 b a^3 \rangle$ with homology $\mathbb{Z}_{24}$,
7. $\langle a, b \mid b = a^3 b a^3, a^4 = b^2 a^{-1} b^2 \rangle$ with homology $\mathbb{Z}_{24}$,
8. $\langle a, b \mid a^2 b a^{-1} = b^{-1} a b^2, b^{-1} a^2 b^{-1} = a b^{-2} a \rangle$ with homology $\mathbb{Z}$,
9. $\langle a, b \mid a^3 = b^{-1} a b^2 a b^{-1}, a b^{-1} a b = b^{-1} a^{-1} b a^{-1} \rangle$ with homology $\mathbb{Z}$,
10. $\langle a, b \mid a^3 = (ab)^3 = (ab^2)^3 \rangle$ with homology $\mathbb{Z} \times \mathbb{Z}_3$,
11. $\langle a, b \mid a^3 = b^2 a^{-1} b^2, b^3 = a^2 b^{-1} a^2$ with homology $\mathbb{Z} \times \mathbb{Z}_4$,
12. $\langle a, b \mid a b^{-2} a^2 = b^2 a^{-2} b, b a^{-2} b = a^{-1} b^2 a^{-1} \rangle$ with homology $\mathbb{Z} \times \mathbb{Z}_5$.

Groups 2, 3 and 10 can be interpreted as some particular subgroups of the 3-dimensional affine group over $\mathbb{R}$.

Chapter 5

Concluding remarks

The categorisation of 6-tuples by fundamental groups of represented manifolds leads us to more sophisticated questions about observed manifolds. Some of them are included in this chapter.

Q1: Is there a coincidence between the homotopy classes, we have identified, and homeomorphism classes of represented 3-manifolds?

The answer was done by direct computations provided by Paola Bandieri and Carlo Gagliardi (University of Modena). They have used the software developed at Modena University. The program is called DUKE and allows us to provide many operations on 4-edge-coloured graphs (crystallisations) such as dipole moves, check colour-preserving graph isomorphisms and much more computations. Using this program the one-to-one correspondence between isomorphism classes of fundamental groups and homeomorphism classes of represented 3-manifolds was confirmed.

The result leads us to the following statement

Theorem 5.1 There are exactly 78 prime 3-manifolds of genus 2 represented by the crystallisations of complexity up to 21.

The proof of the above statement can be completed with some effort formalising the results obtained from DUKE software. Alternatively, it can be checked using different approach, for instance using algorithms developed by Matveev and others [25].

Generally, we can ask whether the prime genus two 3-manifolds are classified by their fundamental groups. Is seems to be that no counterexamples are known at all.

An irreducible boundary irreducible 3-manifold $\mathcal{M}$ with boundary pattern B is called *Haken*, if either $\mathcal{M}$ is *sufficiently large* or $B \neq \emptyset$ and $\mathcal{M}$ is a handlebody, but not a 3-ball. The manifold is *sufficiently large*, if it contains a closed connected surface which is different from S^2, RP^2 and is incompressible and two-sided.

Moreover, as noted by Sergei Matveev [24], the following implications hold.

1. If $\mathcal{M}$ is Haken, then it is determined by the fundamental group [25].
2. If $\mathcal{M}$ is hyperbolic, then the same is true.
3. If $\mathcal{M}$ is not Haken and not hyperbolic, then it is Seifert over the sphere with three exceptional fibers.

We have two possibilities:

a.) $\mathcal{M}$ has a finite fundamental group, hence it is elliptic. Then it is determined by the fundamental group;

b.) $\mathcal{M}$ has an infinite fundamental group. Then its homotopy type is determined by the fundamental group. It is not clear how to convert this homotopy statemet into topological one.

The proof of Theorem 5.1 was achieved by using dipole-move equivalence, the second question naturally arises:

Q2: Is there an algorithm to decide whether two 4-edge-coloured graphs are dipole-move equivalent?

An answer to this question is not known. Although, short chains of reductions of dipoles gives us some results in particular cases, there is no idea to extend the method in general. This problem may be similar to Post correspondency problem in some attributes. A reduction to Post corresponding problem was not done.

Let us come back to our list of isomorphism classes. We have proved that there are 78 isomorphism classes of fundamental groups of prime 3-manifolds of genus 2 and among them there are 39 isomorphism classes with finite fundamental group. By Theorem 2.13 finite fundamental group implies the 3-manifold to be the factor of S^3 (see Chapter 2). Such manifolds are called *elliptic* or spherical, respectively. The classification of finite nonabelian fundamental groups freely acting on S^3 is due to Milnor [25, 26]. The are:

- Finite cyclic groups*,
- Generalised quaternion groups $Q_{4n},\ n \geq 2$,
- Dihedral groups $D_{2^k(2n+1)},\ k \geq 3, n \geq 1$,
- Groups of symmetry of Platonic solids, P_{24}, P_{48}, P_{120} and $P'_{8\cdot 3^k}, k \geq 2$,
- Direct product of any of these groups with a cyclic group of coprime order.

The subscripts in the group symbols above show the orders of groups. Presentations of these groups and homology classes coinciding with the first homology group are the following:

- $Q_{4n} = \langle a, b \mid a^n = b^2 = (ab)^2 \rangle$; $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ if $2 \mid n$, $H_1 = \mathbb{Z}_4$ if $2 \nmid n$,
- $D_{2^k(2n+1)} = \langle a, b \mid a^{2^k} = b^{2n+1} = 1, b^a = b^{-1} \rangle$; $H_1 = \mathbb{Z}_{2^k}$,
- $P_{24} = \langle a, b \mid a^3 = b^2 = (ab)^3, b^4 = 1 \rangle$; $H_1 = \mathbb{Z}_3$,
- $P_{48} = \langle a, b \mid a^4 = b^2 = (ab)^3, b^4 = 1 \rangle$; $H_1 = \mathbb{Z}_2$,
- $P_{120} = \langle a, b \mid a^5 = b^2 = (ab)^3, b^4 = 1 \rangle$; $H_1 = 1$,
- $P'_{8\cdot 3^k} = \langle a, b, c \mid a^2 = b^2 = (ab)^2, a^c = b, b^c = ab, c^{3^k} = 1 \rangle^\dagger$; $H_1 = \mathbb{Z}_{3^k}$

*fundamental groups of lens spaces, we omit them further

†these groups can be also presented as two generator groups [16]

In fact, small representatives of all the previous types of groups appear in our catalogue (see Appendices C and B). Following question arises.

Q3: Is every elliptic 3-manifold of genus at most two?

The question was answered in the affirmative, as we was informed by Sergei Matveev [24]. Since any 3-manifold of genus at most two can be represented by a 6-tuple [7], nice problem can be formulated.

Problem: For each elliptic 3-manifold $\mathcal{M}$ with given Milnor's group find an admissible 6-tuple representing $\mathcal{M}$.

Some particular results in this direction were established yet. For instance, we have found "canonical" representatives for elliptic 3-manifolds given by generalised quaternion groups. Further particular questions related to **Q1** are of interest.

Q4: Given admissible 6-tuple $f \in \mathcal{F}_2$ is there an algorithm to decide finiteness of $\pi_1(f)$?

In Theorem 4.18 we derived the representatives of all Euclidean 3-manifolds of genus 2.

Q5: Can we classify (prime) orientable 3-manifolds of genus two with infinite fundamental group?

Partial result is contained in [25, Chapter 6] for *Haken 3-manifolds*.

> *There exists an algorithmic classification of Haken 3-manifolds*[25, p. 216].

These results may help us while we can recognise which crystallisations represents Haken 3-manifolds of genus two. It is suitable to observe, which 6-tuples represents Haken 3-manifolds and answer the following question.

Q6: How to recognise, if given 6-tuple represents an Haken 3-manifold?

Solving these and similar questions can give us an opportunity to complete our view on 3-manifolds represented by admissible 6-tuples. The success in the further research depends on creating the catalogues of 6-tuples of bigger complexity than we have done. In [25] other approach to solve the homeomorphism problem using so-called Turaev-Viro invariants is described. We have got an information that the author solved the homeomorphism problem for crystallisations up to complexity 24 by using his algorithm.

Chapter 6

Software notes

All software developed to solve the isomorphism problem of fundamental groups is designed as a batch of small programs solving some particular problems and the output of the some program is the input of other one. Programs were developed for the UNIX-like (Linux) operating system using standard GNU compiler set (gcc version 3.2), Python (version 2.3) and GAP [16], which is extremely useful for computations on abstract groups. There should not be any problem to run these programs (after recompilation) under other OS's.

Data shared between steps of batch are stored in usual text files, one item on every line. Each program gather the line of the file, does some computations and resulting data are written into the output file. One can develop some shell script to automatise the batch as possible. Note that the results of the batch are prepared for ad-hoc computations on the groups. These data does not contain full information about groups in general.

Programs are based on algorithms described in Chapters 3 and 4. We describe the most important programs of the bundle. We also describe step-by-step process to create the final catalogue, similar to Appendix C.

Generator of manifolds This task is realised by the program `00_generator`. This program is written in C and simply generates the list of admissible 3-manifolds (see Definition 9). The presentation of fundamental group is computed for every 6-tuple by using Gagliardi's algorithm (see Construction 3.12). The fundamental group is written in compressed form. All groups are considered to be two-generator on generators a and b. The symbols are written in the following form:

- a as `A`
- b as `B`
- a^{-1} as `a`
- b^{-1} as `b`
- power of the generator in multiplicative form (e.g. a^3 as `AAA`)

The program takes only one numerical argument – the highest complexity of 6-tuples included in catalogue. This program creates one file with name `M???.adm` in the current directory. (i.e `/home/user/M021.adm`). Question marks are replaced by highest complexity given as command-line parameter. If is the program ran without the argument, the highest complexity 17 is automatically considered.

Computing $\mathcal{G}$-orbits We use the results of the Chapter 3 and the paper [21] to compute the minimal representatives of $\mathcal{G}$-orbits in our catalogue before we do other considerations. Note, that we do not exclude non-frames (see Definition 10). Thus the new catalogue is more rich as the catalogue by M. R. Casali [6]. It is no problem to add such a test into the script in the future.

This step is realised by Python script named `01_gorbits.py`. This program takes one necessary command-line argument – the name of input text file, one 6-tuple on the line. The minimal representative of $\mathcal{G}$-orbit is computed and if it is equal with the 6-tuple on input, it is written to the output file. The name of output file is `filename.gow` (i.e `/home/user/M021.adm.gow`)

This program also take one optional command-line argument – `-c` – which must be written after filename. In this case the behaviour of program is a bit changed. Program tries to read the presentation (the relators) of the fundamental group on the same line as the 6-tuple and checks if one of relators does not contain unique generator (e.g `AbAAAAA`), which implies the cyclic group. In this case the 6-tuple is not processed further. Otherwise, the 6-tuple is processed as is described higher. The output of the program is written in to the file `filename.goc` (i.e `/home/user/M021.adm.goc`)

Homology group test This step is provided by the Python script `02_homology.py`. We try to construct and check the structure of homology group given by the fundamental group. Since it can be done only in the case of infinite homology group (see Lemma 4.3), we split the list of 6-tuples into two sublists – the first one contains the 6-tuples with finite homology group and the second one includes 6-tuples with infinite homology group. GAP is also not able to compute with such infinite groups.

The orders of generators are computed and appended to each line in the form $\{\mathtt{m}, \mathtt{n}\}$. Zeros in this set indicates the equality of generators; $\{\mathtt{n}, \mathtt{0}\}$ means, that the presentation contains the relator $a = b$. Further, the set $\{\mathtt{0}, \mathtt{0}\}$ can appear; in this case the same powers of both generators equals (e.g. $a^3 = b^3$) and there no other relation in the presentation.

The input of the program is traditional - the name of file from the previous step. Output is generated into two files – `filename.fhg` (`/home/user/M021.adm.gow.fhg`) and `filename.ihg`. The contents of `filename.ihg` can be easy processed by using Lemma 4.3, the second file is prepared for the next step.

Preparing GAP input The data with recognised finite homology group is further processed by the the Python script named `03_prepare-gap.py`. This script creates the script, which can be directly processed in the GAP by using the command `Read`. Input is the file `*.fhg` created in previous step, output is the file `filename.gap`. Note that we use only the catalogue of 6-tuples with finite homology groups.

GAP processing In this step are the simplified presentations of fundamental groups and homology groups computed by the GAP. The input is GAP script, which can be executed by the command `Read`. This script is created in previous step and has the name e.g `M021.adm.gow.fhg.gap`. Part of this script (but functional) follows:

```
# formatting of output was cut !!!
f:=FreeGroup(2);
A:=f.1;
B:=f.2;
pr:=[
["  1,  3,  3,  2,  2,  0",[A*B^-1*B^-1*A,A*A,B*B],"2,2"],
```

```
# many lines
];
for p in pr do
        # for every group in list
        # we gather the presentation
        g:=f/p[2];
        # and create the factor group of
        # free group of rank 2
        t:=SimplifiedFpGroup(g);
        # we compute the simplified presentation
        h:=CommutatorFactorGroup(g);
        # then we create the homology group
        i:=0;
        # since homology group is a factor group
        # of Z x Z
        for x in h do
                if i<Order(x) then
                        i:=Order(x);
                fi;
                # we compute the highest order of element
                # this is the order of the first factor group
        od;
        # then we compute the order of second factor group
        # of direct product
        j:=Order(h)/i
od;
```

The output of GAP script is named (in our case) `M021.adm.gow.fhg.fgroups`. It is prepared to use in last automated part of process

Creating resources The result of computations in GAP we categorise by the script `04_selector.py`. The output are the following files:

- `M???_triv.res` – contains all 6-tuples with trivial fundamental group
- `M???_cyc.res` – contains 6-tuples with cyclic fundamental groups and its presentations
- `M???_prod.res` – contains 6-tuples with fundamental groups, which are free products. It contains also its presentations.
- `M???_fg.res` – contains unresolved cases of fundamental groups with t heir presentations and homology groups.

All files are written in human-readable form, which can be also processed by other programs such as formatters of TEX source etc. The file `M???_fg.res` naturally contains the most of 6-tuples and it is the input of next step. Note that due to limitations of GAP by computing with presentations some cyclic groups and free products may appear in the list of unresolved and must be moved to other resource files by hand.

Creating of catalogue This is the contents of Chapter 4. One interesting tool was developed to make the proofing process more effective. This Python script is called `05_knuth-bendix.py` and contains the program to extract the members of homology group from the file `M???_fg.res` for further processing in GAP.

A lot of fundamental groups in the list of unresolved and there is a problem to find the orders of groups by using standard GAP command `Order()`. The function of this command is based on Todd-Coxeter coset enumeration, which is not an algorithm, but procedure only. Similarly, the identification of the group in GAP library by using `IdSmallGroup()` command is also based on coset enumeration. Thus it is not possible use this command in general situation.

Fortunately, there is standalone GAP package named `KBMAG` [20], which contains tools for more sophisticated dealing finitely presented groups and monoids. Its functionality is based on Knuth-Bendix finite state automata and allow us to test some properties of groups in better way. Details on this package and Knuth-Bendix automata can be found in [20] and [10].

In a few words, if the finite state Knuth-Bendix automaton can be created for the finitely presented group, the script tries to find the order of group, considering the group to be finite. In this case the group is identified in GAP library. In other cases, the group is considered to be infinite and the rest of process is left to human to solve the group in other way.

Knuth-Bendix automaton is the synonym of Knuth-Bendix rewriting system, which is reducing, finitely terminating, confluent term rewriting system whose reductions preserves identities.

One script created by `05_knuth-bendix.py` from homology class $\mathbb{Z}_5$ follows:

```
RequirePackage("kbmag");
F:=FreeGroup("a","b");
a:=F.1;
b:=F.2;
prezent:=[
["(  4,  8,  8,  5,  5, 13)", [a*b^-4*a*b,a*b^-1*a^-1*b^-1*a^3]],
["(  3,  9,  9,  4,  4, 16)", [a*b^-4*a*b,a*b^-1*a^-1*b^-1*a^3]],
["(  5,  5, 11,  4,  8,  4)", [a*b^-1*a^-1*b^-1*a^2*b*a, ....
["(  7,  7,  7,  4,  4,  8)", [b^-1*a^-1*b*a*b*a^-1, .....
];
# shortened

for prez in prezent do
        # for every group in homology class
        G:=F/prez[2];
        # we create finite presented group
        R:=KBMAGRewritingSystem(G);
        # further we create Knuth-Bendix rewriting
        # system
        Print(prez[1]);
        Print(" ");
        KnuthBendix(R);
        # create Knuth-Bendix finite state automaton,
        # if possible, also check the confluence of
        # rewriting system
        if IsConfluent(R)=true then
                # if rewriting system is confluent,
                # we have a chance to find the order and
```

```
                # type of the group
                Print(Size(R));
                Print("  ");
                Print(IdSmallGroup(G));
                # we print the identification of the group
                # in the GAP library
                Print("\n");
        else
                # if rewriting system is in-confluent
                # we deduce the group to be infinite,
                # but it may not to be true
                Print("infinite ?\n");
        fi;

od;
```

There were developed some other utilities for formatting outputs from files becoming in process, checking the partial results and for some other tasks. All of these tools are written in Python.

The point of view in process of programming all the software was on the functionality, not optimisation of programs. But, in fact, the time of processing the catalogues by computer is very short comparing to time of computations by hand.

Bibliography

[1] Amos Altshuler, *Construction and enumeration of regular maps on the torus*, Discrete Math. **4** (1973), 201–217. MR MR0321797 (48 #164)

[2] Paola Bandieri, Maria Rita Casali, and Carlo Gagliardi, *Representing manifolds by crystallization theory: foundations, improvements and related results*, Atti Sem. Mat. Fis. Univ. Modena **49** (2001), no. suppl., 283–337, Dedicated to the memory of Professor M. Pezzana (Italian). MR MR1881101 (2003b:57033)

[3] Paola Bandieri, Carlo Gagliardi, and Laura Ricci, *Classifying genus two 3-manifolds up to 34 tetrahedra*, to appear.

[4] Joan S. Birman, F. González-Acuña, and José M. Montesinos, *Heegaard splittings of prime* 3*-manifolds are not unique*, Michigan Math. J. **23** (1976), no. 2, 97–103. MR MR0431175 (55 #4177)

[5] Benjamin A. Burton, *Minimal triangulations and normal surfaces*, Ph. D. dissertation, 2003, Available from World Wide Web (`http://regina.sourceforge.net/`).

[6] Maria Rita Casali, *A catalogue of the genus two* 3*-manifolds*, Atti Sem. Mat. Fis. Univ. Modena **37** (1989), no. 1, 207–236. MR MR994066 (90h:57017)

[7] Maria Rita Casali and Luigi Grasselli, 2*-symmetric crystallizations and* 2*-fold branched coverings of* S^3, Discrete Math. **87** (1991), no. 1, 9–22. MR MR1090185 (91m:57018)

[8] Harold Scott MacDonald Coxeter and William O. J. Moser, *Porozhdayushchie elementy i opredelyayushchie sootnosheniya diskretnykh grupp*, "Nauka", Moscow, 1980, Translated from the third English edition by V. A. Čurkin, With a supplement by Yu. I. Merzlyakov. MR MR609520 (82c:20002)

[9] Vivien Easson, *Geometrisation of 3-manifolds with Heegaard genus two*, Junior Geometry Seminar, Oxford, 2002.

[10] D. B. A. Epstein, J. W. Cameron, D. F. Holt, M. S. Paterson, and W. P. Thurston, *Word processing and group theory*, Jones and Bartlett, 1992.

[11] M. Ferri, C. Gagliardi, and L. Grasselli, *A graph-theoretical representation of PL-manifolds—a survey on crystallizations*, Aequationes Math. **31** (1986), no. 2-3, 121–141. MR MR867510 (88a:05057)

[12] Massimo Ferri and Carlo Gagliardi, *Crystallisation moves*, Pacific J. Math. **100** (1982), no. 1, 85–103. MR MR661442 (83i:57011)

[13] Carlo Gagliardi, *Personal communication.*

[14] Carlo Gagliardi, *How to deduce the fundamental group of a closed n-manifold from a contracted triangulation*, J. Combin. Inform. System Sci. **4** (1979), no. 3, 237–252. MR MR586313 (81m:57014)

[15] ______, *Extending the concept of genus to dimension n*, Proc. Amer. Math. Soc. **81** (1981), no. 3, 473–481. MR MR597666 (82a:57004)

[16] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.3*, 2002, (`http://www.gap-system.org`).

[17] Luigi Grasselli, Michele Mulazzani, and Roman Nedela, *2-symmetric transformations for 3-manifolds of genus 2*, J. Combin. Theory Ser. B **79** (2000), no. 2, 105–130. MR MR1769216 (2001g:57046)

[18] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR MR1867354 (2002k:55001)

[19] John Hempel, 3-*Manifolds*, Princeton University Press, Princeton, N. J., 1976, Ann. of Math. Studies, No. 86. MR MR0415619 (54 #3702)

[20] Derek F. Holt, *GAP Package — KBMAG*, 2002.

[21] Ján Karabáš and Roman Nedela, *Minimal representatives of $\mathcal{G}$-classes of 3-manifolds of genus two*, Acta Univ. M. Belii Ser. Math. (2003), no. 10, 21–42. MR MR2010521 (2004h:57003)

[22] Sóstenes Lins, *Gems, computers and attractors for* 3-*manifolds*, Series on Knots and Everything, vol. 5, World Scientific Publishing Co. Inc., River Edge, NJ, 1995. MR MR1370443 (97f:57012)

[23] William S. Massey, *A basic course in algebraic topology*, Graduate Texts in Mathematics, vol. 127, Springer-Verlag, New York, 1991. MR MR1095046 (92c:55001)

[24] Sergei Matveev, *Personal communication.*

[25] Sergei Matveev, *Algorithmic topology and classification of 3-manifolds*, Algorithms and Computation in Mathematics, vol. 9, Springer-Verlag, Berlin, 2003. MR MR1997069 (2004i:57026)

[26] John Milnor, *Groups which act on S^n without fixed points*, Amer. J. Math. **79** (1957), 623–630. MR MR0090056 (19,761d)

[27] ______, *A unique decomposition theorem for* 3-*manifolds*, Amer. J. Math. **84** (1962), 1–7. MR MR0142125 (25 #5518)

[28] Edwin E. Moise, *Affine structures in* 3-*manifolds. V. The triangulation theorem and Hauptvermutung*, Ann. of Math. (2) **56** (1952), 96–114. MR MR0048805 (14,72d)

[29] Mario Pezzana, *Sulla struttura topologica delle varietà compatte*, Ati Sem. Mat. Fis. Univ. Modena **23** (1974), no. 1, 269–277 (1975). MR MR0402792 (53 #6606)

[30] Henri Poincaré, *Second complément à l'analysis sitis*, Proc. London Math. Soc. **32** (1900), 277 – 308.

[31] Derek J. S. Robinson, *A course in the theory of groups*, second ed., Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1996. MR MR1357169 (96f:20001)

[32] Carsten Thomassen, *Tilings of the torus and the Klein bottle and vertex-transitive graphs on a fixed surface*, Trans. Amer. Math. Soc. **323** (1991), no. 2, 605–635. MR MR1040045 (91d:57015)

[33] William P. Thurston, *Three-dimensional geometry and topology. Vol. 1*, Princeton Mathematical Series, vol. 35, Princeton University Press, Princeton, NJ, 1997, Edited by Silvio Levy. MR MR1435975 (97m:57016)

[34] Dzh. Vol′f, *Prostranstva postoyannoi krivizny*, "Nauka", Moscow, 1982, Translated from the English and with a supplement by Yu. D. Burago. MR MR685279 (84h:53056)

Appendix A

Catalogues of $\mathcal{G}$-orbits

A.1 Reduced catalogue from [6]

```
+++ z =  7 +++

( 1, 3, 3; 2, 2, 0)

+++ z =  9 +++

( 1, 3, 5; 2, 2, 0)
( 3, 3, 3; 2, 2, 2)

+++ z = 10 +++


+++ z = 11 +++

( 1, 3, 7; 2, 2, 0)
( 1, 5, 5; 2, 2, 0)
( 1, 5, 5; 2, 4, 0)
( 3, 3, 5; 0, 2, 4)
( 3, 3, 5; 2, 2, 4)

+++ z = 12 +++

( 4, 4, 4; 1, 1, 1)
( 4, 4, 4; 1, 1, 5)
( 4, 4, 4; 3, 3, 3)

+++ z = 13 +++

( 1, 3, 9; 2, 2, 0)
( 1, 3, 9; 4, 2, 0)
( 1, 5, 7; 2, 2, 0)
( 1, 5, 7; 2, 4, 0)
( 3, 3, 7; 2, 2, 2)
( 3, 3, 7; 2, 2, 6)
( 3, 5, 5; 2, 4, 0)
( 3, 5, 5; 4, 4, 2)

+++ z = 14 +++

( 4, 4, 6; 1, 1, 1)
( 4, 4, 6; 1, 1, 3)
( 4, 4, 6; 1, 1, 5)
( 4, 4, 6; 1, 1, 7)
( 4, 4, 6; 1, 5, 1)
( 4, 4, 6; 1, 5, 5)
( 4, 4, 6; 3, 1, 5)
( 4, 4, 6; 3, 3, 5)

+++ z = 15 +++

( 1, 3,11; 2, 2, 0)
( 1, 5, 9; 2, 2, 0)
( 1, 5, 9; 2, 4, 0)
( 1, 5, 9; 4, 2, 0)
( 1, 5, 9; 4, 4, 0)
( 1, 7, 7; 2, 2, 0)
( 1, 7, 7; 2, 6, 0)
( 3, 3, 9; 0, 2, 4)
( 3, 3, 9; 0, 2, 8)
( 3, 3, 9; 2, 0, 2)
( 3, 3, 9; 2, 0, 6)
( 3, 3, 9; 2, 2, 8)
( 3, 5, 7; 2, 4, 0)
( 3, 5, 7; 2, 4, 2)
( 3, 5, 7; 4, 4, 0)
( 3, 5, 7; 4, 4,10)
( 5, 5, 5; 0, 4, 4)
( 5, 5, 5; 2, 2, 2)
( 5, 5, 5; 4, 4, 4)

+++ z = 16 +++

( 4, 4, 8; 1, 1, 1)
( 4, 4, 8; 1, 1, 9)
( 4, 4, 8; 1, 5, 1)
( 4, 4, 8; 3, 1, 7)
( 4, 4, 8; 3, 3, 7)
( 4, 4, 8; 3, 7, 3)
( 4, 6, 6; 1, 1, 1)
( 4, 6, 6; 1, 1, 9)
( 4, 6, 6; 1, 7, 1)
( 4, 6, 6; 3, 5,11)
( 4, 6, 6; 5, 5, 3)

+++ z = 17 +++

( 1, 3,13; 2, 2, 0)
```

```
( 1, 3,13; 4, 2, 0)
( 1, 3,13; 6, 2, 0)
( 1, 5,11; 2, 2, 0)
( 1, 5,11; 2, 4, 0)
( 1, 7, 9; 2, 2, 0)
( 1, 7, 9; 2, 6, 0)
( 1, 7, 9; 4, 2, 0)
( 1, 7, 9; 4, 6, 0)
( 3, 3,11; 2, 2, 2)
( 3, 3,11; 2, 2, 4)
( 3, 3,11; 2, 2,10)
( 3, 5, 9; 2, 0, 2)
( 3, 5, 9; 2, 4, 0)
( 3, 5, 9; 4, 2, 0)
( 3, 5, 9; 4, 4,12)
( 3, 5, 9; 4, 6, 0)
( 3, 7, 7; 2, 2, 2)
( 3, 7, 7; 2, 6, 2)
( 3, 7, 7; 4, 4,12)
( 5, 5, 7; 0, 2, 6)
( 5, 5, 7; 0, 4, 6)
( 5, 5, 7; 0, 4,10)
( 5, 5, 7; 2, 0, 4)
( 5, 5, 7; 2, 0, 8)
( 5, 5, 7; 2, 4, 2)
( 5, 5, 7; 2, 4, 6)
( 5, 5, 7; 2, 6, 2)
( 5, 5, 7; 2, 6, 6)
( 5, 5, 7; 4, 0, 6)
( 5, 5, 7; 4, 2, 4)
( 5, 5, 7; 4, 4, 4)
( 5, 5, 7; 4, 4, 6)

+++ z = 18 +++

( 2, 8, 8; 3, 3, 1)
( 2, 8, 8; 3, 5, 1)
( 2, 8, 8; 5, 5, 1)
( 4, 4,10; 1, 1, 1)
( 4, 4,10; 1, 1, 3)
( 4, 4,10; 1, 1, 5)
( 4, 4,10; 1, 1, 7)
( 4, 4,10; 1, 1, 9)
( 4, 4,10; 1, 1,11)
( 4, 4,10; 1, 5, 1)
( 4, 4,10; 1, 5, 7)
( 4, 4,10; 1, 5, 9)
( 4, 4,10; 3, 1, 9)
( 4, 4,10; 3, 3, 3)
( 4, 4,10; 3, 3, 9)
( 4, 6, 8; 1, 1, 1)
( 4, 6, 8; 1, 1, 3)
( 4, 6, 8; 1, 1,11)
( 4, 6, 8; 1, 5, 3)
( 4, 6, 8; 1, 7, 1)
( 4, 6, 8; 3, 5,13)
( 4, 6, 8; 3, 9, 3)
( 4, 6, 8; 3, 9,13)
( 4, 6, 8; 5, 1, 3)
( 4, 6, 8; 5, 5, 3)
( 4, 6, 8; 5, 5,11)
( 6, 6, 6; 1, 1, 1)
( 6, 6, 6; 1, 1, 9)
( 6, 6, 6; 1, 3, 3)
( 6, 6, 6; 1, 3, 7)
( 6, 6, 6; 1, 5, 9)
( 6, 6, 6; 1, 7, 7)
( 6, 6, 6; 3, 3, 5)
( 6, 6, 6; 3, 5, 5)
( 6, 6, 6; 5, 5, 5)

+++ z = 19 +++

( 1, 3,15; 2, 2, 0)
( 1, 3,15; 6, 2, 0)
( 1, 5,13; 2, 2, 0)
( 1, 5,13; 2, 4, 0)
( 1, 5,13; 4, 2, 0)
( 1, 5,13; 4, 4, 0)
( 1, 5,13; 6, 2, 0)
( 1, 5,13; 6, 4, 0)
( 1, 7,11; 2, 2, 0)
( 1, 7,11; 2, 6, 0)
( 1, 9, 9; 2, 2, 0)
( 1, 9, 9; 2, 4, 0)
( 1, 9, 9; 2, 6, 0)
( 1, 9, 9; 2, 8, 0)
( 1, 9, 9; 4, 4, 0)
( 1, 9, 9; 4, 6, 0)
( 3, 3,13; 0, 2, 4)
( 3, 3,13; 0, 2,12)
( 3, 3,13; 2, 0, 2)
( 3, 3,13; 2, 0,10)
( 3, 3,13; 2, 2, 8)
( 3, 3,13; 2, 2,12)
( 3, 5,11; 2, 4, 0)
( 3, 5,11; 2, 4, 2)
( 3, 5,11; 4, 4, 0)
( 3, 5,11; 4, 4, 2)
( 3, 5,11; 4, 4,14)
( 3, 5,11; 6, 4, 0)
( 3, 7, 9; 2, 0, 2)
( 3, 7, 9; 2, 4, 2)
( 3, 7, 9; 4, 2, 0)
( 3, 7, 9; 4, 4, 0)
( 3, 7, 9; 4, 4,14)
( 3, 7, 9; 4, 6, 0)
( 3, 7, 9; 4, 8, 0)
( 3, 7, 9; 4, 8,14)
( 5, 5, 9; 0, 4, 4)
( 5, 5, 9; 0, 4, 6)
( 5, 5, 9; 0, 4,12)
( 5, 5, 9; 2, 0, 2)
( 5, 5, 9; 2, 0,10)
( 5, 5, 9; 2, 2, 2)
( 5, 5, 9; 4, 0, 4)
( 5, 5, 9; 4, 0, 8)
( 5, 5, 9; 4, 4, 8)
( 5, 5, 9; 4, 8, 4)
( 5, 7, 7; 0, 4,12)
( 5, 7, 7; 2, 2, 4)
( 5, 7, 7; 2, 6, 0)
( 5, 7, 7; 2, 8, 2)
( 5, 7, 7; 4, 4, 2)
( 5, 7, 7; 4, 6, 4)
( 5, 7, 7; 4, 6,12)
( 5, 7, 7; 6, 6, 4)

+++ z = 20 +++

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( 4, 4,12; 1, 1, 5)
( 4, 4,12; 1, 1, 9)
( 4, 4,12; 1, 1,13)
( 4, 4,12; 1, 5, 1)
( 4, 4,12; 1, 5, 5)
( 4, 4,12; 3, 1,11)
( 4, 4,12; 3, 3,11)
( 4, 6,10; 1, 1, 1)
( 4, 6,10; 1, 1,13)
( 4, 6,10; 1, 7, 1)
( 4, 6,10; 3, 5, 3)
( 4, 6,10; 3, 5,15)
( 4, 6,10; 3, 9,15)
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( 4, 6,10; 5, 3,15)
( 4, 6,10; 5, 5, 1)
( 4, 6,10; 5, 5,13)
( 4, 6,10; 5, 9, 3)
( 4, 6,10; 7, 1, 1)
( 4, 6,10; 7, 3,15)
( 4, 8, 8; 1, 1, 1)
( 4, 8, 8; 1, 1,13)
( 4, 8, 8; 1, 7, 3)
( 4, 8, 8; 1, 9, 1)
( 4, 8, 8; 1, 9,13)
( 4, 8, 8; 3, 3, 1)
( 4, 8, 8; 5, 5,13)
( 4, 8, 8; 5, 7, 3)
( 6, 6, 8; 1, 1, 1)
( 6, 6, 8; 1, 1, 3)
( 6, 6, 8; 1, 1, 5)
( 6, 6, 8; 1, 1, 7)
( 6, 6, 8; 1, 1, 9)
( 6, 6, 8; 1, 1,11)
( 6, 6, 8; 1, 5, 5)
( 6, 6, 8; 1, 5,11)
( 6, 6, 8; 1, 9, 1)
( 6, 6, 8; 1, 9, 7)
( 6, 6, 8; 3, 1, 9)
( 6, 6, 8; 3, 7, 7)
( 6, 6, 8; 3,11, 3)
( 6, 6, 8; 3,11, 9)
( 6, 6, 8; 5, 1, 7)
( 6, 6, 8; 5, 3, 7)
( 6, 6, 8; 5, 5, 7)
( 6, 6, 8; 5,11, 7)

+++ z = 21 +++

( 1, 3,17; 2, 2, 0)
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( 1, 3,17; 8, 2, 0)
( 1, 5,15; 2, 2, 0)
( 1, 5,15; 2, 4, 0)
( 1, 5,15; 6, 2, 0)
( 1, 5,15; 6, 4, 0)
( 1, 7,13; 2, 2, 0)
( 1, 7,13; 2, 6, 0)
( 1, 7,13; 4, 2, 0)
( 1, 7,13; 4, 6, 0)
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( 1, 7,13; 6, 6, 0)
( 1, 9,11; 2, 2, 0)
( 1, 9,11; 2, 4, 0)
( 1, 9,11; 2, 6, 0)
( 1, 9,11; 2, 8, 0)
( 3, 3,15; 2, 2, 2)
( 3, 3,15; 2, 2, 6)
( 3, 3,15; 2, 2,14)
( 3, 5,13; 2, 0, 2)
( 3, 5,13; 2, 4, 0)
( 3, 5,13; 4, 2, 0)
( 3, 5,13; 4, 4,16)
( 3, 5,13; 4, 6, 0)
( 3, 5,13; 6, 4, 0)
( 3, 5,13; 8, 4, 2)
( 3, 7,11; 2, 2, 2)
( 3, 7,11; 2, 6, 2)
( 3, 7,11; 4, 2, 2)
( 3, 7,11; 4, 4,16)
( 3, 7,11; 4, 6, 2)
( 3, 7,11; 4, 8,16)
( 3, 9, 9; 0, 2, 2)
( 3, 9, 9; 2, 4, 0)
( 3, 9, 9; 2, 8, 0)
( 3, 9, 9; 4, 4,16)
( 5, 5,11; 0, 4,10)
( 5, 5,11; 0, 4,14)
( 5, 5,11; 2, 0, 8)
( 5, 5,11; 2, 0,12)
( 5, 5,11; 2, 2, 8)
( 5, 5,11; 2, 4, 2)
( 5, 5,11; 2, 4,10)
( 5, 5,11; 2, 6, 2)
( 5, 5,11; 2, 6,10)
( 5, 5,11; 4, 0,10)
( 5, 5,11; 4, 2, 4)
( 5, 5,11; 4, 2, 8)
( 5, 5,11; 4, 4, 8)
( 5, 5,11; 4, 4,10)
( 5, 5,11; 4, 8, 4)
( 5, 7, 9; 0, 2, 4)
( 5, 7, 9; 0, 2,12)
( 5, 7, 9; 0, 4,14)
( 5, 7, 9; 0, 6, 4)
( 5, 7, 9; 2, 0, 2)
( 5, 7, 9; 2, 2, 4)
( 5, 7, 9; 2, 4, 4)
( 5, 7, 9; 2, 6, 0)
( 5, 7, 9; 4, 0, 2)
( 5, 7, 9; 4, 0,14)
( 5, 7, 9; 4, 2, 2)
( 5, 7, 9; 4, 6, 0)
( 5, 7, 9; 4, 6, 2)
( 5, 7, 9; 4, 6,14)
( 5, 7, 9; 4,10, 4)
( 5, 7, 9; 4,10,14)
( 5, 7, 9; 6, 4, 4)
( 5, 7, 9; 6, 6, 0)
( 5, 7, 9; 6, 6,12)
( 5, 7, 9; 6, 6,14)
( 5, 7, 9; 6, 8, 2)
( 5, 7, 9; 6, 8,12)
( 5, 7, 9; 6,10,14)
( 7, 7, 7; 2, 2, 2)
( 7, 7, 7; 2, 2, 6)
( 7, 7, 7; 2, 2,10)
( 7, 7, 7; 2, 6,10)
( 7, 7, 7; 2, 8, 8)
( 7, 7, 7; 4, 4, 4)
( 7, 7, 7; 4, 4, 8)
( 7, 7, 7; 6, 6, 6)
```

A.2 Our version of catalogue [21]

```
+++ z =  7 +++

( 1, 3, 3; 2, 2, 0)

+++ z =  9 +++

( 1, 1, 7; 2, 0, 2)
( 1, 3, 5; 2, 2, 0)
( 3, 3, 3; 2, 2, 2)

+++ z = 11 +++
```

```
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( 1, 5, 5; 2, 2, 0)
( 1, 5, 5; 2, 4, 0)
( 3, 3, 5; 0, 2, 4)
( 3, 3, 5; 2, 2, 4)

+++ z = 12 +++

( 2, 4, 6; 3, 3, 1)
( 4, 4, 4; 1, 1, 1)
( 4, 4, 4; 1, 1, 5)
( 4, 4, 4; 3, 3, 3)

+++ z = 13 +++

( 1, 1,11; 2, 0, 2)
( 1, 3, 9; 2, 2, 0)
( 1, 3, 9; 4, 2, 0)
( 1, 5, 7; 2, 2, 0)
( 1, 5, 7; 2, 4, 0)
( 3, 3, 7; 2, 2, 2)
( 3, 3, 7; 2, 2, 6)
( 3, 5, 5; 2, 4, 0)
( 3, 5, 5; 4, 4, 2)

+++ z = 14 +++

( 2, 2,10; 3, 1, 3)
( 2, 6, 6; 3, 3, 1)
( 2, 6, 6; 3, 5, 1)
( 4, 4, 6; 1, 1, 1)
( 4, 4, 6; 1, 1, 3)
( 4, 4, 6; 1, 1, 5)
( 4, 4, 6; 1, 1, 7)
( 4, 4, 6; 1, 5, 1)
( 4, 4, 6; 1, 5, 5)
( 4, 4, 6; 3, 1, 5)
( 4, 4, 6; 3, 3, 3)
( 4, 4, 6; 3, 3, 5)

+++ z = 15 +++

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( 1, 5, 9; 2, 2, 0)
( 1, 5, 9; 2, 4, 0)
( 1, 5, 9; 4, 2, 0)
( 1, 5, 9; 4, 4, 0)
( 1, 7, 7; 2, 2, 0)
( 1, 7, 7; 2, 6, 0)
( 3, 3, 9; 0, 2, 4)
( 3, 3, 9; 0, 2, 8)
( 3, 3, 9; 2, 0, 2)
( 3, 3, 9; 2, 0, 4)
( 3, 3, 9; 2, 0, 6)
( 3, 3, 9; 2, 2, 8)
( 3, 5, 7; 2, 4, 0)
( 3, 5, 7; 2, 4, 2)
( 3, 5, 7; 4, 4, 0)
( 3, 5, 7; 4, 4,10)
( 5, 5, 5; 0, 4, 4)
( 5, 5, 5; 2, 2, 2)
( 5, 5, 5; 2, 2, 6)
( 5, 5, 5; 4, 4, 4)

+++ z = 16 +++

( 2, 4,10; 3, 3, 1)
( 2, 4,10; 5, 3,13)
( 2, 6, 8; 3, 3, 1)
( 2, 6, 8; 3, 5,13)
( 2, 6, 8; 5, 3, 1)
( 4, 4, 8; 1, 1, 1)
( 4, 4, 8; 1, 1, 9)
( 4, 4, 8; 1, 3, 5)
( 4, 4, 8; 1, 5, 1)
( 4, 4, 8; 3, 1, 7)
( 4, 4, 8; 3, 3, 7)
( 4, 4, 8; 3, 7, 3)
( 4, 6, 6; 1, 1, 1)
( 4, 6, 6; 1, 1, 9)
( 4, 6, 6; 1, 7, 1)
( 4, 6, 6; 3, 5, 3)
( 4, 6, 6; 3, 5,11)
( 4, 6, 6; 5, 5, 3)

+++ z = 17 +++

( 1, 1,15; 2, 0, 2)
( 1, 3,13; 2, 2, 0)
( 1, 3,13; 4, 2, 0)
( 1, 3,13; 6, 2, 0)
( 1, 5,11; 2, 2, 0)
( 1, 5,11; 2, 4, 0)
( 1, 7, 9; 2, 2, 0)
( 1, 7, 9; 2, 6, 0)
( 1, 7, 9; 4, 2, 0)
( 1, 7, 9; 4, 6, 0)
( 3, 3,11; 2, 2, 2)
( 3, 3,11; 2, 2, 4)
( 3, 3,11; 2, 2,10)
( 3, 5, 9; 2, 0, 2)
( 3, 5, 9; 2, 4, 0)
( 3, 5, 9; 4, 0, 2)
( 3, 5, 9; 4, 2, 0)
( 3, 5, 9; 4, 4, 2)
( 3, 5, 9; 4, 4,12)
( 3, 5, 9; 4, 6, 0)
( 3, 7, 7; 2, 2, 2)
( 3, 7, 7; 2, 6, 2)
( 3, 7, 7; 4, 4, 0)
( 3, 7, 7; 4, 4,12)
( 3, 7, 7; 4, 6, 0)
( 5, 5, 7; 0, 2, 4)
( 5, 5, 7; 0, 2, 6)
( 5, 5, 7; 0, 2, 8)
( 5, 5, 7; 0, 4, 6)
( 5, 5, 7; 0, 4,10)
( 5, 5, 7; 2, 0, 2)
( 5, 5, 7; 2, 0, 4)
( 5, 5, 7; 2, 0, 6)
( 5, 5, 7; 2, 0, 8)
( 5, 5, 7; 2, 2, 8)
( 5, 5, 7; 2, 4, 2)
( 5, 5, 7; 2, 4, 6)
( 5, 5, 7; 2, 6, 2)
( 5, 5, 7; 2, 6, 6)
( 5, 5, 7; 4, 0, 6)
( 5, 5, 7; 4, 2, 4)
( 5, 5, 7; 4, 4, 4)
( 5, 5, 7; 4, 4, 6)

+++ z = 18 +++

( 2, 6,10; 3, 3, 1)
( 2, 6,10; 3, 5, 1)
( 2, 6,10; 3, 5,15)
( 2, 6,10; 5, 3,15)
( 2, 6,10; 5, 5,15)
( 2, 8, 8; 3, 3, 1)
( 2, 8, 8; 3, 5, 1)
( 2, 8, 8; 5, 5, 1)
( 4, 4,10; 1, 1, 1)
( 4, 4,10; 1, 1, 3)
( 4, 4,10; 1, 1, 5)
```

```
( 4, 4,10; 1, 1, 7)
( 4, 4,10; 1, 1, 9)
( 4, 4,10; 1, 1,11)
( 4, 4,10; 1, 5, 1)
( 4, 4,10; 1, 5, 7)
( 4, 4,10; 1, 5, 9)
( 4, 4,10; 3, 1, 9)
( 4, 4,10; 3, 3, 3)
( 4, 4,10; 3, 3, 7)
( 4, 4,10; 3, 3, 9)
( 4, 4,10; 3, 7, 7)
( 4, 6, 8; 1, 1, 1)
( 4, 6, 8; 1, 1, 3)
( 4, 6, 8; 1, 1,11)
( 4, 6, 8; 1, 5, 3)
( 4, 6, 8; 1, 7, 1)
( 4, 6, 8; 3, 5,13)
( 4, 6, 8; 3, 9, 3)
( 4, 6, 8; 3, 9,13)
( 4, 6, 8; 5, 1, 3)
( 4, 6, 8; 5, 3, 1)
( 4, 6, 8; 5, 5, 3)
( 4, 6, 8; 5, 5,11)
( 4, 6, 8; 5, 7, 1)
( 4, 6, 8; 5, 7,11)
( 4, 6, 8; 5, 7,13)
( 6, 6, 6; 1, 1, 1)
( 6, 6, 6; 1, 1, 9)
( 6, 6, 6; 1, 3, 3)
( 6, 6, 6; 1, 3, 7)
( 6, 6, 6; 1, 5, 9)
( 6, 6, 6; 1, 7, 7)
( 6, 6, 6; 3, 3, 5)
( 6, 6, 6; 3, 3, 7)
( 6, 6, 6; 3, 5, 5)
( 6, 6, 6; 5, 5, 5)

+++ z = 19 +++

( 1, 3,15; 2, 2, 0)
( 1, 3,15; 6, 2, 0)
( 1, 5,13; 2, 2, 0)
( 1, 5,13; 2, 4, 0)
( 1, 5,13; 4, 2, 0)
( 1, 5,13; 4, 4, 0)
( 1, 5,13; 6, 2, 0)
( 1, 5,13; 6, 4, 0)
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( 1, 7,11; 2, 6, 0)
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( 1, 9, 9; 2, 8, 0)
( 1, 9, 9; 4, 4, 0)
( 1, 9, 9; 4, 6, 0)
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( 3, 3,13; 0, 2,12)
( 3, 3,13; 2, 0, 2)
( 3, 3,13; 2, 0,10)
( 3, 3,13; 2, 2, 8)
( 3, 3,13; 2, 2,12)
( 3, 3,13; 4, 2, 4)
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( 3, 5,11; 4, 4, 2)
( 3, 5,11; 4, 4,14)
( 3, 5,11; 6, 4, 0)
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( 3, 7, 9; 4, 2, 0)
( 3, 7, 9; 4, 4, 0)
( 3, 7, 9; 4, 4, 2)
( 3, 7, 9; 4, 4,14)
( 3, 7, 9; 4, 6, 0)
( 3, 7, 9; 4, 6, 2)
( 3, 7, 9; 4, 8, 0)
( 3, 7, 9; 4, 8,14)
( 5, 5, 9; 0, 4, 4)
( 5, 5, 9; 0, 4, 6)
( 5, 5, 9; 0, 4,12)
( 5, 5, 9; 2, 0, 2)
( 5, 5, 9; 2, 0,10)
( 5, 5, 9; 2, 2, 2)
( 5, 5, 9; 2, 2,10)
( 5, 5, 9; 2, 6, 2)
( 5, 5, 9; 4, 0, 4)
( 5, 5, 9; 4, 0, 8)
( 5, 5, 9; 4, 4, 4)
( 5, 5, 9; 4, 4, 8)
( 5, 5, 9; 4, 8, 4)
( 5, 7, 7; 0, 2, 2)
( 5, 7, 7; 0, 2,10)
( 5, 7, 7; 0, 4,12)
( 5, 7, 7; 2, 2, 4)
( 5, 7, 7; 2, 2,10)
( 5, 7, 7; 2, 6, 0)
( 5, 7, 7; 2, 6, 4)
( 5, 7, 7; 2, 8, 0)
( 5, 7, 7; 2, 8, 2)
( 5, 7, 7; 4, 4, 2)
( 5, 7, 7; 4, 6, 4)
( 5, 7, 7; 4, 6,12)
( 5, 7, 7; 6, 6, 2)
( 5, 7, 7; 6, 6, 4)

+++ z = 20 +++

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( 2, 6,12; 7, 3, 1)
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( 2, 8,10; 3, 5, 1)
( 2, 8,10; 3, 7, 1)
( 2, 8,10; 5, 3,17)
( 2, 8,10; 5, 5,17)
( 2, 8,10; 5, 7,17)
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( 4, 4,12; 1, 1, 9)
( 4, 4,12; 1, 1,13)
( 4, 4,12; 1, 5, 1)
( 4, 4,12; 1, 5, 5)
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( 4, 4,12; 3, 1,11)
( 4, 4,12; 3, 3, 3)
( 4, 4,12; 3, 3,11)
( 4, 4,12; 3, 7, 3)
( 4, 6,10; 1, 1, 1)
( 4, 6,10; 1, 1,13)
( 4, 6,10; 1, 7, 1)
( 4, 6,10; 3, 3, 3)
( 4, 6,10; 3, 5, 3)
( 4, 6,10; 3, 5,15)
( 4, 6,10; 3, 9,15)
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( 4, 6,10; 5, 3,15)
```

```
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( 4, 6,10; 5, 9, 3)
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( 4, 6,10; 7, 3,15)
( 4, 6,10; 7, 5, 3)
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( 4, 8, 8; 1, 1,13)
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( 4, 8, 8; 1, 9, 1)
( 4, 8, 8; 1, 9,13)
( 4, 8, 8; 3, 3, 1)
( 4, 8, 8; 3, 5, 1)
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( 4, 8, 8; 5, 5,13)
( 4, 8, 8; 5, 7, 1)
( 4, 8, 8; 5, 7, 3)
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( 6, 6, 8; 1, 1, 3)
( 6, 6, 8; 1, 1, 5)
( 6, 6, 8; 1, 1, 7)
( 6, 6, 8; 1, 1, 9)
( 6, 6, 8; 1, 1,11)
( 6, 6, 8; 1, 3, 7)
( 6, 6, 8; 1, 3, 9)
( 6, 6, 8; 1, 5, 5)
( 6, 6, 8; 1, 5,11)
( 6, 6, 8; 1, 7, 7)
( 6, 6, 8; 1, 9, 1)
( 6, 6, 8; 1, 9, 7)
( 6, 6, 8; 1, 9, 9)
( 6, 6, 8; 3, 1, 9)
( 6, 6, 8; 3, 3, 9)
( 6, 6, 8; 3, 5, 7)
( 6, 6, 8; 3, 7, 3)
( 6, 6, 8; 3, 7, 7)
( 6, 6, 8; 3,11, 3)
( 6, 6, 8; 3,11, 5)
( 6, 6, 8; 3,11, 7)
( 6, 6, 8; 3,11, 9)
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( 6, 6, 8; 5, 3, 5)
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( 6, 6, 8; 5, 5, 7)
( 6, 6, 8; 5,11, 7)

+++ z = 21 +++

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( 1, 1,19; 2, 0, 6)
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( 1, 3,17; 4, 2, 0)
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( 1, 5,15; 2, 4, 0)
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( 1, 5,15; 6, 4, 0)
( 1, 7,13; 2, 2, 0)
( 1, 7,13; 2, 6, 0)
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( 1, 7,13; 4, 6, 0)
( 1, 7,13; 6, 2, 0)
( 1, 7,13; 6, 6, 0)
( 1, 9,11; 2, 2, 0)
( 1, 9,11; 2, 4, 0)
( 1, 9,11; 2, 6, 0)
( 1, 9,11; 2, 8, 0)
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( 3, 3,15; 2, 2,14)
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( 3, 5,13; 2, 4, 0)
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( 3, 5,13; 6, 4, 0)
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( 3, 7,11; 2, 6, 2)
( 3, 7,11; 4, 2, 2)
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( 3, 7,11; 4, 4,16)
( 3, 7,11; 4, 6, 0)
( 3, 7,11; 4, 6, 2)
( 3, 7,11; 4, 8,16)
( 3, 7,11; 6, 2, 2)
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( 3, 9, 9; 6, 6, 2)
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( 5, 5,11; 0, 2,12)
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( 5, 5,11; 0, 4,14)
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( 5, 5,11; 2, 0, 6)
( 5, 5,11; 2, 0, 8)
( 5, 5,11; 2, 0,10)
( 5, 5,11; 2, 0,12)
( 5, 5,11; 2, 2, 8)
( 5, 5,11; 2, 2,12)
( 5, 5,11; 2, 4, 2)
( 5, 5,11; 2, 4, 6)
( 5, 5,11; 2, 4,10)
( 5, 5,11; 2, 6, 2)
( 5, 5,11; 2, 6, 8)
( 5, 5,11; 2, 6,10)
( 5, 5,11; 4, 0,10)
( 5, 5,11; 4, 2, 4)
( 5, 5,11; 4, 2, 8)
( 5, 5,11; 4, 4, 8)
( 5, 5,11; 4, 4,10)
( 5, 5,11; 4, 8, 4)
( 5, 5,11; 4, 8, 8)
( 5, 7, 9; 0, 2, 2)
( 5, 7, 9; 0, 2, 4)
( 5, 7, 9; 0, 2,12)
( 5, 7, 9; 0, 4,14)
( 5, 7, 9; 0, 6, 4)
( 5, 7, 9; 2, 0, 2)
( 5, 7, 9; 2, 0,12)
( 5, 7, 9; 2, 2, 4)
( 5, 7, 9; 2, 2,12)
( 5, 7, 9; 2, 4, 4)
( 5, 7, 9; 2, 6, 0)
( 5, 7, 9; 2, 8, 0)
( 5, 7, 9; 2, 8, 2)
( 5, 7, 9; 4, 0, 2)
( 5, 7, 9; 4, 0,14)
( 5, 7, 9; 4, 2, 2)
( 5, 7, 9; 4, 6, 0)
( 5, 7, 9; 4, 6, 2)
( 5, 7, 9; 4, 6, 4)
( 5, 7, 9; 4, 6,14)
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( 5, 7, 9; 4,10,14)
( 5, 7, 9; 6, 0, 4)
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( 5, 7, 9; 6, 6, 0)
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( 7, 7, 7; 6, 6, 6)
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Appendix B

Index of isomorphism classes

The Appendix contains a list of fundamental groups coded by admissible 6-tuples (see Appendix C). The following format of entries is used. Each entry fills up two lines. First item is the code of the isomorphism class; **L** is the code for lens space (including $S^1 \times S^2$), **S** is the code for decomposable 3-manifold of genus two and **P** is the code for prime genus two 3-manifold. The second item determines the presentation of fundamental group of given class with the respective GAP code; if the group is infinite, the code is always [∞,?]. Third item is the respective homology group. Indexes of 6-tuples representing the isomorphism class (see Appendix C) resides in the second line of the entry. Third line of a record in the sublist of acyclic fundamental groups contains a 6-tuple with with the least index in the list of representatives.

Cyclic groups

L.1 $\pi_1(f) = 1 \cong [1, 1]$ $H_1(f) = 1$
$f = 348$
L.2 $\pi_1(f) = \mathbb{Z}_2 \cong [2, 1]$ $H_1(f) = \mathbb{Z}_2$
$f = 89, 198, 252$
L.3 $\pi_1(f) = \mathbb{Z}_4 \cong [4, 1]$ $H_1(f) = \mathbb{Z}_4$
$f = 353$
L.4 $\pi_1(f) = \mathbb{Z}_5 \cong [5, 1]$ $H_1(f) = \mathbb{Z}_5$
$f = 52, 96, 229, 307, 362, 424$
L.5 $\pi_1(f) = \mathbb{Z}_6 \cong [6, 2]$ $H_1(f) = \mathbb{Z}_6$
$f = 204, 346$
L.6 $\pi_1(f) = \mathbb{Z}_7 \cong [7, 1]$ $H_1(f) = \mathbb{Z}_7$
$f = 213, 422$
L.7 $\pi_1(f) = \mathbb{Z}_9 \cong [9, 1]$ $H_1(f) = \mathbb{Z}_9$
$f = 365$
L.8 $\pi_1(f) = \mathbb{Z}_{11} \cong [11, 1]$ $H_1(f) = \mathbb{Z}_{11}$
$f = 205, 352$
L.9 $\pi_1(f) = \mathbb{Z}_{12} \cong [12, 2]$ $H_1(f) = \mathbb{Z}_{12}$
$f = 28, 46, 106, 138, 306, 386$

L.10 $\pi_1(f) = \mathbb{Z}_{13} \cong [13, 1]$ $H_1(f) = \mathbb{Z}_{13}$
$f = 27, 99, 103, 293, 298, 351$

L.11 $\pi_1(f) = \mathbb{Z}_{16} \cong [16, 1]$ $H_1(f) = \mathbb{Z}_{16}$
$f = 122, 129, 183, 206, 370$

L.12 $\pi_1(f) = \mathbb{Z}_{17} \cong [17, 1]$ $H_1(f) = \mathbb{Z}_{17}$
$f = 400$

L.13 $\pi_1(f) = \mathbb{Z}_{18} \cong [18, 2]$ $H_1(f) = \mathbb{Z}_{18}$
$f = 105, 305$

L.14 $\pi_1(f) = \mathbb{Z}_{19} \cong [19, 1]$ $H_1(f) = \mathbb{Z}_{19}$
$f = 123, 126, 147, 363, 366, 409$

L.15 $\pi_1(f) = \mathbb{Z}_{21} \cong [21, 2]$ $H_1(f) = \mathbb{Z}_{21}$
$f = 124, 156, 212, 421$

L.16 $\pi_1(f) = \mathbb{Z}_{24} \cong [24, 2]$ $H_1(f) = \mathbb{Z}_{24}$
$f = 288, 369$

L.17 $\pi_1(f) = \mathbb{Z}_{25} \cong [25, 1]$ $H_1(f) = \mathbb{Z}_{25}$
$f = 287$

L.18 $\pi_1(f) = \mathbb{Z}_{26} \cong [26, 2]$ $H_1(f) = \mathbb{Z}_{26}$
$f = 388$

L.19 $\pi_1(f) = \mathbb{Z} \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}$
$f = 98, 153, 292$

Free products of finite cyclic groups

S.1 $\pi_1(f) = \mathbb{Z}_2 * \mathbb{Z}_2 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}_2$
$f = 1, 10, 24, 60, 91, 195, 239, 281, 358, 360$

S.2 $\pi_1(f) = \mathbb{Z}_2 * \mathbb{Z}_3 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}_3$
$f = 3, 8, 25, 56, 57, 58, 63, 100, 102, 113, 117, 121, 148, 201, 238, 241, 244, 291, 295, 355, 375$

S.3 $\pi_1(f) = \mathbb{Z}_2 * \mathbb{Z}_4 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}_4$
$f = 5, 21, 29, 151, 233, 234, 235, 279, 361, 391$

S.4 $\pi_1(f) = \mathbb{Z}_2 * \mathbb{Z}_5 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}_5$
$f = 15, 16, 48, 50, 119, 237, 280, 283$

S.5 $\pi_1(f) = \mathbb{Z}_2 * \mathbb{Z}_6 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}_6$
$f = 35, 88, 222, 236$

S.6 $\pi_1(f) = \mathbb{Z}_2 * \mathbb{Z}_7 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}_7$
$f = 75, 76, 77, 187, 189, 192, 397, 403, 412$

S.7 $\pi_1(f) = \mathbb{Z}_2 * \mathbb{Z}_8 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}_8$
$f = 164, 165, 337, 341$

S.8 $\pi_1(f) = \mathbb{Z}_2 * \mathbb{Z}_9 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}_9$
$f = 315, 316, 317$

S.9 $\pi_1(f) = \mathbb{Z}_3 * \mathbb{Z}_3 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_3 \times \mathbb{Z}_3$
$f = 6, 7, 42, 43, 118, 120, 240, 245$

S.10 $\pi_1(f) = \mathbb{Z}_3 * \mathbb{Z}_4 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_3 \times \mathbb{Z}_4$
$f = 17, 18, 90, 93, 180, 181, 242, 243$

S.11 $\pi_1(f) = \mathbb{Z}_3 * \mathbb{Z}_5 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_3 \times \mathbb{Z}_5$
$f = 36, 37, 38, 39, 196, 197, 200, 202$

S.12 $\pi_1(f) = \mathbb{Z}_3 * \mathbb{Z}_6 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_3 \times \mathbb{Z}_6$
$f = 78, 79, 356, 357$

S.13 $\pi_1(f) = \mathbb{Z}_3 * \mathbb{Z}_7 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_3 \times \mathbb{Z}_7$
$f = 166, 167, 168, 169, 170, 171$

S.14 $\pi_1(f) = \mathbb{Z}_3 * \mathbb{Z}_8 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_3 \times \mathbb{Z}_8$
$f = 318, 319, 320, 321$

S.15 $\pi_1(f) = \mathbb{Z}_4 * \mathbb{Z}_4 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_4 \times \mathbb{Z}_4$
$f = 40, 41, 338, 340$

S.16 $\pi_1(f) = \mathbb{Z}_4 * \mathbb{Z}_5 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_4 \times \mathbb{Z}_5$
$f = 80, 81, 82, 83$

S.17 $\pi_1(f) = \mathbb{Z}_4 * \mathbb{Z}_6 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_4 \times \mathbb{Z}_6$
$f = 172, 173$

S.18 $\pi_1(f) = \mathbb{Z}_4 * \mathbb{Z}_7 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_4 \times \mathbb{Z}_7$
$f = 322, 323, 324, 325, 326, 327$

S.19 $\pi_1(f) = \mathbb{Z}_5 * \mathbb{Z}_5 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_5 \times \mathbb{Z}_5$
$f = 174, 175, 176, 177, 178, 179$

S.20 $\pi_1(f) = \mathbb{Z}_5 * \mathbb{Z}_6 \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_5 \times \mathbb{Z}_6$
$f = 328, 329, 330, 331$

S.21 $\pi_1(f) = \mathbb{Z}_2 * \mathbb{Z} \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}$
$f = 2, 23, 45, 186, 368$

S.22 $\pi_1(f) = \mathbb{Z}_3 * \mathbb{Z} \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_3 \times \mathbb{Z}$
$f = 14, 231, 232, 332$

S.23 $\pi_1(f) = \mathbb{Z}_4 * \mathbb{Z} \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_4 \times \mathbb{Z}$
$f = 74$

S.24 $\pi_1(f) = \mathbb{Z}_5 * \mathbb{Z} \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_5 \times \mathbb{Z}$
$f = 313, 314$

Acyclic fundamental groups

P.1 $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab)^2 \rangle \cong [120, 5]$ $H_1(f) = 1$
$f = 55, 92, 66, 262$
`(5,5,5,4,4,4)`

P.2 $\pi_1(f) = \langle a, b \mid a^7 = b^3 = (ab)^2 \rangle \cong [\infty, ?]$ $H_1(f) = 1$
$f = 215, 230, 255, 339, 433$
`(5,5,9,4,4,8)`

P.3 $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab)^2 \rangle \cong [48, 28]$ $H_1(f) = \mathbb{Z}_2$
$f = 34, 51, 144, 265, 300, 395$
`(4,4,6,3,3,5)`

P.4 $\pi_1(f) = \langle a, b \mid a^5 = b^4 = (ab)^2 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_2$
$f = 150, 199, 303$
`(4,6,8,3,3,11)`

P.5 $\pi_1(f) = \langle a, b \mid a^8 = b^3 = (ab)^2 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_2$
$f = 384, 413$
`(5,5,11,4,4,10)`

P.6 $\pi_1(f) = \langle a, b \mid a^2 = b^2 = (ab)^2 \rangle \cong [8,4]$ $H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}_2$
$f = 4, 33, 54, 214, 302, 309$
`(3,3,3,2,2,2)`

P.7 $\pi_1(f) = \langle a, b \mid a^4 = b^2 = (ab)^2 \rangle \cong [16,9]$ $H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}_2$
$f = 20, 133, 152, 210$
`(3,3,7,2,2,6)`

P.8 $\pi_1(f) = \langle a, b \mid a^6 = b^2 = (ab)^2 \rangle \cong [24,4]$ $H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}_2$
$f = 86$
`(3,3,11,2,2,10)`

P.9 $\pi_1(f) = \langle a, b \mid a^8 = b^2 = (ab)^2 \rangle \cong [32,20]$ $H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}_2$
$f = 335$
`(3,3,15,2,2,14)`

P.10 $\pi_1(f) = \langle a, b \mid a^6 = b^4 = (ab)^2 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}_2$
$f = 267, 347$
`(4,6,10,5,5,13)`

P.11 $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^2 \rangle \cong [24,3]$ $H_1(f) = \mathbb{Z}_3$
$f = 13, 22, 72, 161, 221, 272, 294, 432$
`(4,4,4,3,3,3)`

P.12 $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab^{-2})^{-3} \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_3$
$f = 228, 379, 429$
`(5,7,7,4,6,12)`

P.13 $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^3 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_3 \times \mathbb{Z}_3$
$f = 69, 111, 284, 404$
`(4,6,6,1,1,9)`

P.14 $\pi_1(f) = \langle a, b \mid a^3 = b^2 = (ab)^2 \rangle \cong [12,1]$ $H_1(f) = \mathbb{Z}_4$
$f = 9, 31, 71, 108, 136, 158, 301, 378, 405, 410$
`(3,3,5,2,2,4)`

P.15 $\pi_1(f) = \langle a, b \mid a^5 = b^2 = (ab)^2 \rangle \cong [20,1]$ $H_1(f) = \mathbb{Z}_4$
$f = 47, 268, 373$
`(3,3,9,2,2,8)`

P.16 $\pi_1(f) = \langle a, b \mid a^7 = b^2 = (ab)^2 \rangle \cong [28,1]$ $H_1(f) = \mathbb{Z}_4$
$f = 185$
`(3,3,13,2,2,12)`

P.17 $\pi_1(f) = \langle a, b \mid a^4 = b^2a^{-1}b^2, b^3 = a^3b^{-1}a^3 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_4$
$f = 428$
`(7,7,7,2,6,10)`

P.18 $\pi_1(f) = \langle a, b \mid a^2 = (ab^2)^2, b^2 = (a^2b)^2 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_4 \times \mathbb{Z}_4$
$f = 53$
`(5,5,5,2,2,2)`

P.19 $\pi_1(f) = \langle a, b \mid a^2 = (a^{-1}b^2)^2, b^2 = (a^2b)^2 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_4 \times \mathbb{Z}_4$
$f = 109, 216$
`(5,5,7,2,4,2)`

P.20 $\pi_1(f) = \langle a, b \mid a^5 = b^5 = (ab)^2 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_5$
$f = 282, 359$
`(4,8,8,5,5,13)`

P.21 $\pi_1(f) = \langle a, b \mid a^7 = b^2 = (a^{-2}b)^3 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_5$
$f = 385, 431$
`(5,5,11,4,8,4)`

P.22 $\pi_1(f) = \langle a, b \mid a^3b^{-1}a = ba^{-1}b^3 = (a^2b^2)^2\rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_5 \times \mathbb{Z}_5$
$f = 423, 430$
`(7,7,7,2,2,2)`

P.23 $\pi_1(f) = \langle a, b \mid a^5 = b^4 = (a^2b)^2\rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_6$
$f = 261, 310, 349, 416$
`(4,6,10,3,5,3)`

P.24 $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab^{-2})^{-3}\rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_6$
$f = 406$
`(5,7,9,4,6,14)`

P.25 $\pi_1(f) = \langle a, b \mid a^2 = b^2 = (ab)^{-2}\rangle \cong [24, 11]$ $H_1(f) = \mathbb{Z}_6 \times \mathbb{Z}_2$
$f = 11, 44, 104, 256$
`(4,4,4,1,1,1)`

P.26 $\pi_1(f) = \langle a, b \mid a^4 = b^2 = (ab^{-1})^2\rangle \cong [48, 27]$ $H_1(f) = \mathbb{Z}_6 \times \mathbb{Z}_2$
$f = 30, 84, 107, 224$
`(4,4,6,1,5,1)`

P.27 $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (a^2b)^3\rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_6 \times \mathbb{Z}_3$
$f = 269, 275, 382, 402$
`(4,6,10,5,9,3)`

P.28 $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (a^2b)^2\rangle \cong [840, 13]$ $H_1(f) = \mathbb{Z}_7$
$f = 135, 162, 190, 227, 381, 414$
`(4,4,10,3,3,3)`

P.29 $\pi_1(f) = \langle a, b \mid a^3 = b^2 = (a^2b)^2\rangle \cong [24, 1]$ $H_1(f) = \mathbb{Z}_8$
$f = 12, 19, 32, 101, 211, 223, 254, 260, 380, 426$
`(4,4,4,1,1,5)`

P.30 $\pi_1(f) = \langle a, b \mid a^5 = b^2 = (a^2b)^2\rangle \cong [40, 1]$ $H_1(f) = \mathbb{Z}_8$
$f = 85, 110, 115, 251$
`(3,3,11,2,2,4)`

P.31 $\pi_1(f) = \langle a, b \mid a^7 = (a^4b)^2, b^2 = (a^2b)^2\rangle \cong [56, 1]$ $H_1(f) = \mathbb{Z}_8$
$f = 334, 376, 383$
`(3,3,15,2,2,6)`

P.32 $\pi_1(f) = \langle a, b \mid a^4 = b^4 = (ab)^3\rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_8$
$f = 274, 427$
`(4,8,8,1,1,13)`

P.33 $\pi_1(f) = \langle a, b \mid a^4 = b^4 = (a^{-1}b)^2\rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_8 \times \mathbb{Z}_2$
$f = 146, 344, 371$
`(4,6,8,3,9,13)`

P.34 $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (a^{-1}b)^2\rangle \cong [72, 3]$ $H_1(f) = \mathbb{Z}_9$
$f = 29, 49, 65, 67, 149, 218, 266, 364, 399, 417$
`(4,4,6,1,1,7)`

P.35 $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab)^3\rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_9$
$f = 141, 225$
`(4,6,8,1,1,11)`

P.36 $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab)^3\rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_9$
$f = 258, 408$
`(4,6,10,1,1,13)`

P.37 $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^{-3}\rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_9 \times \mathbb{Z}_3$
$f = 154$
`(6,6,6,1,1,1)`

P.38 $\pi_1(f) = \langle a, b \mid a^4 = b^2 = (ab)^3 \rangle \cong [240, 102]$ $H_1(f) = \mathbb{Z}_{10}$
$f = 62, 95, 134, 145, 393$
(4,4,8,1,1,9)

P.39 $\pi_1(f) = \langle a, b \mid a^2 = b^2 = (a^3b^3)^2 \rangle \cong [40, 11]$ $H_1(f) = \mathbb{Z}_{10} \times \mathbb{Z}_2$
$f = 247, 278$
(4,4,12,1,1,5)

P.40 $\pi_1(f) = \langle a, b \mid a^4 = b^2 = (ab)^{-2} \rangle \cong [80, 27]$ $H_1(f) = \mathbb{Z}_{10} \times \mathbb{Z}_2$
$f = 61, 193, 387$
(4,4,8,1,1,1)

P.41 $\pi_1(f) = \langle a, b \mid {}^6 = b^2 = (ab^{-1})^2 \rangle \cong [120, 21]$ $H_1(f) = \mathbb{Z}_{10} \times \mathbb{Z}_2$
$f = 131, 390$
(4,4,10,1,5,1)

P.42 $\pi_1(f) = \langle a, b \mid a^5 = b^{-2} = (ab^{-1})^3 \rangle \cong [1320, 14]$ $H_1(f) = \mathbb{Z}_{11}$
$f = 130, 203, 253, 277, 304$
(4,4,10,1,1,11)

P.43 $\pi_1(f) = \langle a, b \mid a^5 = b^2 = (a^2b^{-1})^2 \rangle \cong [60, 2]$ $H_1(f) = \mathbb{Z}_{12}$
$f = 132, 157, 184, 308, 415$
(4,4,10,1,5,7)

P.44 $\pi_1(f) = \langle a, b \mid a^6 = b^3 = (a^{-1}b)^2 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_{12}$
$f = 249, 350$
(4,4,12,1,1,13)

P.45 $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (a^2b^{-1})^2 \rangle \cong [1560, 13]$ $H_1(f) = \mathbb{Z}_{13}$
$f = 271, 299, 342$
(4,6,10,7,3,15)

P.46 $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab^{-1})^2 \rangle \cong [336, 115]$ $H_1(f) = \mathbb{Z}_{14}$
$f = 70, 188, 208, 389, 420$
(4,6,6,1,7,1)

P.47 $\pi_1(f) = \langle a, b \mid a^6 = b^2 = (ab)^{-2} \rangle \cong [168, 29]$ $H_1(f) = \mathbb{Z}_{14} \times \mathbb{Z}_2$
$f = 246$
(4,4,12,1,1,1)

P.48 $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (a^{-2}b)^2 \rangle \cong [120, 15]$ $H_1(f) = \mathbb{Z}_{15}$
$f = 94, 114, 127, 142, 160, 226$
(3,7,7,2,2,2)

P.49 $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab^{-1})^3 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_{15}$
$f = 155, 377$
(6,6,6,1,1,9)

P.50 $\pi_1(f) = \langle a, b \mid a^3 = b^2 = (ab)^{-2} \rangle \cong [48, 1]$ $H_1(f) = \mathbb{Z}_{16}$
$f = 26, 87, 182, 217, 367$
(4,4,6,1,1,1)

P.51 $\pi_1(f) = \langle a, b \mid a^5 = b^2 = (ab^{-1})^2 \rangle \cong [80, 1]$ $H_1(f) = \mathbb{Z}_{16}$
$f = 64, 219, 333, 398$
(4,4,8,1,5,1)

P.52 $\pi_1(f) = \langle a, b \mid a^4 = b^4 = (a^{-1}b)^3 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_{16}$
$f = 290$
(6,6,8,1,1,1)

P.53 $\pi_1(f) = \langle a, b \mid a^4 = b^4 = (ab)^{-2} \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_{16} \times \mathbb{Z}_2$
$f = 273$
(4,4,8,1,1,1)

P.54 $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (a^2b^2)^2 \rangle \cong [2040, ?]$ $H_1(f) = \mathbb{Z}_{17}$
$f = 345, 411$
`(3,7,11,4,2,2)`

P.55 $\pi_1(f) = \langle a, b \mid a^5 = b^4 = (a^{-1}b)^2 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_{18}$
$f = 263$
`(4,6,10,3,9,15)`

P.56 $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab^{-1})^2 \rangle \cong [2280, ?]$ $H_1(f) = \mathbb{Z}_{19}$
$f = 143, 401$
`(4,6,8,1,7,1)`

P.57 $\pi_1(f) = \langle a, b \mid a^3 = b^2 = (a^2b)^{-2} \rangle \cong [60, 1]$ $H_1(f) = \mathbb{Z}_{20}$
$f = 128, 140, 194, 248$
`(4,4,10,1,1,7)`

P.58 $\pi_1(f) = \langle a, b \mid a^2 = b^2ab^2, a^3 = ba^{-3}b^3 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_{20}$
$f = 396, 407, 418$
`(5,7,9,2,4,4)`

P.59 $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^{-2} \rangle \cong [168, 22]$ $H_1(f) = \mathbb{Z}_{21}$
$f = 68, 207, 336, 419$
`(4,6,6,1,1,1)`

P.60 $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab^{-1})^3 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_{21}$
$f = 296$
`(6,6,8,1,9,1)`

P.61 $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab^{-2})^2 \rangle \cong [528, 87]$ $H_1(f) = \mathbb{Z}_{22}$
$f = 264, 289, 343$
`(4,6,10,5,1,1)`

P.62 $\pi_1(f) = \langle a, b \mid a^5 = b^4 = (ab^{-1})^2 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_{22}$
$f = 276$
`(4,8,8,1,9,1)`

P.63 $\pi_1(f) = \langle a, b \mid a^5 = b^2 = (ab)^{-2} \rangle \cong [120, 2]$ $H_1(f) = \mathbb{Z}_{24}$
$f = 125, 354$
`(4,4,10,1,1,1)`

P.64 $\pi_1(f) = \langle a, b \mid a^7 = b^2 = (ab^{-1})^2 \rangle \cong [168, 4]$ $H_1(f) = \mathbb{Z}_{24}$
$f = 250$
`(4,4,12,1,5,1)`

P.65 $\pi_1(f) = \langle a, b \mid a^6 = b^3 = (ab^{-1})^2 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_{24}$
$f = 259$
`(4,6,10,1,7,1)`

P.66 $\pi_1(f) = \langle a, b \mid a = b^2a^2b^2, b = a^3ba^3 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_{24}$
$f = 209, 220, 372, 394$
`(5,5,9,2,2,2)`

P.67 $\pi_1(f) = \langle a, b \mid b = a^3ba^3, a^4 = b^2a^{-1}b^2 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_{24}$
$f = 374, 425$
`(5,5,11,2,4,2)`

P.68 $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab)^{-2} \rangle \cong [624, 131]$ $H_1(f) = \mathbb{Z}_{26}$
$f = 139, 392$
`(4,6,8,1,1,1)`

P.69 $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (a^2b)^{-2} \rangle \cong [216, 3]$ $H_1(f) = \mathbb{Z}_{27}$
$f = 270, 286$
`(4,6,10,7,1,1)`

P.70 $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab)^{-2} \rangle \cong [3720, ?]$ $H_1(f) = \mathbb{Z}_{31}$
$f = 257$
`(4,6,10,1,1,1)`

P.71 $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab)^{-3} \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}_{33}$
$f = 285$
`(6,6,8,1,1,1)`

P.72 $\pi_1(f) = \langle a, b \mid a^6 = b^3 = (ab)^2 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}$
$f = 116, 137, 191$
`(5,5,7,4,4,6)`

P.73 $\pi_1(f) = \langle a, b \mid a^2ba^{-1} = b^{-1}ab^2, b^{-1}a^2b^{-1} = ab^{-2}a \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}$
$f = 163$
`(6,6,6,5,5,5)`

P.74 $\pi_1(f) = \langle a, b \mid a^3 = b^{-1}ab^2ab^{-1}, ab^{-1}ab = b^{-1}a^{-1}ba^{-1} \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z}$
$f = 311$
`(6,6,8,5,5,7)`

P.75 $\pi_1(f) = \langle a, b \mid a^4 = b^4 = (ab)^2 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z} \times \mathbb{Z}_2$
$f = 73, 97$
`(4,6,6,5,5,3)`

P.76 $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^3 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z} \times \mathbb{Z}_3$
$f = 112, 297$
`(5,5,7,2,6,6)`

P.77 $\pi_1(f) = \langle a, b \mid a^3 = b^2a^{-1}b^2, b^3 = a^2b^{-1}a^2 \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z} \times \mathbb{Z}_4$
$f = 159$
`(6,6,6,1,7,7)`

P.78 $\pi_1(f) = \langle a, b \mid ab^{-2}a^2 = b^2a^{-2}b, ba^{-2}b = a^{-1}b^2a^{-1} \rangle \cong [\infty, ?]$ $H_1(f) = \mathbb{Z} \times \mathbb{Z}_5$
$f = 312$
`(6,6,8,5,11,7)`

Appendix C

List of representatives

The Appendix contains full list of admissible 6-tuples coding 3-manifolds of genus at most two. This list is the result of the full procedure described in the Chapter 4 and it is based of the list of representatives of $\mathcal{G}$-classes in Appendix A (the second catalogue in the Appendix A). The following format of entries is used. First entry contains the number of 6-tuple in the catalogue (see also Appendix B). The second entry contains the respective $\mathcal{G}$-minimal representative. The presentation of fundamental group follows as the last item in the first line. In the second line reside the homology group (as first item), the center of fundamental group, the factor of fundamental group by the center and the respective epimorphism onto the representative of isomorphism class. Some notes about the 3-manifold, in particular the information about the space or fundamental group, are written on the third line of entry. Question marks are written to assign unknown data. The epimorphisms were not computed in the case of genus one (or less), decomposable 3-manifolds of genus two and representatives of isomorphism classes.

1. `( 1, 3, 3, 2, 2, 0)` $G_1 = \mathbb{Z}_2 * \mathbb{Z}_2$
   $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = 1$ $G_1/\zeta = \mathbb{Z}_2 * \mathbb{Z}_2$ —
   connected sum $\mathcal{L}(2,1)\#\mathcal{L}(2,1)$; representative
2. `( 1, 1, 7, 2, 0, 2)` $G_2 = \mathbb{Z}_2 * \mathbb{Z}$
   $H_1 = \mathbb{Z}_2 \times \mathbb{Z}$ $\zeta = 1$ $G_2/\zeta = \mathbb{Z}_2 * \mathbb{Z}$ —
   connected sum $\mathcal{L}(2,1)\#S^1 \times S^2$; representative
3. `( 1, 3, 5, 2, 2, 0)` $G_3 = \mathbb{Z}_2 * \mathbb{Z}_3$
   $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_3/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
   connected sum $\mathcal{L}(2,1)\#\mathcal{L}(3,1)$
4. `( 3, 3, 3, 2, 2, 2)` $G_4 = \langle a, b \mid a^2 = b^2 = (ab)^2 \rangle$
   $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_4/\zeta = D_4$ —
   representative, GAP code for the group is `[8,4]`
5. `( 1, 3, 7, 2, 2, 0)` $G_5 = \mathbb{Z}_2 * \mathbb{Z}_4$
   $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_4$ $\zeta = 1$ $G_5/\zeta = \mathbb{Z}_2 * \mathbb{Z}_4$ —
   connected sum $\mathcal{L}(2,1)\#\mathcal{L}(4,1)$; representative

6. `( 1, 5, 5, 2, 2, 0)` $G_6 = \mathbb{Z}_3 * \mathbb{Z}_3$
   $H_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$ $\zeta = 1$ $G_6/\zeta = \mathbb{Z}_3 * \mathbb{Z}_3$ —
   connected sum $\mathcal{L}(3,1)\#\mathcal{L}(3,1)$; representative
7. `( 1, 5, 5, 2, 4, 0)` $G_7 = \mathbb{Z}_3 * \mathbb{Z}_3$
   $H_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$ $\zeta = 1$ $G_7/\zeta = \mathbb{Z}_3 * \mathbb{Z}_3$ —
   connected sum $\mathcal{L}(3,1)\#\mathcal{L}(3,1)$
8. `( 3, 3, 5, 0, 2, 4)` $G_8 = \mathbb{Z}_2 * \mathbb{Z}_3$
   $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_8/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
   connected sum $\mathcal{L}(2,1)\#\mathcal{L}(3,1)$
9. `( 3, 3, 5, 2, 2, 4)` $G_9 = \langle a, b \mid a^3 = b^2 = (ab)^2 \rangle$
   $H_1 = \mathbb{Z}_4$ $\zeta = \mathbb{Z}_2$ $G_9/\zeta = D_6$ —
   representative; GAP code for the group is `[12,1]`
10. `( 2, 4, 6, 3, 3, 1)` $G_{10} = \mathbb{Z}_2 * \mathbb{Z}_2$
    $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = 1$ $G_{10}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_2$ —
    connected sum $\mathcal{L}(2,1)\#\mathcal{L}(2,1)$
11. `( 4, 4, 4, 1, 1, 1)` $G_{11} = \langle a, b \mid a^2 = b^2 = (ab)^{-2} \rangle$
    $H_1 = \mathbb{Z}_6 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_6$ $G_{11}/\zeta = D_4$ —
    representative; GAP code for this group is `[24,11]`
12. `( 4, 4, 4, 1, 1, 5)` $G_{12} = \langle a, b \mid ab^2a = 1, abab^{-1}ab^{-1} = 1 \rangle$
    $H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}_4$ $G_{12}/\zeta = D_6$ $\phi_{12} : a^{-1}b \mapsto G_{19}.a,\ ab^{-1}a \mapsto G_{19}.b$
    representative; GAP code for this group is `[24,1]`
13. `( 4, 4, 4, 3, 3, 3)` $G_{13} = \langle a, b \mid a^3 = b^3 = (ab)^2 \rangle$
    $H_1 = \mathbb{Z}_3$ $\zeta = \mathbb{Z}_2$ $G_{13}/\zeta = A_4$ —
    representative; GAP code for this group is `[24,3]`
14. `( 1, 1,11, 2, 0, 2)` $G_{14} = \mathbb{Z}_3 * \mathbb{Z}$
    $H_1 = \mathbb{Z}_3 \times \mathbb{Z}$ $\zeta = 1$ $G_{14}/\zeta = \mathbb{Z}_3 * \mathbb{Z}$ —
    connected sum $\mathcal{L}(3,1)\#S^1 \times S^2$; representative
15. `( 1, 3, 9, 2, 2, 0)` $G_{15} = \mathbb{Z}_2 * \mathbb{Z}_5$
    $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{15}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_5$ —
    connected sum $\mathcal{L}(2,1)\#\mathcal{L}(5,1)$; representative
16. `( 1, 3, 9, 4, 2, 0)` $G_{16} = \mathbb{Z}_2 * \mathbb{Z}_5$
    $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_5$ $\zeta =$ $G_{16}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_5$ —
    connected sum $\mathcal{L}(2,1)\#\mathcal{L}(5,2)$; representative
17. `( 1, 5, 7, 2, 2, 0)` $G_{17} = \mathbb{Z}_3 * \mathbb{Z}_4$
    $H_1 = \mathbb{Z}_3 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{17}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_4$ —
    connected sum $\mathcal{L}(3,1)\#\mathcal{L}(4,1)$; representative
18. `( 1, 5, 7, 2, 4, 0)` $G_{18} = \mathbb{Z}_3 * \mathbb{Z}_4$
    $H_1 = \mathbb{Z}_3 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{18}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_4$ —
    connected sum $\mathcal{L}(3,1)\#\mathcal{L}(4,1)$
19. `( 3, 3, 7, 2, 2, 2)` $G_{19} = \langle a, b \mid a^3 = b^2 = (a^2b)^2 \rangle$
    $H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}_4$ $G_{19}/\zeta = D_6$ —
20. `( 3, 3, 7, 2, 2, 6)` $G_{20} = \langle a, b \mid a^4 = b^2 = (ab)^2 \rangle$
    $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{20}/\zeta = D_8$ —
    representative; GAP code for the group is `[16,9]`

21. `( 3, 5, 5, 2, 4, 0)` $G_{21} = \mathbb{Z}_2 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{21}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_4$ —
connected sum

22. `( 3, 5, 5, 4, 4, 2)` $G_{22} = \langle a, b \mid ab^{-1}a^{-1}b^{-1}a = 1, ab^{-2}ab = 1\rangle$
$H_1 = \mathbb{Z}_3$ $\zeta = \mathbb{Z}_2$ $G_{22}/\zeta = A_4$ $\phi_{22} : a \mapsto G_{13}.a,\ b \mapsto G_{13}.b$

23. `( 2, 2,10, 3, 1, 3)` $G_{23} = \langle a, b \mid a^2b^{-1}a^{-2}ba^{-2}b^{-1}a^2b = 1, a^2 = 1\rangle$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}$ $\zeta = 1$ $G_{23}/\zeta = \mathbb{Z}_2 * \mathbb{Z}$ $\phi_{23} : a \mapsto G_2.a,\ b \mapsto G_2.b$

24. `( 2, 6, 6, 3, 3, 1)` $G_{24} = \mathbb{Z}_2 * \mathbb{Z}_2$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = 1$ $G_{24}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_2$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(2,1)$

25. `( 2, 6, 6, 3, 5, 1)` $G_{25} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{25}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(3,1)$

26. `( 4, 4, 6, 1, 1, 1)` $G_{26} = \langle a, b \mid a^3 = b^2 = (ab)^{-2}\rangle$
$H_1 = \mathbb{Z}_{16}$ $\zeta = \mathbb{Z}_8$ $G_{26}/\zeta = D_6$ —
representative; GAP code for the group is `[48,1]`

27. `( 4, 4, 6, 1, 1, 3)` $G_{27} = \mathbb{Z}_{13}$
$H_1 = \mathbb{Z}_{13}$ $\zeta = \mathbb{Z}_{13}$ $G_{27}/\zeta = 1$ —
lens space $\mathcal{L}(13, ?)$

28. `( 4, 4, 6, 1, 1, 5)` $G_{28} = \mathbb{Z}_{12}$
$H_1 = \mathbb{Z}_{12}$ $\zeta = \mathbb{Z}_{12}$ $G_{28}/\zeta = 1$ —
lens space $\mathcal{L}(12, ?)$

29. `( 4, 4, 6, 1, 1, 7)` $G_{29} = \langle a, b \mid a^3b^2 = 1, abab^{-1}ab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_9$ $\zeta = \mathbb{Z}_6$ $G_{29}/\zeta = A_4$ $\phi_{29} : a^{30}ba \mapsto G_{49}.a,\ b^{-1}a^{30} \mapsto G_{49}.b$
representative; GAP code for the group is `[72,3]`

30. `( 4, 4, 6, 1, 5, 1)` $G_{30} = \langle a, b \mid a^4 = b^2 = (ab^{-1})^2\rangle$
$H_1 = \mathbb{Z}_6 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_6$ $G_{30}/\zeta = D_8$ —
representative; GAP code for the group is `[48,27]`

31. `( 4, 4, 6, 1, 5, 5)` $G_{31} = \langle a, b \mid b^{-1}aba = 1, b^{-1}a^{-3}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_4$ $\zeta = \mathbb{Z}_2$ $G_{31}/\zeta = D_6$ $\phi_{31} : a^{-1} \mapsto G_9.a,\ b \mapsto G_9.b$

32. `( 4, 4, 6, 3, 1, 5)` $G_{32} = \langle a, b \mid aba^{-1}b = 1, ba^2bab^{-1}a = 1\rangle$
$H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}_4$ $G_{32}/\zeta = D_6$ $\phi_{32} : ba^{-1}b^2a^{-1} \mapsto G_{19}.a,\ b^2a^{-1} \mapsto G_{19}.b$

33. `( 4, 4, 6, 3, 3, 3)` $G_{33} = \langle a, b \mid b^2a^2 = 1, ab^{-1}a^{-1}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{33}/\zeta = D_4$ $\phi_{33} : a \mapsto G_4.a,\ b^{-1} \mapsto G_4.b$

34. `( 4, 4, 6, 3, 3, 5)` $G_{34} = \langle a, b \mid b^2ab^{-2}a = 1, ab^{-1}a^{-1}ba^{-1}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{34}/\zeta = S_4$ $\phi_{34} : b^{-1} \mapsto G_{51}.a,\ ab^{-1} \mapsto G_{51}.b$
representative, GAP code for the group is `[48,28]`

35. `( 1, 3,11, 2, 2, 0)` $G_{35} = \mathbb{Z}_2 * \mathbb{Z}_6$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_6$ $\zeta = 1$ $G_{35}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_6$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(6,1)$; representative

36. `( 1, 5, 9, 2, 2, 0)` $G_{36} = \mathbb{Z}_3 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{36}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_5$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(5,1)$; representative

37. `( 1, 5, 9, 2, 4, 0)` $G_{37} = \mathbb{Z}_3 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{37}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_5$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(5,1)$

38. `( 1, 5, 9, 4, 2, 0)` $G_{38} = \mathbb{Z}_3 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{38}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_5$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(5,2)$

39. `( 1, 5, 9, 4, 4, 0)` $G_{39} = \mathbb{Z}_3 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{39}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_5$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(5,2)$

40. `( 1, 7, 7, 2, 2, 0)` $G_{40} = \mathbb{Z}_4 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{40}/\zeta = \mathbb{Z}_4 * \mathbb{Z}_4$ —
connected sum $\mathcal{L}(4,1)\#\mathcal{L}(4,1)$; representative

41. `( 1, 7, 7, 2, 6, 0)` $G_{41} = \mathbb{Z}_4 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{41}/\zeta = \mathbb{Z}_4 * \mathbb{Z}_4$ —
connected sum $\mathcal{L}(4,1)\#\mathcal{L}(4,1)$; representative

42. `( 3, 3, 9, 0, 2, 4)` $G_{42} = \mathbb{Z}_3 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{42}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(3,1)$

43. `( 3, 3, 9, 0, 2, 8)` $G_{43} = \mathbb{Z}_3 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{43}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(3,1)$

44. `( 3, 3, 9, 2, 0, 2)` $G_{44} = \langle a, b \mid ab^2a = 1, ab^{-1}ab^{-3} = 1\rangle$
$H_1 = \mathbb{Z}_6 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_6$ $G_{44}/\zeta = D_4$ $\phi_{44} : a \mapsto G_{11}.a,\ b \mapsto G_{11}.b$

45. `( 3, 3, 9, 2, 0, 4)` $G_{45} = \langle a, b \mid ab^{-1}a^2baba^2b^{-1} = 1, a^2b^{-1}a^2b = 1, a^2 = 1\rangle$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}$ $\zeta = 1$ $G_{45}/\zeta = \mathbb{Z}_2 * \mathbb{Z}$ $\phi_{45} : a \mapsto G_2.a,\ b \mapsto G_2.b$

46. `( 3, 3, 9, 2, 0, 6)` $G_{46} = \mathbb{Z}_{12}$
$H_1 = \mathbb{Z}_{12}$ $\zeta = \mathbb{Z}_{12}$ $G_{46}/\zeta = 1$ —
lens space $\mathcal{L}(12, ?)$

47. `( 3, 3, 9, 2, 2, 8)` $G_{47} = \langle a, b \mid a^5 = b^2 = (ab)^2\rangle$
$H_1 = \mathbb{Z}_4$ $\zeta = \mathbb{Z}_2$ $G_{47}/\zeta = D_{10}$ —
representative; GAP code for the group is `[20,1]`

48. `( 3, 5, 7, 2, 4, 0)` $G_{48} = \mathbb{Z}_2 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{48}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_5$ —
connected sum

49. `( 3, 5, 7, 2, 4, 2)` $G_{49} = \langle a, b \mid a^3 = b^3 = (a^{-1}b)^2\rangle$
$H_1 = \mathbb{Z}_9$ $\zeta = \mathbb{Z}_6$ $G_{49}/\zeta = A_4$ —

50. `( 3, 5, 7, 4, 4, 0)` $G_{50} = \mathbb{Z}_2 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{50}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_5$ —
connected sum

51. `( 3, 5, 7, 4, 4,10)` $G_{51} = \langle a, b \mid a^4 = b^3 = (ab)^2\rangle$
$H_1 = \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{51}/\zeta = S_4$ —

52. `( 5, 5, 5, 0, 4, 4)` $G_{52} = \mathbb{Z}_5$
$H_1 = \mathbb{Z}_5$ $\zeta = \mathbb{Z}_5$ $G_{52}/\zeta = 1$ —
lens space $\mathcal{L}(5, ?)$

53. `( 5, 5, 5, 2, 2, 2)` $G_{53} = \langle a, b \mid a^2 = (ab^2)^2, b^2 = (a^2b)^2 \rangle$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$ ζ ? G_{53}/ζ ? —
representative; Eucleidean 3-manifold

54. `( 5, 5, 5, 2, 2, 6)` $G_{54} = \langle a, b \mid ba^2b = 1, ab^{-1}a^{-1}b^{-1} = 1 \rangle$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{54}/\zeta = D_4$ $\phi_{54} : a \mapsto G_4.a,\ b^{-1} \mapsto G_4.b$

55. `( 5, 5, 5, 4, 4, 4)` $G_{55} = \langle a, b \mid a^5 = b^3 = (ab)^2 \rangle$
$H_1 = 1$ $\zeta = \mathbb{Z}_2$ $G_{55}/\zeta = A_5$ —
representative; GAP code for the group is `[120,5]`

56. `( 2, 4,10, 3, 3, 1)` $G_{56} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{56}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(3,1)$

57. `( 2, 4,10, 5, 3,13)` $G_{57} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1\mathbb{Z}_2 \times \mathbb{Z}_3 =$ $\zeta = 1$ $G_{57}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(3,1)$

58. `( 2, 6, 8, 3, 3, 1)` $G_{58} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{58}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(3,1)$

59. `( 2, 6, 8, 3, 5,13)` $G_{59} = \mathbb{Z}_2 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{59}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_4$ —
connected sum

60. `( 2, 6, 8, 5, 3, 1)` $G_{60} = \mathbb{Z}_2 * \mathbb{Z}_2$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = 1$ $G_{60}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_2$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(2,1)$

61. `( 4, 4, 8, 1, 1, 1)` $G_{61} = \langle a, b \mid a^4 = b^2 = (ab)^{-2} \rangle$
$H_1 = \mathbb{Z}_{10} \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_{10}$ $G_{61}/\zeta = D_8$ —
representative; GAP code for the group is `[80,27]`

62. `( 4, 4, 8, 1, 1, 9)` $G_{62} = \langle a, b \mid a^4 = b^2 = (ab)^3 \rangle$
$H_1 = \mathbb{Z}_{10}$ $\zeta = \mathbb{Z}_{10}$ $G_{62}/\zeta = S_4$ —
representative; GAP code for the group is `[240,102]`

63. `( 4, 4, 8, 1, 3, 5)` $G_{63} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{63}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(3,1)$

64. `( 4, 4, 8, 1, 5, 1)` $G_{64} = \langle a, b \mid a^5 = b^2 = (ab^{-1})^2 \rangle$
$H_1 = \mathbb{Z}_{16}$ $\zeta = \mathbb{Z}_8$ $G_{64}/\zeta = D_{10}$ —
representative; GAP code for the group is `[80,1]`

65. `( 4, 4, 8, 3, 1, 7)` $G_{65} = \langle a, b \mid ab^{-1}a^{-2}b^{-1} = 1, ab^{-1}a^2b^{-1}ab = 1 \rangle$
$H_1 = \mathbb{Z}_9$ $\zeta = \mathbb{Z}_6$ $G_{65}/\zeta = A_4$ $\phi_{65} : b^3ab^2ab^{-2} \mapsto G_{49}.a,\ a^{-1}b^{-5}ab^{-2} \mapsto G_{49}.b$

66. `( 4, 4, 8, 3, 3, 7)` $G_{66} = \langle a, b \mid ab^{-1}a^{-1}ba^{-1}b^{-1} = 1, b^3ab^{-2}a = 1 \rangle$
$H_1 = 1$ $\zeta = \mathbb{Z}_2$ $G_{66}/\zeta = A_5$ $\phi_{66} : b^{-1} \mapsto G_{55}.a,\ ab^{-1} \mapsto G_{55}.b$

67. `( 4, 4, 8, 3, 7, 3)` $G_{67} = \langle a, b \mid ab^{-1}a^{-1}b^{-1}a = 1, a^2b^3a = 1 \rangle$
$H_1 = \mathbb{Z}_9$ $\zeta = \mathbb{Z}_6$ $G_{67}/\zeta = A_4$ $\phi_{67} : b^{-30}ab^5 \mapsto G_{49}.a,\ a^{-1}b^{-5}ab^{-2} \mapsto G_{49}.b$

68. `( 4, 6, 6, 1, 1, 1)` $G_{68} = \langle a, b \mid a^3 = b^3 = (ab)^{-2} \rangle$
$H_1 = \mathbb{Z}_{21}$ $\zeta = \mathbb{Z}_{14}$ $G_{68}/\zeta = A_4$ —
representative; GAP code for the group is `[168,22]`

69. `( 4, 6, 6, 1, 1, 9)` $G_{69} = \langle a, b \mid a^3 = b^3 = (ab)^3 \rangle$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$ $\zeta = \mathbb{Z}$ $G_{69}/\zeta = \Delta^+(3,3,3)$ —
representative

70. `( 4, 6, 6, 1, 7, 1)` $G_{70} = \langle a, b \mid a^4 = b^3 = (ab^{-1})^2 \rangle$
$H_1 = \mathbb{Z}_{14}$ $\zeta = \mathbb{Z}_{14}$ $G_{70}/\zeta = S_4$ —
representative; GAP code for the group is `[336,115]`

71. `( 4, 6, 6, 3, 5, 3)` $G_{71} = \langle a, b \mid ab^{-1}a^{-1}b^{-1} = 1, b^3a^2 = 1 \rangle$
$H_1 = \mathbb{Z}_4$ $\zeta = \mathbb{Z}_2$ $G_{71}/\zeta = D_6$ $\phi_{71} : b \mapsto G_9.a,\ a^{-1} \mapsto G_9.b$

72. `( 4, 6, 6, 3, 5,11)` $G_{72} = \langle a, b \mid b^{-1}a^2b^{-1}a^{-1} = 1, ab^{-2}ab = 1 \rangle$
$H_1 = \mathbb{Z}_3$ $\zeta = \mathbb{Z}_2$ $G_{72}/\zeta = A_4$ $\phi_{72} : a \mapsto G_{13}.a,\ b \mapsto G_{13}.b$

73. `( 4, 6, 6, 5, 5, 3)` $G_{73} = \langle a, b \mid a^4 = b^4 = (ab)^2 \rangle$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}$ $\zeta = \mathbb{Z}$ $G_{73}/\zeta = \Delta^+(4,4,2)$ —
Eucleidean manifold; representative

74. `( 1, 1,15, 2, 0, 2)` $G_{74} = \mathbb{Z}_4 * \mathbb{Z}$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}$ $\zeta = 1$ $G_{74}/\zeta = \mathbb{Z}_4 * \mathbb{Z}$ —
connected sum $\mathcal{L}(4,1)\#S^1 \times S^2$; representative

75. `( 1, 3,13, 2, 2, 0)` $G_{75} = \mathbb{Z}_2 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{75}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_7$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(7,1)$; representative

76. `( 1, 3,13, 4, 2, 0)` $G_{76} = \mathbb{Z}_2 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{76}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_7$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(7,2)$; representative

77. `( 1, 3,13, 6, 2, 0)` $G_{77} = \mathbb{Z}_2 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{77}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_7$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(7,3)$; representative

78. `( 1, 5,11, 2, 2, 0)` $G_{78} = \mathbb{Z}_3 * \mathbb{Z}_6$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_6$ $\zeta = 2$ $G_{78}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_6$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(6,1)$; representative

79. `( 1, 5,11, 2, 4, 0)` $G_{79} = \mathbb{Z}_3 * \mathbb{Z}_6$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_6$ $\zeta = 1$ $G_{79}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_6$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(6,1)$

80. `( 1, 7, 9, 2, 2, 0)` $G_{80} = \mathbb{Z}_4 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{80}/\zeta = \mathbb{Z}_4 * \mathbb{Z}_5$ —
connected sum $\mathcal{L}(4,1)\#\mathcal{L}(5,1)$; representative

81. `( 1, 7, 9, 2, 6, 0)` $G_{81} = \mathbb{Z}_4 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{81}/\zeta = \mathbb{Z}_4 * \mathbb{Z}_5$ —
connected sum $\mathcal{L}(4,1)\#\mathcal{L}(5,1)$

82. `( 1, 7, 9, 4, 2, 0)` $G_{82} = \mathbb{Z}_4 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{82}/\zeta = \mathbb{Z}_4 * \mathbb{Z}_5$ —
connected sum $\mathcal{L}(4,1)\#\mathcal{L}(5,2)$; representative

83. `( 1, 7, 9, 4, 6, 0)` $G_{83} = \mathbb{Z}_4 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{83}/\zeta = \mathbb{Z}_4 * \mathbb{Z}_5$ —
connected sum $\mathcal{L}(4,1)\#\mathcal{L}(5,2)$

84. `( 3, 3,11, 2, 2, 2)` $G_{84} = \langle a, b \mid ab^{-3}ab^{-1} = 1, ab^{-1}a^{-3}b^{-1} = 1 \rangle$
$H_1 = \mathbb{Z}_6 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_6$ $G_{84}/\zeta = D_8$ $\phi_{84} : a \mapsto G_{30}.a,\ b \mapsto G_{30}.b$

85. `( 3, 3,11, 2, 2, 4)` $G_{85} = \langle a, b \mid a^5 = b^2(a^2b)^2\rangle$
$H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}_4$ $G_{85}/\zeta = D_{10}$ —
representative; the GAP code for the group is `[40,1]`

86. `( 3, 3,11, 2, 2,10)` $G_{86} = \langle a, b \mid a^6 = b^2 = (ab)^2\rangle$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{86}/\zeta = D_{12}$ —
representative; the GAP code for the group is `[24,4]`

87. `( 3, 5, 9, 2, 0, 2)` $G_{87} = \langle a, b \mid ab^3a = 1, ab^{-1}ab^{-1}a^2 = 1\rangle$
$H_1 = \mathbb{Z}_{16}$ $\zeta = \mathbb{Z}_8$ $G_{87}/\zeta = D_6$ $\phi_{87} : b \mapsto G_{26}.a,\ ab^{-1} \mapsto G_{26}.b$

88. `( 3, 5, 9, 2, 4, 0)` $G_{88} = \mathbb{Z}_2 * \mathbb{Z}_6$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_6$ $\zeta = 1$ $G_{88}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_6$ —
connected sum

89. `( 3, 5, 9, 4, 0, 2)` $G_{89} = \mathbb{Z}_2$
$H_1 = \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{89}/\zeta = 1$ —
lens space $\mathcal{L}(2,1)$; representative

90. `( 3, 5, 9, 4, 2, 0)` $G_{90} = \mathbb{Z}_3 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{90}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_4$ —
connected sum

91. `( 3, 5, 9, 4, 4, 2)` $G_{91} = \mathbb{Z}_2 * \mathbb{Z}_2$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = 1$ $G_{91}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_2$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(2,1)$

92. `( 3, 5, 9, 4, 4,12)` $G_{92} = \langle a, b \mid a^5 = b^3 = (ab)^2\rangle$
$H_1 = 1$ $\zeta = \mathbb{Z}_2$ $G_{92}/\zeta = A_5$ $\phi_{92} : a \mapsto G_{55}.a, b \mapsto G_{55}.b$

93. `( 3, 5, 9, 4, 6, 0)` $G_{93} = \mathbb{Z}_3 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{93}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_4$ —
connected sum

94. `( 3, 7, 7, 2, 2, 2)` $G_{94} = \langle a, b \mid a^3 = b^3 = (a^{-2}b)^2\rangle$
$H_1 = \mathbb{Z}_{15}$ $\zeta = \mathbb{Z}_{10}$ $G_{94}/\zeta = A_4$ —
representative; GAP code for the group is `[120,15]`

95. `( 3, 7, 7, 2, 6, 2)` $G_{95} = \langle a, b \mid ab^{-1}a^{-2}b^{-1} = 1, ab^{-4}a^2 = 1\rangle$
$H_1 = \mathbb{Z}_{10}$ $\zeta = \mathbb{Z}_{10}$ $G_{95}/\zeta = S_4$ $\phi_{95} : ab^{-1} \mapsto G_{62}.a,\ b \mapsto G_{62}.b$

96. `( 3, 7, 7, 4, 4, 0)` $G_{96} = \mathbb{Z}_5$
$H_1 = \mathbb{Z}_5$ $\zeta = \mathbb{Z}_5$ $G_{96}/\zeta = 1$ —
lens space $\mathcal{L}(5,?)$

97. `( 3, 7, 7, 4, 4,12)` $G_{97} = \langle a, b \mid ab^{-3}ab = 1, a^4b^{-4}, a^3b^{-1}a^{-1}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}$ $\zeta = \mathbb{Z}$ $G_{97}/\zeta = \Delta^+(4,4,2)$ $\phi_{97} : a \mapsto G_{73}.a,\ b \mapsto G_{73}.b$

98. `( 3, 7, 7, 4, 6, 0)` $G_{98} = \mathbb{Z}$
$H_1 = \mathbb{Z}$ $\zeta = \mathbb{Z}$ $G_{98}/\zeta = 1$ —
the space $S^1 \times S^2$; representative

99. `( 5, 5, 7, 0, 2, 4)` $G_{99} = \mathbb{Z}_{13}$
$H_1 = \mathbb{Z}_{13}$ $\zeta = \mathbb{Z}_{13}$ $G_{99}/\zeta = 1$ —
lens space $\mathcal{L}(13,?)$

100. `( 5, 5, 7, 0, 2, 6)` $G_{100} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{100}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(3,1)$

101. `( 5, 5, 7, 0, 2, 8)` $G_{101} = \langle a, b \mid a^2b^2 = 1, abab^{-1}ab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}_4$ $G_{101}/\zeta = D_6$ $\phi_{101} : a^{-1}b \mapsto G_{19}.a,\ ab^{-1}a \mapsto G_{19}.b$

102. `( 5, 5, 7, 0, 4, 6)` $G_{102} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{102}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(3,1)$

103. `( 5, 5, 7, 0, 4,10)` $G_{103} = \mathbb{Z}_{13}$
$H_1 = \mathbb{Z}_{13}$ $\zeta = \mathbb{Z}_{13}$ $G_{103}/\zeta = 1$ —
lens space $\mathcal{L}(13, ?)$

104. `( 5, 5, 7, 2, 0, 2)` $G_{104} = \langle a, b \mid ab^2a = 1, b^{-2}ab^{-1}ab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_6 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_6$ $G_{104}/\zeta = D_4$ $\phi_{104} : a \mapsto G_{11}.a,\ b \mapsto G_{11}.b$

105. `( 5, 5, 7, 2, 0, 4)` $G_{105} = \mathbb{Z}_{18}$
$H_1 = \mathbb{Z}_{18}$ $\zeta = \mathbb{Z}_{18}$ $G_{105}/\zeta = 1$ —
lens space $\mathcal{L}(18, ?)$

106. `( 5, 5, 7, 2, 0, 6)` $G_{106} = \mathbb{Z}_{12}$
$H_1 = \mathbb{Z}_{12}$ $\zeta = \mathbb{Z}_{12}$ $G_{106}/\zeta = 1$ —
lens space $\mathcal{L}(12, ?)$

107. `( 5, 5, 7, 2, 0, 8)` $G_{107} = \langle a, b \mid a^2b^2 = 1, ab^{-1}ab^{-1}a^{-1}b^{-1}ab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_6 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_6$ $G_{107}/\zeta = D_8$ $\phi_{107} : ab^{-1} \mapsto G_{30}.a,\ b^{-1} \mapsto G_{30}.b$

108. `( 5, 5, 7, 2, 2, 8)` $G_{108} = \langle a, b \mid ab^{-1}ab = 1, b^{-1}a^{-3}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_4$ $\zeta = \mathbb{Z}_2$ $G_{108}/\zeta = D_6$ $\phi_{108} : a^{-1} \mapsto G_9.a,\ b \mapsto G_9.b$

109. `( 5, 5, 7, 2, 4, 2)` $G_{109} = \langle a, b \mid a^2 = (a^{-1}b^2)^2, b^2 = (a^2b)^2\rangle$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$ ζ ? G_{109}/ζ ? —
representative

110. `( 5, 5, 7, 2, 4, 6)` $G_{110} = \langle a, b \mid ab^{-1}ab = 1, b^{-1}a^{-3}b^{-2}a^{-2}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}_4$ $G_{110}/\zeta = D_{10}$ $\phi_{110} : ba^{-1}ba \mapsto G_{85}.a,\ b^{-1}ab \mapsto G_{85}.b$

111. `( 5, 5, 7, 2, 6, 2)` $G_{111} = \langle a, b \mid ab^{-3}a^2 = 1, ab^{-1}ab^{-1}a^{-2}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$ $\zeta = \mathbb{Z}$ $G_{111}/\zeta = \Delta^+(3,3,3)$ $\phi_{111} : a \mapsto G_{69}.a,\ b^{-1} \mapsto G_{69}.b$

112. `( 5, 5, 7, 2, 6, 6)` $G_{112} = \langle a, b \mid a^3 = (ab)^3 = (ab^2)^3\rangle$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}$ $\zeta =?$ $G_{112}/\zeta =?$ —
Eucleidean manifold; representative

113. `( 5, 5, 7, 4, 0, 6)` $G_{113} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{113}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(3,1)$

114. `( 5, 5, 7, 4, 2, 4)` $G_{114} = \langle a, b \mid ab^{-1}a^{-1}b^{-1}a^{-1}b^{-1} = 1, ab^{-2}a^2b^{-1}a = 1\rangle$
$H_1 = \mathbb{Z}_{15}$ $\zeta = \mathbb{Z}_{10}$ $G_{114}/\zeta = A_4$ $\phi_{114} : a^{-1}b^{-1}a^{-168} \mapsto G_{94}.a,$
$b^{-1}a^{-170}b^{-1}a^{-176} \mapsto G_{94}.b$

115. `( 5, 5, 7, 4, 4, 4)` $G_{115} = \langle a, b \mid b^{-1}aba = 1, b^{-1}a^{-2}b^{-2}a^{-3}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}_4$ $G_{115}/\zeta = D_{10}$ $\phi_{115} : b^2a \mapsto G_{85}.a,\ b^{-3}a \mapsto G_{85}.b$

116. `( 5, 5, 7, 4, 4, 6)` $G_{116} = \langle a, b \mid a^6 = b^3 = (ab)^2\rangle$
$H_1 = \mathbb{Z}$ $\zeta = \mathbb{Z}$ $G_{116}/\zeta = \Delta^+(6,3,2)$ —
Eucleidean manifold; representative

117. `( 2, 6,10, 3, 3, 1)` $G_{117} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{117}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(3,1)$

118. `( 2, 6,10, 3, 5, 1)` $G_{118} = \mathbb{Z}_3 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{118}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(3,1)$

119. `( 2, 6,10, 3, 5,15)` $G_{119} = \mathbb{Z}_2 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{119}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_5$ —
connected sum

120. `( 2, 6,10, 5, 3,15)` $G_{120} = \mathbb{Z}_3 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{120}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(3,1)$

121. `( 2, 6,10, 5, 5,15)` $G_{121} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{121}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(3,1)$

122. `( 2, 8, 8, 3, 3, 1)` $G_{122} = \mathbb{Z}_{16}$
$H_1 = \mathbb{Z}_{16}$ $\zeta = \mathbb{Z}_{16}$ $G_{122}/\zeta = 1$ —
lens space $\mathcal{L}(16,?)$

123. `( 2, 8, 8, 3, 5, 1)` $G_{123} = \mathbb{Z}_{19}$
$H_1 = \mathbb{Z}_{19}$ $\zeta = \mathbb{Z}_{19}$ $G_{123}/\zeta = 1$ —
lens space $\mathcal{L}(19,?)$

124. `( 2, 8, 8, 5, 5, 1)` $G_{124} = \mathbb{Z}_{21}$
$H_1 = \mathbb{Z}_{21}$ $\zeta = \mathbb{Z}_{21}$ $G_{124}/\zeta = 1$ —
lens space $\mathcal{L}(21,?)$

125. `( 4, 4,10, 1, 1, 1)` $G_{125} = \langle a, b \mid a^5 = b^2 = (ab)^{-2} \rangle$
$H_1 = \mathbb{Z}_{24}$ $\zeta = \mathbb{Z}_{12}$ $G_{125}/\zeta = D_{10}$ —
representative; GAP code for the group is `[120,2]`

126. `( 4, 4,10, 1, 1, 3)` $G_{126} = \mathbb{Z}_{19}$
$H_1 = \mathbb{Z}_{19}$ $\zeta = \mathbb{Z}_{19}$ $G_{126}/\zeta = 1$ —
lens space $\mathcal{L}(19,?)$

127. `( 4, 4,10, 1, 1, 5)` $G_{127} = \langle a, b \mid ab^{-1}a^{-2}b^{-1}ab^{-1} = 1, a^3baba = 1 \rangle$
$H_1 = \mathbb{Z}_{15}$ $\zeta = \mathbb{Z}_{10}$ $G_{127}/\zeta = A_4$ $\phi_{127} : b^{-1}a^{-1}b^2 \mapsto G_{94}.a,\ b^{-2}a^{-1}b^2 \mapsto G_{94}.b$

128. `( 4, 4,10, 1, 1, 7)` $G_{128} = \langle a, b \mid bab^2a^{-1}b = 1, aba^4b = 1 \rangle$
$H_1 = \mathbb{Z}_{20}$ $\zeta = \mathbb{Z}_{10}$ $G_{128}/\zeta = D_6$ $\phi_{128} : b^3ab \mapsto G_{194}.a,\ a^{-1}b^{-1} \mapsto G_{194}.b$
representative; GAP code for the group is `[60,1]`

129. `( 4, 4,10, 1, 1, 9)` $G_{129} = \mathbb{Z}_{16}$
$H_1 = \mathbb{Z}_{16}$ $\zeta = \mathbb{Z}_{16}$ $G_{129}/\zeta = 1$ —
lens space $\mathcal{L}(16,?)$

130. `( 4, 4,10, 1, 1,11)` $G_{130} = \langle a, b \mid a^5 = b^{-2} = (ab^{-1})^3 \rangle$
$H_1 = \mathbb{Z}_{11}$ $\zeta = \mathbb{Z}_{22}$ $G_{130}/\zeta = A_5$ —
representative; GAP code for the group is `[1320,14]`

131. `( 4, 4,10, 1, 5, 1)` $G_{131} = \langle a, b \mid a^6 = b^2 = (ab^{-1})^2 \rangle$
$H_1 = \mathbb{Z}_{10} \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_{10}$ $G_{131}/\zeta = D_{12}$ —
rerpesentative; GAP code for the group is `[120,21]`

132. `( 4, 4,10, 1, 5, 7)` $G_{132} = \langle a, b \mid b^{-1}a^{-1}b^{-2}ab^{-1} = 1, aba^{-4}b = 1 \rangle$
$H_1 = \mathbb{Z}_{12}$ $\zeta = \mathbb{Z}_6$ $G_{132}/\zeta = D_{10}$ $\phi_{132} : a \mapsto G_{415}.a,\ ba^{-1} \mapsto G_{415}.b$
representative; GAP code for the group is `[60,2]`

133. `( 4, 4,10, 1, 5, 9)` $G_{133} = \langle a, b \mid b^{-1}aba = 1, b^{-1}a^{-4}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{133}/\zeta = D_8$ $\phi_{133} : a \mapsto G_{20}.a,\ b \mapsto G_{20}.b$

134. `( 4, 4,10, 3, 1, 9)` $G_{134} = \langle a, b \mid ab^{-1}a^{-3}b^{-1} = 1, ab^{-1}abab^{-1}a = 1\rangle$
$H_1 = \mathbb{Z}_{10}$ $\zeta = \mathbb{Z}_{10}$ $G_{134}/\zeta = S_4$ $\phi_{134} : ab^{-1} \mapsto G_{62}.a,\ b \mapsto G_{62}.b$

135. `( 4, 4,10, 3, 3, 3)` $G_{135} = \langle a, b \mid a^5 = b^3 = (a^2b)^2\rangle$
$H_1 = \mathbb{Z}_7$ $\zeta = \mathbb{Z}_{14}$ $G_{135}/\zeta = A_5$ —
representative; GAP code for the group is `[840,13]`

136. `( 4, 4,10, 3, 3, 7)` $G_{136} = \langle a, b \mid b^{-1}aba = 1, b^{-1}a^{-3}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_4$ $\zeta = \mathbb{Z}_2$ $G_{136}/\zeta = D_6$ $\phi_{136} : a^{-1} \mapsto G_9.a,\ b \mapsto G_9.b$

137. `( 4, 4,10, 3, 3, 9)` $G_{137} = \langle a, b \mid ab^{-1}a^{-1}ba^{-1}b^{-1} = 1, a^2b^{-1}a^{-4}b^{-1} = 1, ab^{-1}a^{-3}b^{-1}ab = 1\rangle$
$H_1 = \mathbb{Z}$ $\zeta = \mathbb{Z}$ $G_{137}/\zeta = \Delta^+(6, 3, 2)$ $\phi_{137} : a \mapsto G_{116}.a,\ ab^{-1} \mapsto G_{116}.b$

138. `( 4, 4,10, 3, 7, 7)` $G_{138} = \mathbb{Z}_{12}$
$H_1 = \mathbb{Z}_{12}$ $\zeta = \mathbb{Z}_{12}$ $G_{138}/\zeta = 1$ —
lens space $\mathcal{L}(12, ?)$

139. `( 4, 6, 8, 1, 1, 1)` $G_{139} = \langle a, b \mid a^4 = b^3 = (ab)^{-2}\rangle$
$H_1 = \mathbb{Z}_{26}$ $\zeta = \mathbb{Z}_{26}$ $G_{139}/\zeta = S_4$ —
representative; GAP code for the group is `[624,131]`

140. `( 4, 6, 8, 1, 1, 3)` $G_{140} = \langle a, b \mid ab^{-1}a^3b^{-1} = 1, aba^2b^2a = 1\rangle$
$H_1 = \mathbb{Z}_{20}$ $\zeta = \mathbb{Z}_{10}$ $G_{140}/\zeta = D_6$ $\phi_{140} : b^{-1}a^2b^{-1} \mapsto G_{194}.a,\ ab^{-2}a^2b^{-1} \mapsto G_{194}.b$

141. `( 4, 6, 8, 1, 1,11)` $G_{141} = \langle a, b \mid a^4 = b^3 = (ab)^3\rangle$
$H_1 = \mathbb{Z}_9$ $\zeta = \mathbb{Z}$ $G_{141}/\zeta = \Delta^+(4, 3, 3)$ —
representative

142. `( 4, 6, 8, 1, 5, 3)` $G_{142} = \langle a, b \mid ab^{-2}a^2 = 1, ab^{-1}a^{-2}b^{-1}a^{-2}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_{15}$ $\zeta = \mathbb{Z}_{10}$ $G_{142}/\zeta = A_4$ $\phi_{142} : a^{-1}b^{40}a^{-1}b^{42} \mapsto G_{94}.a,\ a^{-1}b^{43} \mapsto G_{94}.b$

143. `( 4, 6, 8, 1, 7, 1)` $G_{143} = \langle a, b \mid a^5 = b^3 = (ab^{-1})^2\rangle$
$H_1 = \mathbb{Z}_{19}$ $\zeta = \mathbb{Z}_{38}$ $G_{143}/\zeta = A_5$ —
representative; this group is isomorphic to the group of order 2280,
which is an extension of $\mathbb{Z}_{19} \times A_5$

144. `( 4, 6, 8, 3, 5,13)` $G_{144} = \langle a, b \mid ab^{-2}ab = 1, b^{-1}a^3b^{-1}a^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{144}/\zeta = S_4$ $\phi_{144} : a \mapsto G_{51}.a,\ b \mapsto G_{51}.b$

145. `( 4, 6, 8, 3, 9, 3)` $G_{145} = \langle a, b \mid ab^{-1}a^{-1}b^{-1}a = 1, a^2b^4a = 1\rangle$
$H_1 = \mathbb{Z}_{10}$ $\zeta = \mathbb{Z}_{10}$ $G_{145}/\zeta = S_4$ $\phi_{145} : ab^{-1} \mapsto G_{62}.a,\ b \mapsto G_{62}.b$

146. `( 4, 6, 8, 3, 9,13)` $G_{146} = \langle a, b \mid a^4 = b^4 = (a^{-1}b)^2\rangle$
$H_1 = \mathbb{Z}_8 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}$ $G_{146}/\zeta = \Delta^+(4, 4, 2)$ —
representative

147. `( 4, 6, 8, 5, 1, 3)` $G_{147} = \mathbb{Z}_{19}$
$H_1 = \mathbb{Z}_{19}$ $\zeta = \mathbb{Z}_{19}$ $G_{147}/\zeta = 1$ —
lens space $\mathcal{L}(19, ?)$

148. `( 4, 6, 8, 5, 3, 1)` $G_{148} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{148}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2, 1)\#\mathcal{L}(3, 1)$

149. `( 4, 6, 8, 5, 5, 3)` $G_{149} = \langle a, b \mid ab^{-2}a^2 = 1, b^{-1}a^{-1}b^{-1}a^{-1}ba^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_9$ $\zeta = \mathbb{Z}_6$ $G_{149}/\zeta = A_4$ $\phi_{149} : a^{-1}b^{-60}a^{-1}b^{-58} \mapsto G_{49}.a,$
$a^{-1}b^{-60}a^{-1}b^{-57} \mapsto G_{49}.b$

150. `( 4, 6, 8, 5, 5,11)` $G_{150} = \langle a, b \mid a^5 = b^4 = (ab)^2 \rangle$
$H_1 = \mathbb{Z}_2$ $\zeta = \mathbb{Z}$ $G_{150}/\zeta = \Delta^+(5,4,2)$ —
representative

151. `( 4, 6, 8, 5, 7, 1)` $G_{151} = \mathbb{Z}_2 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{151}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_4$ —
connected sum

152. `( 4, 6, 8, 5, 7,11)` $G_{152} = \langle a, b \mid ab^{-1}ab = 1, b^{-1}a^{-1}b^{-1}a^3 = 1 \rangle$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{152}/\zeta = D_8$ $\phi_{152} : a \mapsto G_{20}.a,\ b \mapsto G_{20}.b$

153. `( 4, 6, 8, 5, 7,13)` $G_{153} = \langle a, b \mid a^3b^{-1}a^3b^{-2}a^{-1}b^{-2} = 1, a^3b^{-3} = 1, a^2b^{-2} = 1 \rangle$
$H_1 = \mathbb{Z}$ $\zeta = \mathbb{Z}$ $G_{153}/\zeta = 1$ $\phi_{153} : a \mapsto G_{98}.a, b \mapsto G_{98}.b$
note that the group is two-generated, but $a = b$!

154. `( 6, 6, 6, 1, 1, 1)` $G_{154} = \langle a, b \mid a^3 = b^3 = (ab)^{-3} \rangle$
$H_1 = \mathbb{Z}_9 \times \mathbb{Z}_3$ $\zeta = \mathbb{Z}$ $G_{154}/\zeta = \Delta^+(3,3,3)$ —
representative

155. `( 6, 6, 6, 1, 1, 9)` $G_{155} = \langle a, b \mid ab^3a^2 = 1, ab^2ab^{-1}ab^{-1}ab^{-1} = 1 \rangle$
$H_1 = \mathbb{Z}_{15}$ $\zeta = \mathbb{Z}$ $G_{155}/\zeta = \Delta^+(4,3,3)$ $\phi_{155} : ab^{-1} \mapsto G_{377}.a,\ a \mapsto G_{377}.b$
representative

156. `( 6, 6, 6, 1, 3, 3)` $G_{156} = \mathbb{Z}_{21}$
$H_1 = \mathbb{Z}_{21}$ $\zeta = \mathbb{Z}_{21}$ $G_{156}/\zeta = 1$ —
lens space $\mathcal{L}(21, ?)$

157. `( 6, 6, 6, 1, 3, 7)` $G_{157} = \langle a, b \mid ab^2a = 1, abab^{-1}ab^{-1}a^{-1}b^{-1}ab^{-1} = 1 \rangle$
$H_1 = \mathbb{Z}_{12}$ $\zeta = \mathbb{Z}_6$ $G_{157}/\zeta = D_{10}$ $\phi_{157} : a^{-7}b \mapsto G_{415}.a,\ a^{-1}b^{-1}a^7 \mapsto G_{415}.b$

158. `( 6, 6, 6, 1, 5, 9)` $G_{158} = \langle a, b \mid a^2b^{-2} = 1, a^{-1}ba^{-1}ba^{-1}b^{-1} = 1 \rangle$
$H_1 = \mathbb{Z}_4$ $\zeta = \mathbb{Z}_2$ $G_{158}/\zeta = D_6$ $\phi_{158} : a^{-1}b \mapsto G_9.a,\ a \mapsto G_9.b$

159. `( 6, 6, 6, 1, 7, 7)` $G_{159} = \langle a, b \mid a^3 = b^2a^{-1}b^2, b^3 = a^2b^{-1}a^2 \rangle$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}$ ζ ? G_{159}/ζ ? —
representative

160. `( 6, 6, 6, 3, 3, 5)` $G_{160} = \langle a, b \mid ab^{-1}a^{-1}b^{-1}ab^{-2} = 1, ab^{-1}abab^{-1}a = 1 \rangle$
$H_1 = \mathbb{Z}_{15}$ $\zeta = \mathbb{Z}_{10}$ $G_{160}/\zeta = A_4$ $\phi_{160} : b^{-1}a^{56}b^{-1}a^{55} \mapsto G_{94}.a,$
$a^{-63}bab^{-1}a^{55} \mapsto G_{94}.b$

161. `( 6, 6, 6, 3, 3, 7)` $G_{161} = \langle a, b \mid ab^{-1}a^{-2}b^{-1} = 1, ab^{-1}ab^2 = 1 \rangle$
$H_1 = \mathbb{Z}_3$ $\zeta = \mathbb{Z}_2$ $G_{161}/\zeta = A_4$ $\phi_{161} : a \mapsto G_{13}.a,\ b^{-1} \mapsto G_{13}.b$

162. `( 6, 6, 6, 3, 5, 5)` $G_{162} = \langle a, b \mid ab^{-2}ab = 1, ab^{-1}a^{-1}b^{-1}a^{-1}bab = 1,$
$a^2b^{-1}a^{-1}b^{-1}ab^{-1}a^{-1}b^{-1} = 1 \rangle$
$H_1 = \mathbb{Z}_7$ $\zeta = \mathbb{Z}_{14}$ $G_{162}/\zeta = A_5$ ϕ_{162} : ?
GAP code for the group is `[840,13]`

163. `( 6, 6, 6, 5, 5, 5)` $G_{163} = \langle a, b \mid a^2ba^{-1} = b^{-1}ab^2, b^{-1}a^2b^{-1} = ab^{-2}a \rangle$
$H_1 = \mathbb{Z}$ ζ ? G_{163}/ζ ? —
representative

164. `( 1, 3,15, 2, 2, 0)` $G_{164} = \mathbb{Z}_2 * \mathbb{Z}_8$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_8$ $\zeta = 1$ $G_{164}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_8$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(8,1)$; representative

165. `( 1, 3,15, 6, 2, 0)` $G_{165} = \mathbb{Z}_2 * \mathbb{Z}_8$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_8$ $\zeta = 1$ $G_{165}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_8$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(8,3)$; representative

166. `( 1, 5,13, 2, 2, 0)` $G_{166} = \mathbb{Z}_3 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{166}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_7$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(7,1)$; representative

167. `( 1, 5,13, 2, 4, 0)` $G_{167} = \mathbb{Z}_3 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{167}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_7$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(7,1)$

168. `( 1, 5,13, 4, 2, 0)` $G_{168} = \mathbb{Z}_3 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{168}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_7$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(7,2)$; representative

169. `( 1, 5,13, 4, 4, 0)` $G_{169} = \mathbb{Z}_3 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{169}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_7$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(7,2)$

170. `( 1, 5,13, 6, 2, 0)` $G_{170} = \mathbb{Z}_3 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{170}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_7$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(7,3)$

171. `( 1, 5,13, 6, 4, 0)` $G_{171} = \mathbb{Z}_3 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{171}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_7$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(7,3)$

172. `( 1, 7,11, 2, 2, 0)` $G_{172} = \mathbb{Z}_4 * \mathbb{Z}_6$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_6$ $\zeta = 1$ $G_{172}/\zeta = \mathbb{Z}_4 * \mathbb{Z}_6$ —
connected sum $\mathcal{L}(4,1)\#\mathcal{L}(6,1)$; representative

173. `( 1, 7,11, 2, 6, 0)` $G_{173} = \mathbb{Z}_4 * \mathbb{Z}_6$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_6$ $\zeta = 1$ $G_{173}/\zeta = \mathbb{Z}_4 * \mathbb{Z}_6$ —
connected sum $\mathcal{L}(4,1)\#\mathcal{L}(6,1)$

174. `( 1, 9, 9, 2, 2, 0)` $G_{174} = \mathbb{Z}_5 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_5 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{174}/\zeta = \mathbb{Z}_5 * \mathbb{Z}_5$ —
connected sum $\mathcal{L}(5,1)\#\mathcal{L}(5,1)$; representative

175. `( 1, 9, 9, 2, 4, 0)` $G_{175} = \mathbb{Z}_5 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_5 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{175}/\zeta = \mathbb{Z}_5 * \mathbb{Z}_5$ —
connected sum $\mathcal{L}(5,1)\#\mathcal{L}(5,2)$; representative

176. `( 1, 9, 9, 2, 6, 0)` $G_{176} = \mathbb{Z}_5 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_5 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{176}/\zeta = \mathbb{Z}_5 * \mathbb{Z}_5$ —
connected sum $\mathcal{L}(5,1)\#\mathcal{L}(5,3)$; representative

177. `( 1, 9, 9, 2, 8, 0)` $G_{177} = \mathbb{Z}_5 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_5 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{177}/\zeta = \mathbb{Z}_5 * \mathbb{Z}_5$ —
connected sum $\mathcal{L}(5,1)\#\mathcal{L}(5,2)$

178. `( 1, 9, 9, 4, 4, 0)` $G_{178} = \mathbb{Z}_5 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_5 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{178}/\zeta = \mathbb{Z}_5 * \mathbb{Z}_5$ —
connected sum $\mathcal{L}(5,2)\#\mathcal{L}(5,2)$; representative

179. `( 1, 9, 9, 4, 6, 0)` $G_{179} = \mathbb{Z}_5 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_5 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{179}/\zeta = \mathbb{Z}_5 * \mathbb{Z}_5$ —
connected sum $\mathcal{L}(5,2)\#\mathcal{L}(5,3)$; representative

180. `( 3, 3,13, 0, 2, 4)` $G_{180} = \mathbb{Z}_3 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{180}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_4$ —
connected sum

181. `( 3, 3,13, 0, 2,12)` $G_{181} = \mathbb{Z}_3 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{181}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_4$ —
connected sum

182. `( 3, 3,13, 2, 0, 2)` $G_{182} = \langle a, b \mid ab^2a^2 = 1, ab^{-3}ab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_{16}$ $\zeta = \mathbb{Z}_8$ $G_{182}/\zeta = D_6$ $\phi_{182} : a^{-1}b^{16} \mapsto G_{26}.a,\ b \mapsto G_{26}.b$

183. `( 3, 3,13, 2, 0,10)` $G_{183} = \mathbb{Z}_{16}$
$H_1 = \mathbb{Z}_{16}$ $\zeta = \mathbb{Z}_{16}$ $G_{183}/\zeta = 1$ —
lens space $\mathcal{L}(16, ?)$

184. `( 3, 3,13, 2, 2, 8)` $G_{184} = \langle a, b \mid b^{-1}a^{-1}b^{-2}ab^{-1} = 1, aba^{-4}b = 1\rangle$
$H_1 = \mathbb{Z}_{12}$ $\zeta = \mathbb{Z}_6$ $G_{184}/\zeta = D_{10}$ $\phi_{184} : a \mapsto G_{415}.a,\ ba^{-1} \mapsto G_{415}.b$

185. `( 3, 3,13, 2, 2,12)` $G_{185} = \langle a, b \mid a^7 = b^2 = (ab)^2\rangle$
$H_1 = \mathbb{Z}_4$ $\zeta = \mathbb{Z}_2$ $G_{185}/\zeta = D_{14}$ —
representative; GAP code for the group is `[28,1]`

186. `( 3, 3,13, 4, 2, 4)` $G_{186} = \langle a, b \mid a^2b^{-1}a^{-2}ba^2b^{-1}a^2ba^{-2}b^{-1}a^2b = 1, a^2 = 1\rangle$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}$ $\zeta = 1$ $G_{186}/\zeta = \mathbb{Z}_2 * \mathbb{Z}$ $\phi_{186} : a \mapsto G_2.a,\ b \mapsto G_2.b$

187. `( 3, 5,11, 2, 4, 0)` $G_{187} = \mathbb{Z}_2 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{187}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_7$ —
connected sum

188. `( 3, 5,11, 2, 4, 2)` $G_{188} = \langle a, b \mid ab^{-1}a^{-3}b^{-1} = 1, ab^{-4}ab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_{14}$ $\zeta = \mathbb{Z}_{14}$ $G_{188}/\zeta = S_4$ $\phi_{188} : a \mapsto G_{70}.a,\ ab^{-1}a^{109} \mapsto G_{70}.b$

189. `( 3, 5,11, 4, 4, 0)` $G_{189} = \mathbb{Z}_2 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{189}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_7$ —
connected sum

190. `( 3, 5,11, 4, 4, 2)` $G_{190} = \langle a, b \mid ab^{-1}a^{-2}b^{-1}a^2 = 1, ab^{-2}a^2ba = 1\rangle$
$H_1 = \mathbb{Z}_7$ $\zeta = \mathbb{Z}_{14}$ $G_{190}/\zeta = A_5$ $\phi_{190} : a \mapsto G_{135}.a,\ b \mapsto G_{135}.b$

191. `( 3, 5,11, 4, 4,14)` $G_{191} = \langle a, b \mid ab^{-2}ab, a^6b^{-3} = 1, a^5b^{-1}a^{-1}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}$ $\zeta = \mathbb{Z}$ $G_{191}/\zeta = \Delta^+(6, 3, 2)$ $\phi_{191} : a \mapsto G_{116}.a,\ b \mapsto G_{116}.b$

192. `( 3, 5,11, 6, 4, 0)` $G_{192} = \mathbb{Z}_2 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{192}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_7$ —
connected sum

193. `( 3, 7, 9, 2, 0, 2)` $G_{193} = \langle a, b \mid ab^{-1}a^3b^{-1} = 1, ab^4a = 1\rangle$
$H_1 = \mathbb{Z}_{10} \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_{10}$ $G_{193}/\zeta = D_8$ $\phi_{193} : b \mapsto G_{61}.a,\ ab^{-1} \mapsto G_{61}.b$

194. `( 3, 7, 9, 2, 4, 2)` $G_{194} = \langle a, b \mid a^3 = b^2 = (a^2b)^2\rangle$
$H_1 = \mathbb{Z}_{20}$ $\zeta = \mathbb{Z}_{10}$ $G_{194}/\zeta = D_6$ —

195. `( 3, 7, 9, 4, 0, 2)` $G_{195} = \mathbb{Z}_2 * \mathbb{Z}_2$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = 1$ $G_{195}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_2$ —
connected sum $\mathcal{L}(2, 1)\#\mathcal{L}(2, 1)$

196. `( 3, 7, 9, 4, 2, 0)` $G_{196} = \mathbb{Z}_3 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{196}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_5$ —
connected sum

197. `( 3, 7, 9, 4, 4, 0)` $G_{197} = \mathbb{Z}_3 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{197}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_5$ —
connected sum

198. `( 3, 7, 9, 4, 4, 2)` $G_{198} = \mathbb{Z}_2$
$H_1 = \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{198}/\zeta = 1$ —
lens space $\mathcal{L}(2,1)$

199. `( 3, 7, 9, 4, 4,14)` $G_{199} = \langle a, b \mid ab^{-3}ab = 1, ab^{-1}a^{-1}b^{-1}a^3 = 1 \rangle$
$H_1 = \mathbb{Z}_2$ $\zeta = \mathbb{Z}$ $G_{199}/\zeta = \Delta^+(5,4,2)$ $\phi_{199} : a \mapsto G_{150}.a,\ b \mapsto G_{150}.b$

200. `( 3, 7, 9, 4, 6, 0)` $G_{200} = \mathbb{Z}_3 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{200}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_5$ —
connected sum

201. `( 3, 7, 9, 4, 6, 2)` $G_{201} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{201}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(3,1)$

202. `( 3, 7, 9, 4, 8, 0)` $G_{202} = \mathbb{Z}_3 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{202}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_5$ —
connected sum

203. `( 3, 7, 9, 4, 8,14)` $G_{203} = \langle a, b \mid ab^{-1}ab^2 = 1, a^2b^{-3}a^3 = 1 \rangle$
$H_1 = \mathbb{Z}_{11}$ $\zeta = \mathbb{Z}_{22}$ $G_{203}/\zeta = A_5$ $\phi_{203} : a \mapsto G_{130}.a,\ ab^{-1} \mapsto G_{130}.b$

204. `( 5, 5, 9, 0, 4, 4)` $G_{204} = \mathbb{Z}_6$
$H_1 = \mathbb{Z}_6$ $\zeta = \mathbb{Z}_6$ $G_{204}/\zeta = 1$ $\phi_{204} :$
lens space $\mathcal{L}(6,?)$

205. `( 5, 5, 9, 0, 4, 6)` $G_{205} = \mathbb{Z}_{11}$
$H_1 = \mathbb{Z}_{11}$ $\zeta = \mathbb{Z}_{11}$ $G_{205}/\zeta = 1$ —
lens space $\mathcal{L}(11,?)$

206. `( 5, 5, 9, 0, 4,12)` $G_{206} = \mathbb{Z}_{16}$
$H_1 = \mathbb{Z}_{16}$ $\zeta = \mathbb{Z}_{16}$ $G_{206}/\zeta =$ —
lens space $\mathcal{L}(16,?)$

207. `( 5, 5, 9, 2, 0, 2)` $G_{207} = \langle a, b \mid ab^3a = 1, ab^{-1}ab^{-1}a^3b^{-1} = 1 \rangle$
$H_1 = \mathbb{Z}_{21}$ $\zeta = \mathbb{Z}_{14}$ $G_{207}/\zeta = A_4$ $\phi_{207} : b^{-13} \mapsto G_{68}.a,\ b^{-21}ab^{13} \mapsto G_{68}.b$

208. `( 5, 5, 9, 2, 0,10)` $G_{208} = \langle a, b \mid a^3b^2 = 1, ab^{-1}ab^{-1}abab^{-1} = 1 \rangle$
$H_1 = \mathbb{Z}_{14}$ $\zeta = \mathbb{Z}_{14}$ $G_{208}/\zeta = S_4$ $\phi_{208} : b^{-56}ab^{-53}a \mapsto G_{70}.a,\ b^{-56}ab^{-54}a \mapsto G_{70}.b$

209. `( 5, 5, 9, 2, 2, 2)` $G_{209} = \langle a, b \mid a = b^2a^2b^2, b = a^3ba^3 \rangle$
$H_1 = \mathbb{Z}_{24}$ ζ ? G_{209}/ζ ? —
representative

210. `( 5, 5, 9, 2, 2,10)` $G_{210} = \langle a, b \mid ab^{-1}ab = 1, b^{-1}a^{-4}b^{-1} = 1 \rangle$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{210}/\zeta = D_8$ $\phi_{210} : a \mapsto G_{20}.a,\ b \mapsto G_{20}.b$

211. `( 5, 5, 9, 2, 6, 2)` $G_{211} = \langle a, b \mid ab^{-1}a^{-2}b^{-1} = 1, ab^{-2}a^2 = 1 \rangle$
$H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}_4$ $G_{211}/\zeta = D_6$ $\phi_{211} : a \mapsto G_{19}.a,\ b \mapsto G_{19}.b$

212. `( 5, 5, 9, 4, 0, 4)` $G_{212} = \mathbb{Z}_{21}$
$H_1 = \mathbb{Z}_{21}$ $\zeta = \mathbb{Z}_{21}$ $G_{212}/\zeta = 1$ —
lens space $\mathcal{L}(21,?)$

213. `( 5, 5, 9, 4, 0, 8)` $G_{213} = \mathbb{Z}_7$
$H_1 = \mathbb{Z}_7$ $\zeta = \mathbb{Z}_7$ $G_{213}/\zeta = 1$ —
lens space $\mathcal{L}(7,?)$

214. `( 5, 5, 9, 4, 4, 4)` $G_{214} = \langle a, b \mid b^2a^2 = 1, ab^{-1}a^{-1}b^{-1} = 1 \rangle$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{214}/\zeta = D_4$ $\phi_{214} : a \mapsto G_4.a,\ b^{-1} \mapsto G_4.b$

215. `( 5, 5, 9, 4, 4, 8)` $G_{215} = \langle a, b \mid ab^{-1}a^{-1}ba^{-1}b^{-1}a = 1, a^2ba^{-1}b^{-1}a^{-1}ba = 1\rangle$
$H_1 = 1$ $\zeta = \mathbb{Z}$ $G_{215}/\zeta = \Delta^+(7,3,2)$ $\phi_{215} : a \mapsto G_{339}.a,\ a^2b \mapsto G_{339}.b$
representative

216. `( 5, 5, 9, 4, 8, 4)` $G_{216} = \langle a, b \mid ab^{-1}a^2ba = 1, b^{-1}a^{-3}b^{-2}a^{-1}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$ ζ ? G_{216}/ζ ? $\phi_{216} : a^{-1} \mapsto G_{109}.a,\ b \mapsto G_{109}.b$

217. `( 5, 7, 7, 0, 2, 2)` $G_{217} = \langle a, b \mid ab^2a^2 = 1, b^{-2}ab^{-1}ab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_{16}$ $\zeta = \mathbb{Z}_8$ $G_{217}/\zeta = D_6$ $\phi_{217} : a \mapsto G_{26}.a,\ ba^{-9} \mapsto G_{26}.b$

218. `( 5, 7, 7, 0, 2,10)` $G_{218} = \langle a, b \mid a^2b^3 = 1, a^{-1}b^{-1}ab^{-1}ab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_9$ $\zeta = \mathbb{Z}_6$ $G_{218}/\zeta = A_4$ $\phi_{218} : a^{-1}b^{-28} \mapsto G_{49}.a,\ b^{-1} \mapsto G_{49}.b$

219. `( 5, 7, 7, 0, 4,12)` $G_{219} = \langle a, b \mid ab^{-3}ab^{-1} = 1, ba^{-5}b = 1\rangle$
$H_1 = \mathbb{Z}_{16}$ $\zeta = \mathbb{Z}_8$ $G_{219}/\zeta = D_{10}$ $\phi_{219} : a \mapsto G_{64}.a,\ b \mapsto G_{64}.b$

220. `( 5, 7, 7, 2, 2, 4)` $G_{220} = \langle a, b \mid a^{-2}b^{-1}a^{-1}b^{-1}a^{-2}b = 1, b^2a^{-1}b^2aba = 1, ab^{-2}ab^{-1}a^2b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_{24}$ ζ ? G_{220}/ζ ? $\phi_{220} : a^b \mapsto G_{209}.a,\ a \mapsto G_{209}.b$

221. `( 5, 7, 7, 2, 2,10)` $G_{221} = \langle a, b \mid ab^{-1}a^{-2}b^{-1} = 1, ab^{-1}ab^2 = 1\rangle$
$H_1 = \mathbb{Z}_3$ $\zeta = \mathbb{Z}_2$ $G_{221}/\zeta = A_4$ $\phi_{221} : a \mapsto G_{13}.a,\ b^{-1} \mapsto G_{13}.b$

222. `( 5, 7, 7, 2, 6, 0)` $G_{222} = \mathbb{Z}_2 * \mathbb{Z}_6$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_6$ $\zeta = 1$ $G_{222}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_6$ —
connected sum

223. `( 5, 7, 7, 2, 6, 4)` $G_{223} = \langle a, b \mid ab^{-2}a = 1, a^{-1}ba^{-1}b^{-1}a^{-1}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}_4$ $G_{223}/\zeta = D_6$ $\phi_{223} : a^{-1}ba^{-1}ba^2 \mapsto G_{19}.a,\ a \mapsto G_{19}.b$

224. `( 5, 7, 7, 2, 8, 0)` $G_{224} = \langle a, b \mid ba^{-1}ba^3 = 1, ab^{-1}ab^{-3} = 1\rangle$
$H_1 = \mathbb{Z}_6 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_6$ $G_{224}/\zeta = D_8$ $\phi_{224} : a \mapsto G_{30}.a,\ b \mapsto G_{30}.b$

225. `( 5, 7, 7, 2, 8, 2)` $G_{225} = \langle a, b \mid ab^{-1}ab^{-1}a^{-2}b^{-1} = 1, ab^{-4}a^2 = 1\rangle$
$H_1 = \mathbb{Z}_9$ $\zeta = \mathbb{Z}$ $G_{225}/\zeta = \Delta^+(4,3,3)$ $\phi_{225} : b \mapsto G_{141}.a,\ ab^{-1} \mapsto G_{141}.b$

226. `( 5, 7, 7, 4, 4, 2)` $G_{226} = \langle a, b \mid b^{-2}a^{-3}b^{-1} = 1, ab^{-2}a^{-1}b^{-2}a = 1\rangle$
$H_1 = \mathbb{Z}_{15}$ $\zeta = \mathbb{Z}_{10}$ $G_{226}/\zeta = A_4$ $\phi_{226} : b^{-23} \mapsto G_{94}.a,\ b^{260}ab^{-23} \mapsto G_{94}.b$

227. `( 5, 7, 7, 4, 6, 4)` $G_{227} = \langle a, b \mid ab^{-2}ab = 1, ab^{-1}a^{-1}b^{-1}a^{-1}bab = 1,$
$ab^{-1}a^{-1}b^{-1}ab^{-1}a^{-1}b^{-1}a = 1\rangle$
$H_1 = \mathbb{Z}_7$ $\zeta = \mathbb{Z}_{14}$ $G_{227}/\zeta = A_5$ ϕ_{227} : ?
GAP code for the group is `[840,13]`

228. `( 5, 7, 7, 4, 6,12)` $G_{228} = \langle a, b \mid a^4 = b^3 = (ab^{-2})^{-3}\rangle$
$H_1 = \mathbb{Z}_3$ $\zeta = \mathbb{Z}$ $G_{228}/\zeta = \Delta^+(4,3,3)$ —
representative

229. `( 5, 7, 7, 6, 6, 2)` $G_{229} = \mathbb{Z}_5$
$H_1 = \mathbb{Z}_5$ $\zeta = \mathbb{Z}_5$ $G_{229}/\zeta = 1$ —
lens space $\mathcal{L}(5, ?)$

230. `( 5, 7, 7, 6, 6, 4)` $G_{230} = \langle a, b \mid a^{-1}b^{-1}abab^{-1} = 1, b^{-1}a^{-1}b^4a^{-1}b^{-1}a = 1\rangle$
$H_1 = 1$ $\zeta = \mathbb{Z}$ $G_{230}/\zeta = \Delta^+(7,3,2)$ $\phi_{230} : b \mapsto G_{339}.a,\ ab \mapsto G_{339}.b$

231. `( 2, 2,16, 3, 1, 3)` $G_{231} = \langle a, b \mid a^3b^{-1}a^{-3}ba^{-3}b^{-1}a^3b = 1, a^3 = 1\rangle$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}$ $\zeta = 1$ $G_{231}/\zeta = \mathbb{Z}_3 * \mathbb{Z}$ $\phi_{231} : a \mapsto G_{14}.a,\ b \mapsto G_{14}.b$

232. `( 2, 2,16, 3, 1, 9)` $G_{232} = \langle a, b \mid a^3b^{-1}a^{-3}ba^{-3}b^{-1}a^3b = 1, a^3 = 1\rangle$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}$ $\zeta = 1$ $G_{232}/\zeta = \mathbb{Z}_3 * \mathbb{Z}$ $\phi_{232} : a \mapsto G_{14}.a,\ b \mapsto G_{14}.b$

233. `( 2, 4,14, 3, 3, 1)` $G_{233} = \mathbb{Z}_2 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{233}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_4$ —
connected sum

234. `( 2, 4,14, 5, 3,17)` $G_{234} = \mathbb{Z}_2 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{234}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_4$ —
connected sum

235. `( 2, 6,12, 3, 3, 1)` $G_{235} = \mathbb{Z}_2 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{235}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_4$ —
connected sum

236. `( 2, 6,12, 3, 5,17)` $G_{236} = \mathbb{Z}_2 * \mathbb{Z}_6$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_6$ $\zeta = 1$ $G_{236}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_6$ —
connected sum

237. `( 2, 6,12, 5, 3, 1)` $G_{237} = \mathbb{Z}_2 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{237}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_5$ —
connected sum

238. `( 2, 6,12, 5, 5,17)` $G_{238} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{238}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(3,1)$

239. `( 2, 6,12, 7, 3, 1)` $G_{239} = \mathbb{Z}_2 * \mathbb{Z}_2$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = 1$ $G_{239}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_2$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(2,1)$

240. `( 2, 8,10, 3, 3, 1)` $G_{240} = \mathbb{Z}_3 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{240}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(3,1)$

241. `( 2, 8,10, 3, 5, 1)` $G_{241} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{241}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(3,1)$

242. `( 2, 8,10, 3, 7, 1)` $G_{242} = \mathbb{Z}_3 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{242}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_4$ —
connected sum

243. `( 2, 8,10, 5, 3,17)` $G_{243} = \mathbb{Z}_3 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{243}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_4$ —
connected sum

244. `( 2, 8,10, 5, 5,17)` $G_{244} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{244}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(3,1)$

245. `( 2, 8,10, 5, 7,17)` $G_{245} = \mathbb{Z}_3 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{245}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(3,1)$

246. `( 4, 4,12, 1, 1, 1)` $G_{246} = \langle a, b \mid a^6 = b^2 = (ab)^{-2} \rangle$
$H_1 = \mathbb{Z}_{14} \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_{14}$ $G_{246}/\zeta = D_{12}$ —
representative; GAP code for the group is `[168,29]`

247. `( 4, 4,12, 1, 1, 5)` $G_{247} = \langle a, b \mid ab^{-1}a^3b^{-1} = 1, b^{-1}a^{-2}b^{-2}ab^{-1}a^{-1} = 1 \rangle$
$H_1 = \mathbb{Z}_{10} \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_{10}$ $G_{247}/\zeta = D_4$ $\phi_{247} : a^{-6}b \mapsto G_{278}.a,\ a^{-11}b \mapsto G_{278}.b$
representative; GAP code for the group is `[40,11]`

248. `( 4, 4,12, 1, 1, 9)` $G_{248} = \langle a, b \mid ab^{-1}a^{-1}b^{-1}ab^{-1} = 1, a^3ba^2ba = 1\rangle$
$H_1 = \mathbb{Z}_{20}$ $\zeta = \mathbb{Z}_{10}$ $G_{248}/\zeta = D_6$ $\phi_{248} : a^{-1}b^{-11} \mapsto G_{194}.a,\ b^6a \mapsto G_{194}.b$

249. `( 4, 4,12, 1, 1,13)` $G_{249} = \langle a, b \mid ab^{-1}abab^{-1} = 1, b^{-1}a^{-6}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_{12}$ $\zeta = \mathbb{Z}$ $G_{249}/\zeta = \Delta^+(6, 3, 2)$ $\phi_{249} : a \mapsto G_{350}.a,\ ab^{-1} \mapsto G_{350}.b$
representative

250. `( 4, 4,12, 1, 5, 1)` $G_{250} = \langle a, b \mid a^7 = b^2 = (ab^{-1})^2\rangle$
$H_1 = \mathbb{Z}_{24}$ $\zeta = \mathbb{Z}_{12}$ $G_{250}/\zeta = D_{14}$ —
representative; GAP code for the group is `[168,4]`

251. `( 4, 4,12, 1, 5, 5)` $G_{251} = \langle a, b \mid b^{-1}a^2ba^2 = 1, a^3b^{-1}a^{-2}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}_4$ $G_{251}/\zeta = D_{10}$ $\phi_{251} : a \mapsto G_{85}.a,\ b \mapsto G_{85}.b$

252. `( 4, 4,12, 3, 1, 5)` $G_{252} = \mathbb{Z}_2$
$H_1 = \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{252}/\zeta = 1$ —
lens space $\mathcal{L}(2, 1)$

253. `( 4, 4,12, 3, 1,11)` $G_{253} = \langle a, b \mid ab^{-1}a^{-4}b^{-1} = 1, ab^{-1}abab^{-1}a = 1\rangle$
$H_1 = \mathbb{Z}_{11}$ $\zeta = \mathbb{Z}_{22}$ $G_{253}/\zeta = A_5$ $\phi_{253} : a \mapsto G_{130}.a,\ ba^{-1} \mapsto G_{130}.b$

254. `( 4, 4,12, 3, 3, 3)` $G_{254} = \langle a, b \mid ab^{-1}a^{-2}b^{-1} = 1, ab^{-2}a^2 = 1\rangle$
$H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}_4$ $G_{254}/\zeta = D_6$ $\phi_{254} : a \mapsto G_{19}.a,\ b \mapsto G_{19}.b$

255. `( 4, 4,12, 3, 3,11)` $G_{255} = \langle a, b \mid b^{-1}aba^{-1}ba = 1, bab^{-1}a^{-4}b^{-1}a = 1\rangle$
$H_1 = 1$ $\zeta = \mathbb{Z}$ $G_{255}/\zeta = \Delta^+(7, 3, 2)$ $\phi_{255} : a \mapsto G_{339}.a,\ b^{-1}a \mapsto G_{339}.b$

256. `( 4, 4,12, 3, 7, 3)` $G_{256} = \langle a, b \mid ab^2a = 1, ab^{-1}ab^{-3} = 1\rangle$
$H_1 = \mathbb{Z}_6 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_6$ $G_{256}/\zeta = D_4$ $\phi_{256} : a \mapsto G_{11}.a,\ b \mapsto G_{11}.b$

257. `( 4, 6,10, 1, 1, 1)` $G_{257} = \langle a, b \mid a^5 = b^3 = (ab)^{-2}\rangle$
$H_1 = \mathbb{Z}_{31}$ $\zeta = \mathbb{Z}_{62}$ $G_{257}/\zeta = A_5$ —
representative; group of order 3720

258. `( 4, 6,10, 1, 1,13)` $G_{258} = \langle a, b \mid a^5 = b^3 = (ab)^3\rangle$
$H_1 = \mathbb{Z}_9$ $\zeta = \mathbb{Z}$ $G_{258}/\zeta = \Delta^+(5, 3, 3)$ —
representative

259. `( 4, 6,10, 1, 7, 1)` $G_{259} = \langle a, b \mid a^6 = b^3 = (ab^{-1})^2\rangle$
$H_1 = \mathbb{Z}_{24}$ $\zeta = \mathbb{Z}$ $G_{259}/\zeta = \Delta^+(6, 5, 2)$ —
representative

260. `( 4, 6,10, 3, 3, 3)` $G_{260} = \langle a, b \mid aba^{-1}ba = 1, ab^{-2}a^2 = 1\rangle$
$H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}_4$ $G_{260}/\zeta = D_6$ $\phi_{260} : a \mapsto G_{19}.a,\ b \mapsto G_{19}.b$

261. `( 4, 6,10, 3, 5, 3)` $G_{261} = \langle a, b \mid a^5 = b^4 = (a^2b)^2\rangle$
$H_1 = \mathbb{Z}_6$ $\zeta = \mathbb{Z}$ $G_{261}/\zeta = \Delta^+(5, 4, 2)$ $\phi_{261} : a \mapsto G_{310}.a,\ b \mapsto G_{310}.b^3G_{310}.a^{-4}$
representative

262. `( 4, 6,10, 3, 5,15)` $G_{262} = \langle a, b \mid ab^{-2}ab, b^{-1}a^4b^{-1}a^{-1}\rangle$
$H_1 = 1$ $\zeta = Z_2$ $G_{262}/\zeta = A_5$ $\phi_{262} : a \mapsto G_{55}.a, b \mapsto G_{55}.b$

263. `( 4, 6,10, 3, 9,15)` $G_{263} = \langle a, b \mid a^5 = b^4 = (a^{-1}b)^2\rangle$
$H_1 = \mathbb{Z}_{18}$ $\zeta = \mathbb{Z}$ $G_{263}/\zeta = \Delta^+(5, 4, 2)$ —
representative

264. `( 4, 6,10, 5, 1, 1)` $G_{264} = \langle a, b \mid a^4 = b^3 = (ab^{-2})^2\rangle$
$H_1 = \mathbb{Z}_{22}$ $\zeta = \mathbb{Z}_{22}$ $G_{264}/\zeta = S_4$ —
representative; GAP code for the group is `[528,87]`

265. `( 4, 6,10, 5, 3,15)` $G_{265} = \langle a, b \mid ab^{-2}ab = 1, ab^{-1}a^{-1}b^{-1}a^2 = 1\rangle$
$H_1 = \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{265}/\zeta = S_4$ $\phi_{265} : a \mapsto G_{51}.a,\ b \mapsto G_{51}.b$

266. `( 4, 6,10, 5, 5, 1)` $G_{266} = \langle a, b \mid aba^{-1}ba = 1, ab^{-3}a^2 = 1\rangle$
$H_1 = \mathbb{Z}_9$ $\zeta = \mathbb{Z}_6$ $G_{266}/\zeta = A_4$ $\phi_{266} : a^{-13} \mapsto G_{49}.a,\ a^{44}ba^{-13} \mapsto G_{49}.b$

267. `( 4, 6,10, 5, 5,13)` $G_{267} = \langle a, b \mid a^6 = b^4 = (ab)^2\rangle$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}$ $G_{267}/\zeta = \Delta^+(6, 4, 2)$ —
representative

268. `( 4, 6,10, 5, 7,13)` $G_{268} = \langle a, b \mid ab^{-1}ab = 1, b^{-1}a^{-1}b^{-1}a^4 = 1\rangle$
$H_1 = \mathbb{Z}_4$ $\zeta = \mathbb{Z}_2$ $G_{268}/\zeta = D_{10}$ $\phi_{268} : a \mapsto G_{47}.a,\ b \mapsto G_{47}.b$

269. `( 4, 6,10, 5, 9, 3)` $G_{269} = \langle a, b \mid a^3 = b^3 = (a^2b)^3\rangle$
$H_1 = \mathbb{Z}_6 \times \mathbb{Z}_3$ $\zeta = \mathbb{Z}$ $G_{269}/\zeta = \Delta^+(3, 3, 3)$ —
representative

270. `( 4, 6,10, 7, 1, 1)` $G_{270} = \langle a, b \mid a^3 = b^3 = (a^2b)^{-2}\rangle$
$H_1 = \mathbb{Z}_{27}$ $\zeta = \mathbb{Z}_{18}$ $G_{270}/\zeta = A_4$ —
representative; GAP code for the group is `[216,3]`

271. `( 4, 6,10, 7, 3,15)` $G_{271} = \langle a, b \mid a^5 = b^3 = (a^2b^{-1})^2\rangle$
$H_1 = \mathbb{Z}_{13}$ $\zeta = \mathbb{Z}_{26}$ $G_{271}/\zeta = A_5$ —
representative; GAP code for the group is `[1560,13]`

272. `( 4, 6,10, 7, 5, 3)` $G_{272} = \langle a, b \mid ab^{-2}ab = 1, ab^{-1}a^{-1}b^{-1}a = 1\rangle$
$H_1 = \mathbb{Z}_3$ $\zeta = \mathbb{Z}_2$ $G_{272}/\zeta = A_4$ $\phi_{272} : a \mapsto G_{13}.a,\ b \mapsto G_{13}.b$

273. `( 4, 8, 8, 1, 1, 1)` $G_{273} = \langle a, b \mid a^4 = b^4 = (ab)^{-2}\rangle$
$H_1 = \mathbb{Z}_{16} \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}$ $G_{273}/\zeta = \Delta^+(4, 4, 2)$ —
representative

274. `( 4, 8, 8, 1, 1,13)` $G_{274} = \langle a, b \mid a^4 = b^4 = (ab)^3\rangle$
$H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}$ $G_{274}/\zeta = \Delta^+(4, 4, 3)$ —
representative

275. `( 4, 8, 8, 1, 7, 3)` $G_{275} = \langle a, b \mid ab^{-3}a^2 = 1, ab^{-1}a^{-2}b^{-1}a^{-2}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_6 \times \mathbb{Z}_3$ $\zeta = \mathbb{Z}$ $G_{275}/\zeta = \Delta^+(3, 3, 3)$ $\phi_{275} : a^{-1} \mapsto G_{269}.a,\ b{-}1 \mapsto G_{269}.b$

276. `( 4, 8, 8, 1, 9, 1)` $G_{276} = \langle a, b \mid a^5 = b^4 = (ab^{-1})^2\rangle$
$H_1 = \mathbb{Z}_{22}$ $\zeta = \mathbb{Z}$ $G_{276}/\zeta = \Delta^+(5, 4, 2)$ —
representative

277. `( 4, 8, 8, 1, 9,13)` $G_{277} = \langle a, b \mid ab^{-2}ab = 1, a^{-4}b^{-3}a^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_{11}$ $\zeta = \mathbb{Z}_{22}$ $G_{277}/\zeta = A_5$ $\phi_{277} : a \mapsto G_{130}.a,\ ba \mapsto G_{130}.b$

278. `( 4, 8, 8, 3, 3, 1)` $G_{278} = \langle a, b \mid a^2 = b^2 = (a^3b^3)^2\rangle$
$H_1 = \mathbb{Z}_{10} \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_{10}$ $G_{278}/\zeta = D_4$ —

279. `( 4, 8, 8, 3, 5, 1)` $G_{279} = \mathbb{Z}_2 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{279}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_4$ —
connected sum

280. `( 4, 8, 8, 3, 7,15)` $G_{280} = \mathbb{Z}_2 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{280}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_5$ —
connected sum

281. `( 4, 8, 8, 5, 5, 1)` $G_{281} = \mathbb{Z}_2 * \mathbb{Z}_2$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = 1$ $G_{281}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_2$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(2,1)$

282. `( 4, 8, 8, 5, 5,13)` $G_{282} = \langle a, b \mid a^5 = b^5 = (ab)^2 \rangle$
$H_1 = \mathbb{Z}_5$ $\zeta = \mathbb{Z}$ $G_{282}/\zeta = \Delta^+(5,5,2)$ — representative

283. `( 4, 8, 8, 5, 7, 1)` $G_{283} = \mathbb{Z}_2 * \mathbb{Z}_5$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_5$ $\zeta = 1$ $G_{283}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_5$ —
connected sum

284. `( 4, 8, 8, 5, 7, 3)` $G_{284} = \langle a, b \mid ab^{-3}a^2 = 1, a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^2 = 1 \rangle$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$ $\zeta = \mathbb{Z}$ $G_{284}/\zeta = \Delta^+(3,3,3)$ $\phi_{284} : a^{-1} \mapsto G_{69}.a,\ b^{-1} \mapsto G_{69}.b$

285. `( 6, 6, 8, 1, 1, 1)` $G_{285} = \langle a, b \mid a^4 = b^3 = (ab)^{-3} \rangle$
$H_1 = \mathbb{Z}_{33}$ $\zeta = \mathbb{Z}$ $G_{285}/\zeta = \Delta^+(4,3,3)$ —
representative

286. `( 6, 6, 8, 1, 1, 3)` $G_{286} = \langle a, b \mid aba^2b^2a = 1, a^3b^{-1}ab^{-1}ab^{-1} = 1 \rangle$
$H_1 = \mathbb{Z}_{27}$ $\zeta = \mathbb{Z}_{18}$ $G_{286}/\zeta = A_4$ $\phi_{286} : b^{-1}a^{377} \mapsto G_{270}.a,\ b^{-1}a^{398} \mapsto G_{270}.b$

287. `( 6, 6, 8, 1, 1, 5)` $G_{287} = \mathbb{Z}_{25}$
$H_1 = \mathbb{Z}_{25}$ $\zeta = \mathbb{Z}_{25}$ $G_{287}/\zeta = 1$ —
lens space $\mathcal{L}(25, ?)$

288. `( 6, 6, 8, 1, 1, 7)` $G_{288} = \mathbb{Z}_{24}$
$H_1 = \mathbb{Z}_{24}$ $\zeta = \mathbb{Z}_{24}$ $G_{288}/\zeta = 1$ —
lens space $\mathcal{L}(24, ?)$

289. `( 6, 6, 8, 1, 1, 9)` $G_{289} = \langle a, b \mid a^2b^2a^2b = 1, ab^{-1}a^{-1}b^{-1}ab^{-1}ab^{-1} = 1 \rangle$
$H_1 = \mathbb{Z}_{22}$ $\zeta = \mathbb{Z}_{22}$ $G_{289}/\zeta = S_4$ $\phi_{289} : ?$
group is isomorphic to `[528,87]`

290. `( 6, 6, 8, 1, 1,11)` $G_{290} = \langle a, b \mid a^4 = b^4 = (a^{-1}b)^3 \rangle$
$H_1 = \mathbb{Z}_{16}$ $\zeta = \mathbb{Z}$ $G_{290}/\zeta = \Delta^+(4,4,3)$ —
representative

291. `( 6, 6, 8, 1, 3, 7)` $G_{291} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{291}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(3,1)$

292. `( 6, 6, 8, 1, 3, 9)` $G_{292} = \langle a, b \mid ab^{-1}a^{-2}b^{-1}ab^{-1} = 1, ab^{-1}a^{-1}b^{-1} = 1,$
$ab^{-1}a^{-2}b^{-2}a^{-2}b^{-1}ab^{-1}abab^{-1} = 1 \rangle$
$H_1 = \mathbb{Z}$ $\zeta = \mathbb{Z}$ $G_{292}/\zeta = 1$ $\phi_{292} : a \mapsto G_{98}.a, b \mapsto G_{98}.b$
note that the group is two-generated, but $a = b$!

293. `( 6, 6, 8, 1, 5, 5)` $G_{293} = \mathbb{Z}_{13}$
$H_1 = \mathbb{Z}_{13}$ $\zeta =$ $G_{293}/\zeta = \mathbb{Z}_{13}$ —
lens space $\mathcal{L}(13, ?)$

294. `( 6, 6, 8, 1, 5,11)` $G_{294} = \langle a, b \mid ba^{-1}b^{-1}a^{-1}b = 1, a^2b^{-1}a^{-1}b^{-1} = 1 \rangle$
$H_1 = \mathbb{Z}_3$ $\zeta = \mathbb{Z}_2$ $G_{294}/\zeta = A_4$ $\phi_{294} : a \mapsto G_{13}.a,\ b \mapsto G_{13}.b$

295. `( 6, 6, 8, 1, 7, 7)` $G_{295} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{295}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(3,1)$

296. `( 6, 6, 8, 1, 9, 1)` $G_{296} = \langle a, b \mid a^5 = b^3 = (ab^{-1})^3 \rangle$
$H_1 = \mathbb{Z}_{21}$ $\zeta = \mathbb{Z}$ $G_{296}/\zeta = \Delta^+(5,3,3)$ —
representative

297. `( 6, 6, 8, 1, 9, 7)` $G_{297} = \langle a, b \mid ab^{-1}ab^{-1}ab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1} = 1,$
$ab^{-1}ab^{-1}a^{-2}b^{-1} = 1, a^2b^{-1}a^{-1}b^{-1}a^{-1}b^{-1} = 1 \rangle$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}$ $\zeta =?$ $G_{297}/\zeta =?$ $\phi_{297} : a \mapsto G_{112}.a,\ b \mapsto G_{112}.b$

298. `( 6, 6, 8, 1, 9, 9)` $G_{298} = \mathbb{Z}_{13}$
$H_1 = \mathbb{Z}_{13}$ $\zeta = \mathbb{Z}_{13}$ $G_{298}/\zeta = 1$ —
lens space $\mathcal{L}(13, ?)$

299. `( 6, 6, 8, 3, 1, 9)` $G_{299} = \langle a, b \mid a^3b^2 = 1, abab^{-1}ab^{-1}abab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_{13}$ $\zeta = \mathbb{Z}_{26}$ $G_{299}/\zeta = A_5$ $\phi_{299} : ?$
group is isomorphic to `[1560,13]`

300. `( 6, 6, 8, 3, 3, 9)` $G_{300} = \langle a, b \mid ab^{-1}ab^2 = 1, a^{-1}b^{-1}ab^{-1}a^{-2} = 1\rangle$
$H_1 = \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{300}/\zeta = S_4$ $\phi_{300} : a \mapsto G_{51}.a,\ b^{-1} \mapsto G_{51}.b$

301. `( 6, 6, 8, 3, 5, 7)` $G_{301} = \langle a, b \mid b^{-1}aba = 1, b^{-1}a^{-3}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_4$ $\zeta = \mathbb{Z}_2$ $G_{301}/\zeta = D_6$ $\phi_{301} : a^{-1} \mapsto G_9.a,\ b \mapsto G_9.b$

302. `( 6, 6, 8, 3, 7, 3)` $G_{302} = \langle a, b \mid b^2a^2 = 1, ab^{-1}a^{-1}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{302}/\zeta = D_4$ $\phi_{302} : a \mapsto G_4.a,\ b^{-1} \mapsto G_4.b$

303. `( 6, 6, 8, 3, 7, 7)` $G_{303} = \langle a, b \mid aba^{-2}ba = 1, ab^{-1}a^{-1}bababa^{-1}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_2$ $\zeta = \mathbb{Z}$ $G_{303}/\zeta = \Delta^+(5, 4, 2)$ $\phi_{303} : ab \mapsto G_{150}.a,\ a \mapsto G_{150}.b$

304. `( 6, 6, 8, 3,11, 3)` $G_{304} = \langle a, b \mid ab^{-1}a^{-1}b^{-1}a = 1, a^2b^5a = 1\rangle$
$H_1 = \mathbb{Z}_{11}$ $\zeta = \mathbb{Z}_{22}$ $G_{304}/\zeta = A_5$ $\phi_{304} : ?$
the group is isomorphic to `[1320,14]` [16]

305. `( 6, 6, 8, 3,11, 5)` $G_{305} = \mathbb{Z}_{18}$
$H_1 = \mathbb{Z}_{18}$ $\zeta = \mathbb{Z}_{18}$ $G_{305}/\zeta = 1$ —
lens space $\mathcal{L}(18, ?)$

306. `( 6, 6, 8, 3,11, 7)` $G_{306} = \mathbb{Z}_{12}$
$H_1 = \mathbb{Z}_{12}$ $\zeta = \mathbb{Z}_{12}$ $G_{306}/\zeta = 1$ —
lens space $\mathcal{L}(12, ?)$

307. `( 6, 6, 8, 3,11, 9)` $G_{307} = \mathbb{Z}_5$
$H_1 = \mathbb{Z}_5$ $\zeta = \mathbb{Z}_5$ $G_{307}/\zeta = 1$ —
lens space $\mathcal{L}(5, ?)$

308. `( 6, 6, 8, 5, 1, 7)` $G_{308} = \langle a, b \mid aba^{-1}b = 1, ba^2bab^{-1}abab^{-1}a = 1\rangle$
$H_1 = \mathbb{Z}_{12}$ $\zeta = \mathbb{Z}_6$ $G_{308}/\zeta = D_{10}$ $\phi_{308} : a^{-8}b \mapsto G_{415}.b,\ a^{-5}b \mapsto G_{415}.b$

309. `( 6, 6, 8, 5, 3, 5)` $G_{309} = \langle a, b \mid b^2a^2 = 1, ab^{-1}a^{-1}b^{-1}\rangle = 1$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{309}/\zeta = D_4$ $\phi_{309} : a \mapsto G_4.a,\ b^{-1} \mapsto G_4.b$

310. `( 6, 6, 8, 5, 3, 7)` $G_{310} = \langle a, b \mid a^8 = b^5, a^4 = (ab)^2\rangle$
$H_1 = \mathbb{Z}_6$ $\zeta = \mathbb{Z}$ $G_{310}/\zeta = \Delta^+(5.4.2)$ $\phi_{310} : a \mapsto G_{261}.a,\ b \mapsto G_{261}.b^2$

311. `( 6, 6, 8, 5, 5, 7)` $G_{311} = \langle a, b \mid a^3 = b^{-1}ab^2ab^{-1}, ab^{-1}ab = b^{-1}a^{-1}ba^{-1}\rangle$
$H_1 = \mathbb{Z}$ $\zeta\ ?$ $G_{311}/\zeta\ ?$ —
representative

312. `( 6, 6, 8, 5,11, 7)` $G_{312} = \langle a, b \mid ab^{-2}a^2 = b^2a^{-2}b, ba^{-2}b = a^{-1}b^2a^{-1}\rangle$
$H_1 = \mathbb{Z}_5 \times \mathbb{Z}$ $\zeta\ ?$ $G_{312}/\zeta\ ?$ —
representative

313. `( 1, 1,19, 2, 0, 2)` $G_{313} = \mathbb{Z}_5 * \mathbb{Z}$
$H_1 = \mathbb{Z}_5 \times \mathbb{Z}$ $\zeta = 1$ $G_{313}/\zeta = \mathbb{Z}_5 * \mathbb{Z}$ —
connected sum; representative

314. `( 1, 1,19, 2, 0, 6)` $G_{314} = \langle a, b \mid a^5b^{-1}a^5b = 1, a^5 = 1\rangle$
$H_1 = \mathbb{Z}_5 \times \mathbb{Z}$ $\zeta = 1$ $G_{314}/\zeta = \mathbb{Z}_5 * \mathbb{Z}$ $\phi_{314} : a \mapsto G_{313}.a,\ b \mapsto G_{313}.b$

315. `( 1, 3,17, 2, 2, 0)` $G_{315} = \mathbb{Z}_2 * \mathbb{Z}_9$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_9$ $\zeta = 1$ $G_{315}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_9$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(9,1)$; representative

316. `( 1, 3,17, 4, 2, 0)` $G_{316} = \mathbb{Z}_2 * \mathbb{Z}_9$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_9$ $\zeta = 1$ $G_{316}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_9$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(9,2)$; representative

317. `( 1, 3,17, 8, 2, 0)` $G_{317} = \mathbb{Z}_2 * \mathbb{Z}_9$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_9$ $\zeta = 1$ $G_{317}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_9$ —
connected sum $\mathcal{L}(2,1)\#\mathcal{L}(9,4)$; representative

318. `( 1, 5,15, 2, 2, 0)` $G_{318} = \mathbb{Z}_3 * \mathbb{Z}_8$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_8$ $\zeta = 1$ $G_{318}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_8$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(8,1)$; representative

319. `( 1, 5,15, 2, 4, 0)` $G_{319} = \mathbb{Z}_3 * \mathbb{Z}_8$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_8$ $\zeta = 1$ $G_{319}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_8$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(8,1)$

320. `( 1, 5,15, 6, 2, 0)` $G_{320} = \mathbb{Z}_3 * \mathbb{Z}_8$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_8$ $\zeta = 1$ $G_{320}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_8$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(8,3)$; representative

321. `( 1, 5,15, 6, 4, 0)` $G_{321} = \mathbb{Z}_3 * \mathbb{Z}_8$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_8$ $\zeta = 1$ $G_{321}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_8$ —
connected sum $\mathcal{L}(3,1)\#\mathcal{L}(8,3)$

322. `( 1, 7,13, 2, 2, 0)` $G_{322} = \mathbb{Z}_4 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{322}/\zeta = \mathbb{Z}_4 * \mathbb{Z}_7$ —
connected sum $\mathcal{L}(4,1)\#\mathcal{L}(7,1)$; representative

323. `( 1, 7,13, 2, 6, 0)` $G_{323} = \mathbb{Z}_4 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{323}/\zeta = \mathbb{Z}_4 * \mathbb{Z}_7$ —
connected sum $\mathcal{L}(4,1)\#\mathcal{L}(7,1)$

324. `( 1, 7,13, 4, 2, 0)` $G_{324} = \mathbb{Z}_4 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{324}/\zeta = \mathbb{Z}_4 * \mathbb{Z}_7$ —
connected sum $\mathcal{L}(4,1)\#\mathcal{L}(7,2)$

325. `( 1, 7,13, 4, 6, 0)` $G_{325} = \mathbb{Z}_4 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{325}/\zeta = \mathbb{Z}_4 * \mathbb{Z}_7$ —
connected sum $\mathcal{L}(4,1)\#\mathcal{L}(7,2)$

326. `( 1, 7,13, 6, 2, 0)` $G_{326} = \mathbb{Z}_4 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{326}/\zeta = \mathbb{Z}_4 * \mathbb{Z}_7$ —
connected sum $\mathcal{L}(4,1)\#\mathcal{L}(7,3)$; representative

327. `( 1, 7,13, 6, 6, 0)` $G_{327} = \mathbb{Z}_4 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{327}/\zeta = \mathbb{Z}_4 * \mathbb{Z}_7$ —
connected sum $\mathcal{L}(4,1)\#\mathcal{L}(7,3)$

328. `( 1, 9,11, 2, 2, 0)` $G_{328} = \mathbb{Z}_5 * \mathbb{Z}_6$
$H_1 = \mathbb{Z}_5 \times \mathbb{Z}_6$ $\zeta = 1$ $G_{328}/\zeta = \mathbb{Z}_5 * \mathbb{Z}_6$ —
connected sum $\mathcal{L}(5,1)\#\mathcal{L}(6,1)$; representative

329. `( 1, 9,11, 2, 4, 0)` $G_{329} = \mathbb{Z}_5 * \mathbb{Z}_6$
$H_1 = \mathbb{Z}_5 \times \mathbb{Z}_6$ $\zeta = 1$ $G_{329}/\zeta = \mathbb{Z}_5 * \mathbb{Z}_6$ —
connected sum $\mathcal{L}(5,2)\#\mathcal{L}(6,1)$; representative

330. `( 1, 9,11, 2, 6, 0)` $G_{330} = \mathbb{Z}_5 * \mathbb{Z}_6$
$H_1 = \mathbb{Z}_5 \times \mathbb{Z}_6$ $\zeta = 1$ $G_{330}/\zeta = \mathbb{Z}_5 * \mathbb{Z}_6$ —
connected sum $\mathcal{L}(5,3)\#\mathcal{L}(6,1)$; representative

331. `( 1, 9,11, 2, 8, 0)` $G_{331} = \mathbb{Z}_5 * \mathbb{Z}_6$
$H_1 = \mathbb{Z}_5 \times \mathbb{Z}_6$ $\zeta = 1$ $G_{331}/\zeta = \mathbb{Z}_5 * \mathbb{Z}_6$ —
connected sum $\mathcal{L}(5,1)\#\mathcal{L}(6,1)$

332. `( 3, 3,15, 2, 0, 4)` $G_{332} = \langle a, b \mid ab^{-1}a^3ba^2ba^3b^{-1} = 1, a^3b^{-1}a^3b = 1, a^3 = 1\rangle$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}$ $\zeta = 1$ $G_{332}/\zeta = \mathbb{Z}_3 * \mathbb{Z}$ $\phi_{332} : a \mapsto G_{14}.a,\ b \mapsto G_{14}.b$

333. `( 3, 3,15, 2, 2, 2)` $G_{333} = \langle a, b \mid ab^{-3}ab^{-1} = 1, ba^{-5}b = 1\rangle$
$H_1 = \mathbb{Z}_{16}$ $\zeta = \mathbb{Z}_8$ $G_{333}/\zeta = D_{10}$ $\phi_{333} : a \mapsto G_{64}.a,\ b \mapsto G_{64}.b$

334. `( 3, 3,15, 2, 2, 6)` $G_{334} = \langle a, b \mid a^7 = (a^4b)^2, b^2 = (a^2b)^2\rangle$
$H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}_4$ $G_{334}/\zeta = D_{14}$ —
representative; GAP code for the group is `[56,1]`

335. `( 3, 3,15, 2, 2,14)` $G_{335} = \langle a, b \mid a^8 = b^2 = (ab)^2\rangle$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{335}/\zeta = D_{16}$ —
representative; GAP code for the group is `[32,20]`

336. `( 3, 5,13, 2, 0, 2)` $G_{336} = \langle a, b \mid ab^3a^2 = 1, ab^{-4}ab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_{21}$ $\zeta = \mathbb{Z}_{14}$ $G_{336}/\zeta = A_4$ $\phi_{336} : a^{-1} \mapsto G_{68}.a,\ b \mapsto G_{68}.b$

337. `( 3, 5,13, 2, 4, 0)` $G_{337} = \mathbb{Z}_2 * \mathbb{Z}_8$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_8$ $\zeta = 1$ $G_{337}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_8$ —
connected sum

338. `( 3, 5,13, 4, 2, 0)` $G_{338} = \mathbb{Z}_4 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{338}/\zeta = \mathbb{Z}_4 * \mathbb{Z}_4$ —
connected sum

339. `( 3, 5,13, 4, 4,16)` $G_{339} = \langle a, b \mid a^7 = b^3 = (ab)^2\rangle$
$H_1 = 1$ $\zeta = \mathbb{Z}$ $G_{339}/\zeta = \Delta^+(7,3,2)$ —

340. `( 3, 5,13, 4, 6, 0)` $G_{340} = \mathbb{Z}_4 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{340}/\zeta = \mathbb{Z}_4 * \mathbb{Z}_4$ —
connected sum

341. `( 3, 5,13, 6, 4, 0)` $G_{341} = \mathbb{Z}_2 * \mathbb{Z}_8$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_8$ $\zeta = 1$ $G_{341}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_8$ —
connected sum

342. `( 3, 5,13, 8, 4, 2)` $G_{342} = \langle a, b \mid ab^{-1}a^{-3}b^{-1}a = 1, ab^{-3}a^4 = 1\rangle$
$H_1 = \mathbb{Z}_{13}$ $\zeta = \mathbb{Z}_{26}$ $G_{342}/\zeta = A_5$ $\phi_{342} : a \mapsto G_{271}.a,\ b \mapsto G_{271}.b$

343. `( 3, 7,11, 2, 2, 2)` $G_{343} = \langle a, b \mid ab^{-3}a^3 = 1, ab^2a^{-1}b^2a^2 = 1\rangle$
$H_1 = \mathbb{Z}_{22}$ $\zeta = \mathbb{Z}_{22}$ $G_{343}/\zeta = S_4$ $\phi_{343} : a \mapsto G_{264}.a,\ ba^{-88} \mapsto G_{264}.b$

344. `( 3, 7,11, 2, 6, 2)` $G_{344} = \langle a, b \mid ab^{-1}a^{-3}b^{-1} = 1, ab^{-5}ab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_8 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}$ $G_{344}/\zeta = \Delta^+(4,4,2)$ $\phi_{344} : a \mapsto G_{146}.a,\ b \mapsto G_{146}.b$

345. `( 3, 7,11, 4, 2, 2)` $G_{345} = \langle a, b \mid a^5 = b^3 = (a^2b^2)^2\rangle$
$H_1 = \mathbb{Z}_{17}$ $\zeta = \mathbb{Z}_{34}$ $G_{345}/\zeta = A_5$ —
representative; fundamental group is isomorphic to the group of size 2040,
which is an extension of $\mathbb{Z}_{17} \times A_5$

346. `( 3, 7,11, 4, 4, 0)` $G_{346} = \mathbb{Z}_6$
$H_1 = \mathbb{Z}_6$ $\zeta = \mathbb{Z}_6$ $G_{346}/\zeta = 1$ —
lens space $\mathcal{L}(6, ?)$

347. `( 3, 7,11, 4, 4,16)` $G_{347} = \langle a, b \mid ab^{-3}ab = 1, b^{-1}a^5b^{-1}a^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}$ $G_{347}/\zeta = \Delta^+(6, 4, 2)$ $\phi_{347} : a \mapsto G_{267}.a,\ b \mapsto G_{267}.b$

348. `( 3, 7,11, 4, 6, 0)` $G_{348} = 1$
$H_1 = 1$ $\zeta = 1$ $G_{348}/\zeta = 1$ —
homotopy sphere S^3, representative

349. `( 3, 7,11, 4, 6, 2)` $G_{349} = \langle a, b \mid a^3 = ba^2b, b^3 = a^2ba^2\rangle$
$H_1 = \mathbb{Z}_6$ $\zeta = \mathbb{Z}$ $G_{349}/\zeta = \Delta^+(5, 4, 2)$ $\phi_{349} : a \mapsto G_{261}.a,\ b \mapsto G_{261}.b$

350. `( 3, 7,11, 4, 8,16)` $G_{350} = \langle a, b \mid a^6 = b^3 = (a^{-1}b)^2\rangle$
$H_1 = \mathbb{Z}_{12}$ $\zeta = \mathbb{Z}$ $G_{350}/\zeta = \Delta^+(6, 3, 2)$ —

351. `( 3, 7,11, 6, 2, 2)` $G_{351} = \mathbb{Z}_{13}$
$H_1 = \mathbb{Z}_{13}$ $\zeta = \mathbb{Z}_{13}$ $G_{351}/\zeta = 1$ —
lens space $\mathcal{L}(13, ?)$

352. `( 3, 7,11, 6, 4, 0)` $G_{352} = \mathbb{Z}_{11}$
$H_1 = \mathbb{Z}_{11}$ $\zeta = \mathbb{Z}_{11}$ $G_{352}/\zeta = 1$ —
lens space $\mathcal{L}(11, ?)$

353. `( 3, 7,11, 6, 6, 0)` $G_{353} = \mathbb{Z}_4$
$H_1 = \mathbb{Z}_4$ $\zeta = \mathbb{Z}_4$ $G_{353}/\zeta = 1$ —
lens space $\mathcal{L}(4, ?)$

354. `( 3, 9, 9, 0, 2, 2)` $G_{354} = \langle a, b \mid ab^{-3}ab^{-1} = 1, ab^2a^4 = 1\rangle$
$H_1 = \mathbb{Z}_{24}$ $\zeta = \mathbb{Z}_{12}$ $G_{354}/\zeta = D_{10}$ $\phi_{354} : a \mapsto G_{125}.a,\ b \mapsto G_{125}.b$

355. `( 3, 9, 9, 0, 4, 2)` $G_{355} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{355}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2, 1)\#\mathcal{L}(3, 1)$

356. `( 3, 9, 9, 2, 4, 0)` $G_{356} = \mathbb{Z}_3 * \mathbb{Z}_6$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_6$ $\zeta = 1$ $G_{356}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_6$ —
connected sum

357. `( 3, 9, 9, 2, 8, 0)` $G_{357} = \mathbb{Z}_3 * \mathbb{Z}_6$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_6$ $\zeta = 1$ $G_{357}/\zeta = \mathbb{Z}_3 * \mathbb{Z}_6$ —
connected sum

358. `( 3, 9, 9, 4, 4, 2)` $G_{358} = \mathbb{Z}_2 * \mathbb{Z}_2$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = 1$ $G_{358}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_2$ —
connected sum $\mathcal{L}(2, 1)\#\mathcal{L}(2, 1)$

359. `( 3, 9, 9, 4, 4,16)` $G_{359} = \langle a, b \mid ab^{-4}ab = 1, ab^{-1}a^{-1}b^{-1}a^3 = 1\rangle$
$H_1 = \mathbb{Z}_5$ $\zeta = \mathbb{Z}$ $G_{359}/\zeta = \Delta^+(5, 5, 2)$ $\phi_{359} : a \mapsto G_{282}.a,\ b \mapsto G_{282}.b$

360. `( 3, 9, 9, 4, 6, 2)` $G_{360}\ \mathbb{Z}_2 * \mathbb{Z}_2 =$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\zeta = 1$ $G_{360}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_2$ —
connected sum $\mathcal{L}(2, 1)\#\mathcal{L}(2, 1)$

361. `( 3, 9, 9, 4, 8, 2)` $G_{361} = \mathbb{Z}_2 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{361}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_4$ —
connected sum

362. `( 3, 9, 9, 6, 6, 2)` $G_{362} = \mathbb{Z}_5$
$H_1 = \mathbb{Z}_5$ $\zeta = \mathbb{Z}_5$ $G_{362}/\zeta = 1$ —
lens space $\mathcal{L}(5, ?)$

363. `( 5, 5,11, 0, 2, 4)` $G_{363} = \mathbb{Z}_{19}$
$H_1 = \mathbb{Z}_{19}$ $\zeta = \mathbb{Z}_{19}$ $G_{363}/\zeta = 1$ —
lens space $\mathcal{L}(19, ?)$

364. `( 5, 5,11, 0, 2,12)` $G_{364} = \langle a, b \mid a^3b^2 = 1, abab^{-1}ab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_9$ $\zeta = \mathbb{Z}_6$ $G_{364}/\zeta = A_4$ $\phi_{364} : a^{-1}b^4a^{-3}a \mapsto G_{49}.a,\ a^{-1}b^5 \mapsto G_{49}.b$

365. `( 5, 5,11, 0, 4,10)` $G_{365} = \mathbb{Z}_9$
$H_1 = \mathbb{Z}_9$ $\zeta = \mathbb{Z}_9$ $G_{365}/\zeta = 1$ —
lens space $\mathcal{L}(9, ?)$

366. `( 5, 5,11, 0, 4,14)` $G_{366} = \mathbb{Z}_{19}$
$H_1 = \mathbb{Z}_{19}$ $\zeta = \mathbb{Z}_{19}$ $G_{366}/\zeta = 1$ —
lens space $\mathcal{L}(19, ?)$

367. `( 5, 5,11, 2, 0, 2)` $G_{367} = \langle a, b \mid ab^2a^2 = 1, ab^{-3}ab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_{16}$ $\zeta = \mathbb{Z}_8$ $G_{367}/\zeta = D_6$ $\phi_{367} : a \mapsto G_{26}.a,\ ba^{-1} \mapsto G_{26}.b$

368. `( 5, 5,11, 2, 0, 6)` $G_{368} = \langle a, b \mid ab^{-1}a^2bababa^2b^{-1}ab^{-1} = 1, ab^{-1}a^2baba^2b^{-1} = 1, a^2 = 1\rangle$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}$ $\zeta = 1$ $G_{368}/\zeta = \mathbb{Z}_2 * \mathbb{Z}$ $\phi_{368} : a \mapsto G_2.a,\ b \mapsto G_2.b$

369. `( 5, 5,11, 2, 0, 8)` $G_{369} = \mathbb{Z}_{24}$
$H_1 = \mathbb{Z}_{24}$ $\zeta = \mathbb{Z}_{24}$ $G_{369}/\zeta = 1$ —
lens space $\mathcal{L}(24, ?)$

370. `( 5, 5,11, 2, 0,10)` $G_{370} = \mathbb{Z}_{16}$
$H_1 = \mathbb{Z}_{16}$ $\zeta = \mathbb{Z}_{16}$ $G_{370}/\zeta = 1$ —
lens space $\mathcal{L}(16, ?)$

371. `( 5, 5,11, 2, 0,12)` $G_{371} = \langle a, b \mid a^4b^2 = 1, ab^{-1}ab^{-1}abab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_8 \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}$ $G_{371}/\zeta = \Delta^+(4, 4, 2)$ $\phi_{371} : a \mapsto G_{146}.a,\ ab^{-1} \mapsto G_{146}.b$

372. `( 5, 5,11, 2, 2, 8)` $G_{372} = \langle a, b \mid ab^{-1}a^3ba^2 = 1, ab^{-1}a^{-1}b^{-1}a^{-2}b^{-1}a^{-1}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_{24}$ $\zeta\ ?$ $G_{372}/\zeta\ ?$ $\phi_{372} : ab \mapsto G_{209}.a,\ a \mapsto G_{209}.b$

373. `( 5, 5,11, 2, 2,12)` $G_{373} = \langle a, b \mid ab^{-1}ab = 1, b^{-1}a^{-5}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_4$ $\zeta = \mathbb{Z}_2$ $G_{373}/\zeta = D_{10}$ $\phi_{373} : a \mapsto G_{47}.a,\ b \mapsto G_{47}.b$

374. `( 5, 5,11, 2, 4, 2)` $G_{374} = \langle a, b \mid b = a^3ba^3, a^4 = b^2a^{-1}b^2\rangle$
$H_1 = \mathbb{Z}_{24}$ $\zeta\ ?$ $G_{374}/\zeta\ ?$ —
representative

375. `( 5, 5,11, 2, 4, 6)` $G_{375} = \mathbb{Z}_2 * \mathbb{Z}_3$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $\zeta = 1$ $G_{375}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_3$ —
connected sum $\mathcal{L}(2, 1)\#\mathcal{L}(3, 1)$

376. `( 5, 5,11, 2, 4,10)` $G_{376} = \langle a, b \mid ab^{-1}ab = 1, b^{-1}a^{-4}b^{-2}a^{-3}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}_4$ $G_{376}/\zeta = D_{14}$ $\phi_{376} : bab \mapsto G_{334}.a,\ b \mapsto G_{334}.b$

377. `( 5, 5,11, 2, 6, 2)` $G_{377} = \langle a, b \mid a^4 = b^3 = (ab^{-1})^3\rangle$
$H_1 = \mathbb{Z}_{15}$ $\zeta = \mathbb{Z}$ $G_{377}/\zeta = \Delta^+(4, 3, 3)$ —

378. `( 5, 5,11, 2, 6, 8)` $G_{378} = \langle a, b \mid b^{-1}aba = 1, b^{-1}a^{-3}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_4$ $\zeta = \mathbb{Z}_2$ $G_{378}/\zeta = D_6$ $\phi_{378} : a^{-1} \mapsto G_9.a,\ b \mapsto G_9.b$

379. `( 5, 5,11, 2, 6,10)` $G_{379} = \langle a, b \mid ab^{-1}ab^{-1}a^{-3}b^{-1} = 1, ab^{-2}ab^{-1}a^{-1}ba^{-1}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_3$ $\zeta = \mathbb{Z}$ $G_{379}/\zeta = \Delta^+(4, 3, 3)$ $\phi_{379} : a \mapsto G_{228}.a,\ ab^{-1} \mapsto G_{228}.b$

380. `( 5, 5,11, 4, 0,10)` $G_{380} = \langle a, b \mid a^2b^2 = 1, ab^{-1}a^{-1}b^{-1}ab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}_4$ $G_{380}/\zeta = D_6$ $\phi_{380} : a^{-1}b \mapsto G_{19}.a,\ ab^{-1}a \mapsto G_{19}.b$

381. `( 5, 5,11, 4, 2, 4)` $G_{381} = \langle a, b \mid a, b \mid a^3b^{-1}a^{-2}b^{-1} = 1, ab^{-2}a^2ba = 1\rangle$
$H_1 = \mathbb{Z}_7$ $\zeta = \mathbb{Z}_{14}$ $G_{381}/\zeta = A_5$ $\phi_{381} : a \mapsto G_{135}.a,\ b \mapsto G_{135}.b$

382. `( 5, 5,11, 4, 2, 8)` $G_{382} = \langle a, b \mid ab^{-1}a^{-2}b^{-1}a^{-2}b^{-1} = 1, ab^{-1}a^{-1}b^{-1}a^2b^{-1}a = 1\rangle$
$H_1 = \mathbb{Z}_6 \times \mathbb{Z}_3$ $\zeta = \mathbb{Z}$ $G_{382}/\zeta = \Delta^+(3, 3, 3)$ $\phi_{382} : a \mapsto G_{269}.a,\ ba^2 \mapsto G_{269}b$

383. `( 5, 5,11, 4, 4, 8)` $G_{383} = \langle a, b \mid b^{-1}aba = 1, b^{-1}a^{-3}b^{-2}a^{-4}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}_4$ $G_{383}/\zeta = D_{14}$ $\phi_{383} : bab \mapsto G_{334}.a,\ b \mapsto G_{334}.b$

384. `( 5, 5,11, 4, 4,10)` $G_{384} = \langle a, b \mid ab^{-1}a^{-1}ba^{-1}b^{-1}a = 1, a^3ba^{-1}b^{-1}a^{-1}ba = 1\rangle$
$H_1 = \mathbb{Z}_2$ $\zeta = \mathbb{Z}$ $G_{384}/\zeta = \Delta^+(8, 3, 2)$ $\phi_{384} : a \mapsto G_{413}.a,\ a^{-1}b^{-1}a^3 \mapsto G_{413}.b$
representative

385. `( 5, 5,11, 4, 8, 4)` $G_{385} = \langle a, b \mid a^7 = b^2 = (a^{-2}b)^3\rangle$
$H_1 = \mathbb{Z}_5$ $\zeta = \mathbb{Z}$ $G_{385}/\zeta = \Delta^+(7, 3, 2)$ $\phi_{385} : a \mapsto G_{431}.a^{-4}(G_{431}.aG_{431}.b^{-1})^2,$
$b \mapsto G_{431}.bG_{431}.a^{-1}$
representative; group can be also presented as $\langle a, b \mid a^7 = b^{-6}, b^3 = (a^{-1}b)^2\rangle$

386. `( 5, 5,11, 4, 8, 8)` $G_{386} = \mathbb{Z}_{12}$
$H_1 = \mathbb{Z}_{12}$ $\zeta = \mathbb{Z}_{12}$ $G_{386}/\zeta = 1$ —
lens space $\mathcal{L}(12, ?)$

387. `( 5, 7, 9, 0, 2, 2)` $G_{387} = \langle a, b \mid ab^{-3}ab^{-1} = 1, ab^2a^3 = 1\rangle$
$H_1 = \mathbb{Z}_{10} \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_{10}$ $G_{387}/\zeta = D_8$ $\phi_{387} : a \mapsto G_{61}.a,\ ab^{-31} \mapsto G_{61}.b$

388. `( 5, 7, 9, 0, 2, 4)` $G_{388} = \mathbb{Z}_{26}$
$H_1 = \mathbb{Z}_{26}$ $\zeta = \mathbb{Z}_{26}$ $G_{388}/\zeta = 1$ —
lens space $\mathcal{L}(26, ?)$

389. `( 5, 7, 9, 0, 2,12)` $G_{389} = \langle a, b \mid a^3b^2 = 1, abab^{-1}ab^{-1}ab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_{14}$ $\zeta = \mathbb{Z}_{14}$ $G_{389}/\zeta = S_4$ $\phi_{389} : a^{-1}b^{11}a^{-1}b^8 \mapsto G_{70}.a,\ b^{-11}a^{-1}b^{11} \mapsto G_{70}.b$

390. `( 5, 7, 9, 0, 4,14)` $G_{390} = \langle a, b \mid ab^{-3}ab^{-1} = 1, b^2a^{-6} = 1\rangle$
$H_1 = \mathbb{Z}_{10} \times \mathbb{Z}_2$ $\zeta = \mathbb{Z}_{10}$ $G_{390}/\zeta = D_{12}$ $\phi_{390} : a \mapsto G_{131}.a,\ b \mapsto G_{131}.b$

391. `( 5, 7, 9, 0, 6, 4)` $G_{391} = \mathbb{Z}_2 * \mathbb{Z}_4$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_4$ $\zeta = 1$ $G_{391}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_4$ —
connected sum

392. `( 5, 7, 9, 2, 0, 2)` $G_{392} = \langle a, b \mid ab^4a = 1, ab^{-1}ab^{-1}a^3b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_{26}$ $\zeta = \mathbb{Z}_{26}$ $G_{392}/\zeta = S_4$ $\phi_{392} : b \mapsto G_{139}.a,\ ab^{-1} \mapsto G_{139}.b$

393. `( 5, 7, 9, 2, 0,12)` $G_{393} = \langle a, b \mid ab^{-1}abab^{-1} = 1, a^4b^2 = 1\rangle$
$H_1 = \mathbb{Z}_{10}$ $\zeta = \mathbb{Z}_{10}$ $G_{393}/\zeta = S_4$ $\phi_{393} : a^{-1} \mapsto G_{62}.a,\ b^{-1} \mapsto G_{62}.b$

394. `( 5, 7, 9, 2, 2, 4)` $G_{394} = \langle a, b \mid ab^{-2}a^{-1}b^{-2} = 1, a^2ba^2ba^2b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_{24}$ $\zeta\ ?$ $G_{394}/\zeta\ ?$ $\phi_{394} : a^2b \mapsto G_{209}.a,\ a \mapsto G_{209}.b$

395. `( 5, 7, 9, 2, 2,12)` $G_{395} = \langle a, b \mid ab^{-1}ab^2 = 1, ab^{-1}a^{-3}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_2$ $\zeta = \mathbb{Z}_2$ $G_{395}/\zeta = S_4$ $\phi_{395} : a \mapsto G_{51}.a,\ b^{-1} \mapsto G_{51}.b$

396. `( 5, 7, 9, 2, 4, 4)` $G_{396} = \langle a, b \mid ab^{-1}a^{-2}b^{-1}ab^{-2} = 1, ab^{-1}a^2baba = 1\rangle$
$H_1 = \mathbb{Z}_{20}$ $\zeta\ ?$ $G_{396}/\zeta\ ?$ $\phi_{396} : b^{-1} \mapsto G_{407}.a,\ a \mapsto G_{407}.b$
representative

397. `( 5, 7, 9, 2, 6, 0)` $G_{397} = \mathbb{Z}_2 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_2 * \mathbb{Z}_7$ $\zeta = 1$ $G_{397}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_7$ —
connected sum

398. `( 5, 7, 9, 2, 8, 0)` $G_{398} = \langle a, b \mid ab^{-1}ab^{-3} = 1, ba^{-1}ba^4 = 1\rangle$
$H_1 = \mathbb{Z}_{16}$ $\zeta = \mathbb{Z}_8$ $G_{398}/\zeta = D_{10}$ $\phi_{398} : a \mapsto G_{64}.a,\ b \mapsto G_{64}.b$

399. `( 5, 7, 9, 2, 8, 2)` $G_{399} = \langle a, b \mid ab^{-1}a^{-2}b^{-1} = 1, b^3a^{-3} = 1\rangle$
$H_1 = \mathbb{Z}_9$ $\zeta = \mathbb{Z}_6$ $G_{399}/\zeta = A_4$ $\phi_{399} : a^{11} \mapsto G_{49}.a,\ a^{-10}ba^{11} \mapsto G_{49}.b$

400. `( 5, 7, 9, 4, 0, 2)` $G_{400} = \mathbb{Z}_{17}$
$H_1 = \mathbb{Z}_{17}$ $\zeta = \mathbb{Z}_{17}$ $G_{400}/\zeta = 1$ —
lens space $\mathcal{L}(17, ?)$

401. `( 5, 7, 9, 4, 0,14)` $G_{401} = \langle a, b \mid ab^{-1}a^4b^{-1} = 1, ab^{-1}ab^4 = 1\rangle$
$H_1 = \mathbb{Z}_{19}$ $\zeta = \mathbb{Z}_{38}$ $G_{401}/\zeta = A_5$ $\phi_{401} : a \mapsto G_{143}.a,\ b \mapsto G_{143}.b$

402. `( 5, 7, 9, 4, 2, 2)` $G_{402} = \langle a, b \mid ab^{-3}a^2 = 1, a^{-2}ba^{-2}ba^{-2}b^{-2} = 1\rangle$
$H_1 = \mathbb{Z}_6 \times \mathbb{Z}_3$ $\zeta = \mathbb{Z}$ $G_{402}/\zeta = \Delta^+(3, 3, 3)$ $\phi_{402} : a \mapsto G_{269}.a,\ a^{-2}b \mapsto G_{269}.b$

403. `( 5, 7, 9, 4, 6, 0)` $G_{403} = \mathbb{Z}_2 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{403}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_7$ —
connected sum

404. `( 5, 7, 9, 4, 6, 2)` $G_{404} = \langle a, b \mid ab^{-3}a^2 = 1, ab^{-1}ab^{-1}a^{-2}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$ $\zeta = \mathbb{Z}$ $G_{404}/\zeta = \Delta^+(3, 3, 3)$ $\phi_{404} : a \mapsto G_{69}.a,\ b^{-1} \mapsto G_{69}.b$

405. `( 5, 7, 9, 4, 6, 4)` $G_{405} = \langle a, b \mid ab^{-1}a^{-1}b^{-1} = 1, b^3a^2 = 1\rangle$
$H_1 = \mathbb{Z}_4$ $\zeta = \mathbb{Z}_2$ $G_{405}/\zeta = D_6$ $\phi_{405} : a^{-1} \mapsto G_9.a,\ b \mapsto G_9.b$

406. `( 5, 7, 9, 4, 6,14)` $G_{406} = \langle a, b \mid a^5 = b^3 = (ab^{-2})^3\rangle$
$H_1 = \mathbb{Z}_6$ $\zeta = \mathbb{Z}$ $G_{406}/\zeta = \Delta^+(5, 3, 3)$ —
representative

407. `( 5, 7, 9, 4,10, 4)` $G_{407} = \langle a, b \mid a^2 = b^2ab^2, a^3 = ba^{-1}b^3\rangle$
$H_1 = \mathbb{Z}_{20}$ ζ ? G_{407}/ζ ? —

408. `( 5, 7, 9, 4,10,14)` $G_{408} = \langle a, b \mid ab^{-1}ab^2ab^{-1} = 1, a^{-4}b^3a^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_9$ $\zeta = \mathbb{Z}$ $G_{408}/\zeta = \Delta^+(5, 3, 3)$ $\phi_{408} : a^{-1} \mapsto G_{258}.a,\ ab^{-1} \mapsto G_{258}.b$

409. `( 5, 7, 9, 6, 0, 4)` $G_{409} = \mathbb{Z}_{19}$
$H_1 = \mathbb{Z}_{19}$ $\zeta = \mathbb{Z}_{19}$ $G_{409}/\zeta = 1$ —
lens space $\mathcal{L}(19, ?)$

410. `( 5, 7, 9, 6, 4, 0)` $G_{410} = \langle a, b \mid b^{-1}a^{-1}ba^{-1} = 1, b^{-1}a^{-1}b^{-1}a^2 = 1\rangle$
$H_1 = \mathbb{Z}_4$ $\zeta = \mathbb{Z}_2$ $G_{410}/\zeta = D_6$ $\phi_{410} : a \mapsto G_9.a,\ b \mapsto G_9.b$

411. `( 5, 7, 9, 6, 4, 4)` $G_{411} = \langle a, b \mid b^{-2}a^{-1}b^2 = 1, ab^{-1}abab = 1, a^2b^{-2}a^2b^{-2}a = 1\rangle$
$H_1 = \mathbb{Z}_{17}$ $\zeta = \mathbb{Z}_{34}$ $G_{411}/\zeta = A_5$ ϕ_{411} : ?
fundamental group is isomorphic to the group of size 2040,
which is an extension of $\mathbb{Z}_{17} \times A_5$

412. `( 5, 7, 9, 6, 6, 0)` $G_{412} = \mathbb{Z}_2 * \mathbb{Z}_7$
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_7$ $\zeta = 1$ $G_{412}/\zeta = \mathbb{Z}_2 * \mathbb{Z}_7$ —
connected sum

413. `( 5, 7, 9, 6, 6,12)` $G_{413} = \langle a, b \mid a^8 = b^3 = (ab)^2\rangle$
$H_1 = \mathbb{Z}_2$ $\zeta = \mathbb{Z}$ $G_{413}/\zeta = \Delta^+(8, 3, 2)$ —

414. `( 5, 7, 9, 6, 6,14)` $G_{414} = \langle a, b \mid ab^{-1}a^{-1}b^{-1}a = 1, ab^{-1}abab^{-2}ab = 1\rangle$
$H_1 = \mathbb{Z}_7$ $\zeta = \mathbb{Z}_{14}$ $G_{414}/\zeta = A_5$ ϕ_{414} : ?
GAP code for the group is `[840,13]`

415. `( 5, 7, 9, 6, 8, 2)` $G_{415} = \langle a, b \mid a^5 = b^2 = (a^2b^{-1})^2\rangle$
$H_1 = \mathbb{Z}_{12}$ $\zeta = \mathbb{Z}_6$ $G_{415}/\zeta = D_{10}$ —

416. `( 5, 7, 9, 6, 8,12)` $G_{416} = \langle a,b \mid ab^{-1}a^{-1}b^{-1}a^2 = 1, ab^{-1}abab^{-2}ab = 1\rangle$
$H_1 = \mathbb{Z}_6$ $\zeta = \mathbb{Z}$ $G_{416}/\zeta = \Delta^+(5,4,2)$ $\phi_{416} : a \mapsto G_{310}.a,\ ab \mapsto G_{310}.b$

417. `( 5, 7, 9, 6,10,12)` $G_{417} = \langle a,b \mid ab^{-2}a^2 = 1, abab^{-1}ab = 1\rangle$
$H_1 = \mathbb{Z}_9$ $\zeta = \mathbb{Z}_6$ $G_{417}/\zeta = A_4$ $\phi_{417} : a^{-1}b^{-60}a^{-1}b^{-58} \mapsto G_{49}.a,$
$a^{-1}b^{-60}a^{-1}b^{-57} \mapsto G_{49}.b$

418. `( 5, 7, 9, 6,10,14)` $G_{418} = \langle a,b \mid ab^{-2}a^{-1}b^{-2}a = 1, ab^{-1}ab^2ab^{-1}a^2 = 1\rangle$
$H_1 = \mathbb{Z}_{20}$ ζ ? G_{418}/ζ ? $\phi_{418} : a^{-1}b^{-1} \mapsto G_{407}.a,\ a \mapsto G_{407}.b$

419. `( 7, 7, 7, 0, 2, 2)` $G_{419} = \langle a,b \mid ab^2a^2 = 1, b^{-2}ab^{-1}ab^{-1}ab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_{21}$ $\zeta = \mathbb{Z}_{14}$ $G_{419}/\zeta = A_4$ $\phi_{419} : ab^{-1} \mapsto G_{68}.a,\ a^{-1} \mapsto G_{68}.b$

420. `( 7, 7, 7, 0, 2,10)` $G_{420} = \langle a,b \mid a^2b^3 = 1, a^{-1}b^{-1}ab^{-1}ab^{-1}ab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_{14}$ $\zeta = \mathbb{Z}_{14}$ $G_{420}/\zeta = S_4$ $\phi_{420} : b^{-1}a^{11}b^{-1}a^8 \mapsto G_{70}.a,\ a^{-11}b^{-1}a^{11} \mapsto G_{70}.b$

421. `( 7, 7, 7, 0, 4, 4)` $G_{421} = \mathbb{Z}_{21}$
$H_1 = \mathbb{Z}_{21}$ $\zeta = \mathbb{Z}_{21}$ $G_{421}/\zeta = 1$ —
lens space $\mathcal{L}(21, ?)$

422. `( 7, 7, 7, 0, 6, 6)` $G_{422} = \mathbb{Z}_7$
$H_1 =$ $\zeta =$ $G_{422}/\zeta =$ $\phi_{422} :$

423. `( 7, 7, 7, 2, 2, 2)` $G_{423} = \langle a,b \mid a^3b^{-1}a = ba^{-1}b^3 = (a^2b^2)^2\rangle$
$H_1 = \mathbb{Z}_5 \times \mathbb{Z}_5$ ζ ? G_{423}/ζ ? $\phi_{423} : a \mapsto G_{430}.a^2G_{430}.b^{-1},$
$b \mapsto G_{430}.aG_{430}.b$
representative

424. `( 7, 7, 7, 2, 2, 4)` $G_{424} = \mathbb{Z}_5$
$H_1 = \mathbb{Z}_5$ $\zeta = \mathbb{Z}_5$ $G_{424}/\zeta = 1$ —
lens space $\mathcal{L}(5, ?)$

425. `( 7, 7, 7, 2, 2, 6)` $G_{425} = \langle a,b \mid ab^{-2}ab^{-1}a^2b^{-1} = 1, a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}ba^{-1}b = 1\rangle$
$H_1 = \mathbb{Z}_{24}$ ζ ? G_{425}/ζ ? $\phi_{425} : ba \mapsto G_{374}.a,\ a \mapsto G_{374}.b$

426. `( 7, 7, 7, 2, 2, 8)` $G_{426} = \langle a,b \mid ab^2a = 1, abab^{-1}ab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}_4$ $G_{426}/\zeta = D_6$ $\phi_{426} : b^{-1}a \mapsto G_{19}.a,\ ba^{-1}b \mapsto G_{19}.b$

427. `( 7, 7, 7, 2, 2,10)` $G_{427} = \langle a,b \mid a^3b^4a = 1, ab^{-1}a^{-3}b^{-1}ab^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_8$ $\zeta = \mathbb{Z}$ $G_{427}/\zeta = \Delta^+(4,4,3)$ $\phi_{427} : a \mapsto G_{274}.a,\ b^{-1} \mapsto G_{274}.b$

428. `( 7, 7, 7, 2, 6,10)` $G_{428} = \langle a,b \mid a^4 = b^2a^{-1}b^2, b^3 = a^3b^{-1}a^3\rangle$
$H_1 = \mathbb{Z}_4$ ζ ? G_{428}/ζ ? —
representative

429. `( 7, 7, 7, 2, 8, 8)` $G_{429} = \langle a,b \mid ba^{-1}b^{-1}a^{-1}b = 1, ab^{-1}a^{-1}b^{-1}a = 1\rangle$
$H_1 = \mathbb{Z}_3$ $\zeta = \mathbb{Z}$ $G_{429}/\zeta = \Delta^+(4,3,3)$ $\phi_{429} : ab \mapsto G_{228}.a,\ a \mapsto G_{228}.b$

430. `( 7, 7, 7, 4, 4, 4)` $G_{430} = \langle a,b \mid ab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}ab^{-2} = 1, ab^{-1}ababab^{-1}a = 1\rangle$
$H_1 = \mathbb{Z}_5 \times \mathbb{Z}_5$ ζ ? G_{430}/ζ ? $\phi_{430} : a \mapsto G_{423}.a^2G_{423}b^2,$
$b \mapsto G_{423}.a^2G_{423}.b^{-1}G_{423}.a$

431. `( 7, 7, 7, 4, 4, 8)` $G_{431} = \langle a,b \mid b^{-1}a^{-1}baba^{-1} = 1, b^{-1}a^{-1}b^2a^{-1}b^{-1}ab^{-1}a^{-2}b^{-1} = 1\rangle$
$H_1 = \mathbb{Z}_5$ $\zeta = \mathbb{Z}$ $G_{431}/\zeta = \Delta^+(7,3,2)$ $\phi_{431} : a \mapsto G_{385}.a^{-2},\ b \mapsto G_{385}.a^{-2}G_{385}.b$

432. `( 7, 7, 7, 4, 6, 6)` $G_{432} = \langle a,b \mid ba^{-1}b^{-1}a^{-1}b = 1, ab^{-1}a^{-1}b^{-1}a = 1\rangle$
$H_1 = \mathbb{Z}_3$ $\zeta = \mathbb{Z}_2$ $G_{432}/\zeta = A_4$ $\phi_{432} : a \mapsto G_{13}.a,\ b \mapsto G_{13}.b$

433. `( 7, 7, 7, 6, 6, 6)` $G_{433} = \langle a, b \mid a^2b^{-1}a^{-1}baba^{-1}b^{-1} = 1, ab^{-1}a^{-1}b^2a^{-1}b^{-1}ab = 1\rangle$
$H_1 = 1$ $\zeta = \mathbb{Z}$ $G_{433}/\zeta = \Delta^+(7, 3, 2)$ ϕ_{433} ?
the famous torus link (3,7), this 3-manifold is homeomorphic with
the 3-manifold represented by (4,4,12,3,3,11). Thus isomorphism with
G_{339} must exist, we did not construct it.

Appendix E

Geometrisation of 3-manifolds with Heegaard genus two

by

Vivien Easson, Oxford, 2002