Volumes of 3D Drawings of Homogenous Product Graphs *
(Extended Abstract)

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Abstract

3-dimensional layout of graphs is a standard model for orthogonal graph drawing. Vertices are mapped into the 3D grid and edges are drawn as the grid edge disjoint paths. The main measure of the efficiency of the drawing is the volume which is motivated by the 3D VLSI design. In this paper we develop a general framework for efficient 3D drawing of product graphs in both 1 active layer and general model. As a consequence we obtain several optimal drawings of product graphs when the factor graphs represent typical networks like CCC, Butterfly, star graph, De Bruijn... This is an analogue of a similar work done by Fernandez and Efe [6] for 2D drawings using a different approach.

1 Introduction

The research on three-dimensional graph drawings has started in seminal works [13, 15] and has been inspired by three-dimensional circuit layouts. Several basic results have been proved since then which show that 3-dimensional layout may essentially reduce material, measured as volume [4, 11]. First models considered only the graphs with vertex degrees at most 6. Only few papers investigate the verticies of arbitrary degrees [2, 3]. In this paper we concentrate ourselves on the volumes of three dimensional drawings of the homogenous cartesian products.

Two models of 3D drawings are considered. One-active-layer (1-AL) model which is a natural generalization of 2D layout, when vertices are placed in the basic plane and edges are routed in the volume above the basic plane in edge-disjoint manner. In general model there are no restrictions on vertices placement and edges are routed in edge-disjoint manner. The main measure of drawing effectivity is its volume.

The cartesian product is well-known operation defined on graphs. When applied, the cartesian product combines a set of "factor" graphs into a "product" graph. Several well-known networks are instances of product networks, including the grid, the hypercube and

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the torus. In this paper, we consider only homogenous products, i.e. the factor graphs are *isomorphic*.

In the first part of this paper we provide a framework for laying out the homogenous product of any factor graphs with known cutwidth and edge forwarding index. We also show when this framework gives asymptotically optimal layouts.

In the second part we check the optimality condtitions for products of several known factor graphs and provide their optimal volumes for both models.

Actually, there is a similar work done by Fernandez and Efe [6] concerning the areas of 2D layouts of homogenous products. We have done the analogue to their work for volumes of 3D layouts using different and more efficient technique. On the other hand our results are generalizations of the optimal 3D drawings of hypercubes [18].

2 Preliminaries

The 3-dimensional 1-active layer layout of a graph G is a mapping of G into the grid such the following conditions are satisfied:

- A vertex of degree d is represented by a square of integer coordinates of side d lying in the *basic plane* given by $z = 0$. The sides of the square are parallel to the x and y axes.
- Two vertices (squares) do not touch.
- Edges are represented as edge disjoint paths in the grid graph in the halfspace above the basic plane. A path touches only two squares which represent the endvertices of the corresponding edge.

The 3-dimensional general layout of a graph G is a mapping of G into the grid such the following conditions are satisfied:

- A vertex of degree d is represented by a cube of integer coordinates of side edge length d. The edges of the cube are parallel to the axes.
- Two vertices (cubes) do not touch.
- Edges are represented as edge disjoint paths in the grid graph. A path touches only two cubes which represent the endvertices of the corresponding graph edge.

Note that in both models the edges can croos but not overlap. The technique presented in this paper is based on graph parameter cutwidth which is defined as follows.

Definition 1. *Let $\phi : V \rightarrow 1, 2, 3, ..., |V|$ be a 1-1 labelling of vertices of a graph $G = (V, E)$. Define*

$$\mathrm{cw}(G, \phi) = \max_i \{|\{uv \in E : \phi(u) \leq i < \phi(v)\}|\}.$$

The cutwidth of the graph G is defined as

$$\mathrm{cw}(G) = \min_\phi \{\mathrm{cw}(G, \phi)\}.$$

The cutwidth is strongly related to collinear layouts and, roughly saying, represents the largest edge cut in a graph which is embedded in a line. The collinear layout of the graph G is defined as follows.

Definition 2. *In the collinear layout the vertex is represented as a square with side size equal to its degree. The vertices are arranged in the line in some ordering. The egdes are routed in edge-disjoint manner in the horizontal tracks above the vertices.*

The example of collinear layout is in the first picture of Figure 1. The height of collinear layout of graph G is defined as $\Delta(G) + t$, where $\Delta(G)$ is maximal vertex degree and t is the number of horizontal tracks of the layout. The optimal collinear layout of G has number of tracks equal to $cw(G)$.

The following theorem provides the general lower bounds for volumes of graph G in both models [18].

Theorem 1. *The optimal volume of 3-dimensional 1-active layer layout of any graph G with cutwidth cw(G) satisfies*

$$VOL_{1-AL}(G) \geq cw(G)\sqrt{\Sigma_{v \in V} deg^2(v)}$$

The optimal volume of 3-dimensional layout of any graph G with cutwidth $cw(G)$ satisfies

$$VOL(G) \geq \left(cw(G) - 4\sqrt{cw(G)}\right)^{\frac{3}{2}}$$

A cartesian product is well-known operation on graphs allowing the construction of product graph from several specified small graphs. The smallest "building" graph of product graph is called a factor graph. Following facts follow from definition of cartesian product.

$$\Delta(G_1 \times G_2) = \Delta(G_1) + \Delta(G_2)$$

$$n = n_1 \times n_2$$

where n, n_1, n_2 denote the numbers of vertices of graphs $G_1 \times G_2, G_1, G_2$ respectively and $\Delta(G)$ is the maximal degree of graph G.

Definition 3. *[10] Let the routing ρ be defined as follows. For every two distinct vertices u, v of G there exist 2 paths, from u to v and from v to u. The number of paths of ρ is then $n(n-1)$. The edge forwarding index of (G, ρ) denoted by $\pi(G, \rho)$ is the maximum number of paths, specified by routing ρ going through any edge of G. More precisely :*

$$\pi(G, \rho) = max\{\pi_e(G, \rho) : e \in E(G)\}$$

and the edge-forwarding index of G is defined as

$$\pi(G) = min\{\pi(G, \rho) : \forall \rho\}.$$

The following lemma [6] provides the lower bound for the bisection width of product graph G^r. Since the bisection width is a lower bound for cutwidth, this lemma is useful for the approximation of the cutwidth of product graph G^r.

Lemma 1. *If the edge forwarding index of factor graph G with n vertices is $\pi(G)$ then the bisection width of product graph G^r satisfies*

$$bw(G^r) \geq \frac{n^{2r} - 1}{2\pi(G)n^{r-1}}$$

3 Layout volumes of cartesian product graphs

In this section we consider the volumes of cartesian product graphs in 1-AL layer and general model. We provide the lower bounds and the construction of the upper bound for the layout volumes in both models.

First we introduce a lemma which offers the relation between the cutwidths of cartesian product graphs and their factors.

Lemma 2. *Let G^r be the homogenous product graph with factor graph G, and let n be the number of vertices of G. Then*

$$cw(G^r) \leq \frac{n^r - 1}{n - 1} cw(G) = O\left(n^{r-1} cw(G)\right)$$

Proof. We proceed by induction on r. The claim is true for $r = 2$ as one can easily see that

$$cw(G_1 \times G_2) \leq cw(G_1) + n_1 cw(G_2) \tag{1}$$

where n_1 is the number of vertices in G_1. Let $r \geq 3$. Assume the claim is true for $r-1 \geq 2$. Then by (1) we have.

$$cw(G^r) \leq cw(G^{r-1}) + |V(G^{r-1})| \cdot cw(G)$$

$$cw(G^{r-1} \times G) \leq \frac{n^{r-1} - 1}{n - 1} cw(G) + n^{r-1} cw(G) = \frac{n^r - 1}{n - 1} cw(G)$$

□

3.1 Optimal collinear layout

In this section we provide the algorithm for constructing the optimal collinear layout of the graph G with cutwidth $cw(G)$. In fact, this problem was solved in several papers, i.e. [9], [5], [12] for hypercube, [1] for complete graph. We provide a different way to obtain optimal collinear layout of any graph G with known cutwidth.

Lemma 3. *For any graph G there exists a collinear layout of the height $cw(G) + \Delta(G)$.*

Proof. We start the construction with optimal ordering of vertices of graph G having cutwidth $cw(G)$. We draw the edges as an arches between two end vertices. For this drawing we construct interval graph IG so that the edges are considered as intervals. The interval graph IG has vertices which represent the intervals (edges) from our graph G. Two vertices in IG are connected if, and only if, the corresponding intervals overlap. Then the minimum number of tracks necessary for routing of the edges of graph G and edge-track assignment is obtained as minimal vertex-coloring of IG. Observe that $cw(G)$ equals the maximal clique of IG. As IG is a prefect graph its chromatic number equals the size of the maximal clique i.e., the cutwidth of G. Note that interval graph can be colored in linear time[8].

3.2 One-active-layer model

Theorem 2. *Let G^r be the homogenous product graph with factor graph G and let r be divisible by 2. Then we have for the volume of the layout in 1-AL model the following.*

$$V_{1-AL}(G^r) = \Omega\left(n^{\frac{r}{2}} r\Delta(G)cw(G^r)\right)$$

$$V_{1-AL}(G^r) = O\left(n^r r\Delta(G)cw(G^{\frac{r}{2}})\right)$$

Proof. The lower bound comes from Theorem 1.

For the upper bound we provide the following construction. We arrange the vertices into the area with $n^{\frac{r}{2}}$ columns and $n^{\frac{r}{2}}$ rows. In every row and every column is a graph $G^{\frac{r}{2}}$. Find its optimal collinear layout with number of tracks equal to $cw(G^{\frac{r}{2}})$ (Lemma 3). All vertices of G are of degree at most $\Delta(G^r)$ and of area $\Delta^2(G^r)$. The total vertices area is of size $n(\Delta(G^r)+1)^2$. We create the 1-AL of graph G^r in three steps (see Fig. 1).

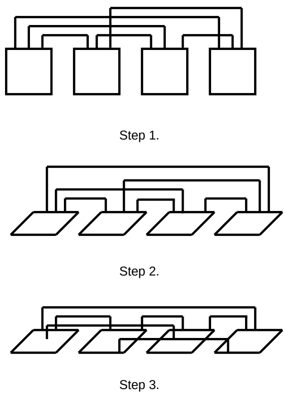

Figure 1: Construction of 1-AL layout from optimal collinear layout

In first step we route the edges as in collinear layouts of $G^{\frac{r}{2}}$. In the second step we turn the plane containing the edges into the space so that it is perpendicular to the basic plane. The height of such a layout is defined by $cw(G^{\frac{r}{2}})$.

In the third step we decrease the layout $\Delta(G^r)$-times by spreading the edges into the routing canal which has width of the node side. This can be done by algorithm described in [18]. The final height is $\frac{cw(G^{\frac{r}{2}})}{\Delta(G^r)}$ which after multiplying by the vertices area of $n(\Delta(G^r)+1)^2$ and with use of $\Delta(G^r) = r\Delta(G)$ it yields claimed volume. □

Observation 1. *The construction of 1-AL layout from proof of theorem 2 is asymptotically optimal if for the cutwidth of the product graph is the following claim true.*

$$cw(G^r) = \Theta\left(n^{r-1}cw(G)\right)$$

Proof. From Lemma 2 we have

$$V_{1-AL} = O\left(n^r r\Delta(G)cw(G^{\frac{r}{2}})\right) = O\left(n^{\frac{3}{2}r-1} r\Delta(G)cw(G)\right)$$

Moreover, if we assume that $cw(G^r) = \Theta(n^{r-1}cw(G))$ we obtain the following claim for lower bound :

$$V_{1-AL} = \Omega\left(n^{\frac{r}{2}} r\Delta(G)n^{r-1}cw(G)\right) = \Omega\left(n^{\frac{3r}{2}-1} r\Delta(G)cw(G)\right)$$

□

3.3 General model

In this section we consider the volume layouts of homogenous products in general model.

Theorem 3. *Let G^r be the homogenous product graph with factor graph G and let r be divisible by 3. Then we have following for the volume of the layout in general model of graph G.*

$$VOL(G^r) = \Omega\left(cw^{\frac{3}{2}}(G^r)\right)$$

$$VOL(G^r) = O\left(n^r(\Delta(G) + \sqrt{cw(G^{\frac{r}{3}})})^3\right)$$

Proof. From Theorem 1 we have for the volume of graph G^r with cutwidth cw in general model the following.

$$VOL(G^r) \geq (cw(G^r) - 4\sqrt{cw(G^r)})^{\frac{3}{2}}$$

$$VOL(G^r) = \Omega\left(cw^{\frac{3}{2}}(G^r)\right)$$

For the upper bound we provide the following construction.

We represent a node in general model by a cube with side size equal to its degree (denoted by Δ). We arrange the nodes of $G^{\frac{r}{3}}$ graphs in the 3D grid so that their collinear layouts are in directions of x, y, z axes. The edges lead only in directions of x, y, z.

Then we enlarge the node volume by the volume neccessary for routing of the edges. The neccessary volume for accommodating the edges around one side of the cube is $cw(G^{\frac{r}{3}})$ since this number of edges lead in one of the directions x, y, z.

We increase the node side size to $\left(\Delta(G^r) + \sqrt{cw(G^{\frac{r}{3}})}\right)$ what is sufficient. The volume of new cube is then $\left(\Delta(G^r) + \sqrt{cw(G^{\frac{r}{3}})}\right)^3$. We put the n^r cubes into the cubic box so that in every direction x, y or z are exactly $n^{\frac{r}{3}}$ cubes. These $n^{\frac{r}{3}}$ cubes form one graph $G^{\frac{r}{3}}$. Whole layout has volume of

$$O\left(n^r(\Delta(G^r) + \sqrt{cw(G^{\frac{r}{3}})})^3\right)$$

With use of $\Delta(G^r) = r\Delta(G)$ we get claimed formula. □

Observation 2. *The construction of the layout in general model from the proof of Theorem 3 is asymptotically optimal if for the cutwidth $cw(G^r)$ are the following claims true.*

$$cw(G^r) = \Theta\left(n^{r-1}cw(G)\right)$$

and

$$\Delta^2(G^r) << cw(G^{\frac{r}{2}})$$

Proof. Assume that $cw(G^r) = \Theta\left(n^{r-1}cw(G)\right)$ and we get

$$VOL(G^r) = \Omega\left(n^{\frac{3r}{2}-\frac{3}{2}}cw^{\frac{3}{2}}(G)\right)$$

If for the factor graph holds $\Delta^2(G^r) << cw(G^{\frac{r}{3}})$ we can write the upper bound formula as

$$VOL(G^r) = O\left(n^r n^{\frac{r}{2}-\frac{3}{2}}cw^{\frac{3}{2}}(G)\right) = O\left(n^{\frac{3r}{2}-\frac{3}{2}}cw^{\frac{3}{2}}(G)\right)$$

and the lower and the upper bound are asymptotically equal. □

Factor graph	n	$\Delta(G)$	$cw(G)$	$\pi(G)$
deBruijn	2^m	4	$\Theta(\frac{2^{m+1}}{m})$ [14]	$\Theta(m2^{m-1})$[16]
Star graph	$m!$	$m-1$	$\Theta(m!)$[1]	$\Theta(m!)$ [7]
Complete transposition graph	$m!$	$m-1$	$\Theta(mm!)$ [17]	$\Theta((m-1)!)$ [7]
Butterfly graph	$m2^m$	4	$\Theta(2^m)$	$\Theta(m^2 2^{m-1})$ [16]
Complete graph	m	$m-1$	$\frac{m^2}{4}$ [1]	2
CCC graph	$m2^m$	m	$\Theta(2^m)$	$\Theta(m^2 2^m)$ [16]
Linear array	m	2	1	$\Theta(m^2)$[10]

Table 1: Input parameters of considered factor graphs

3.4 Volume layouts of some known product graphs

In this section we use previous results to obtain volume bounds for several known product graphs. The key parameters for our framework are cutwidths of the factor graphs and their products. For the cutwidths of product graphs we use an approximation through the lower bound for bisection width. Fernandez and Efe [6] made the lower bounds for bisection widths of product graphs based on the so called edge forwarding index (denoted by $\pi(G)$). We use lemma 1 for obtainig the lower bound of bisection widths of product graphs. The resulting system of inequations looks as follows.

$$\frac{n^{2r}-1}{2\pi(G)n^{r-1}} \leq bw(G^r) \leq cw(G^r) \leq n^{r-1}cw(G) \tag{2}$$

In the following subsections we show that the volumes of considered graphs are asymptotically optimal. It is sufficient to prove that the optimality conditions are satisfied.

For every graph we show in the first step (left column) that $\frac{n^{2r}-1}{2\pi(G)n^{r-1}} = \Theta(n^{r-1}cw(G))$ and in the second step (right column) that $\Delta^2(G^r) << cw(G^{\frac{r}{2}})$. Note that this two steps verify the both optimality conditions from Observation 1 and Observation 2. In fact, it is sufficient to show that $n^{r-1} \cdot cw(G) = O\left(\frac{n^{2r-1}}{2\pi(G)n^{r-1}}\right)$.

Table 1 contains the parameters of the factor graphs used in the following proofs.

3.4.1 Product of de Bruijn graph

The vertex set of de Bruijn graph $B(m)$ is $V = x_1x_2...x_m : x_i \in \{0,1\}, i = 1,2,3...,m$ and the edge set E consists of all edges from one vertex $x_1x_2...x_m$ to 2 other vertices $x_2...x_m\alpha$, where $\alpha \in \{0,1\}$.

$n^{r-1}cw(G) = \Theta(\frac{2^{mr}}{m})$ $\qquad$ $(4r)^2 << \frac{2^{mr}}{2m2^{m-1}2^{\frac{mr}{2}-1}}$

$\frac{n^{2r}-1}{2\pi(G)n^{r-1}} = \Theta(\frac{2^{mr}}{m})$ $\qquad$ $16r^2 << \frac{2^{\frac{mr}{2}}}{m\cdot 2^{m-1}}$

3.4.2 Product of Star graph

An m-dimensional star graph, m-star, is a symmetric graph that has $n = m!$ nodes of degree $m-1$. Each node in an m-star is assigned a label, which is a distinct permutation of the set of m symbols $\{1,2,...,m\}$. Two nodes are connected with a dimension-i link if, and only if, the label can be obtained from the other by interchanging the first and the

i-th symbol.

$$n^{r-1}cw(G) = \Theta((m!)^r) \qquad r^2(m-1)^2 << \frac{(m!)^r}{2m!(m!)^{\frac{r}{2}-1}}$$
$$\frac{n^{2r-1}}{2\pi(G)n^{r-1}} = \Theta((m!)^r) \qquad r^2(m-1)^2 << \frac{(m!)^{\frac{r}{2}}}{2}$$

3.4.3 Product of complete transposition graphs

An m-dimensional complete transposition graph is a symmetric graph that has $n = m!$ nodes of degree $m-1$. Each node of this graph is assigned a label, which is a distinct permutation of the set of m symbols $\{1, 2, ..., m\}$. Two nodes are connected by an edge if, and only if, their labels differ in exactly two positions.

$$n^{r-1}cw(G) = \Theta((m!)^r \cdot m) \qquad (r(m-1))^2 << \frac{(m!)^{r+1}}{2(m-1)!(m!)^{r-1}}$$
$$\frac{n^{2r-1}}{2\pi(G)n^{r-1}} = \Theta\left(\frac{(m!)^r m}{2}\right) \qquad r^2(m-1)^2 << \frac{(m!)^{\frac{r}{2}} m}{2}$$

3.4.4 Product of Butterfly graphs

The m-dimensional butterfly graph, denoted by $BF(m)$, has vertex set $V = \{(x; i) : x \in V(Q_m), 0 \leq i \leq m\}$. Two vertices $(x; i)$ and $(y; j)$ are connected by an edge if and only if $j = i + 1$ and either $x = y$ or x differs from y in precisely the jth bit.

Note that approximation of cutwidth of Butterfly graph from Table 1 can be obtained by placing the graph on the line recursively. The lower bound comes from bisection. From both, the construction and the lower bound, it comes that $cw(B_m) = \Theta(2^m)$.

$$n^{r-1}cw(G) = \Theta(m^{r-1}2^{mr}) \qquad 16r^2 << m^{\frac{r}{2}-1}2^{\frac{mr}{2}}$$
$$\frac{n^{2r-1}}{2\pi(G)n^{r-1}} = \Theta(m^{r-1}2^{mr})$$

3.4.5 Product of Complete graphs

The complete graph K_m has m vertices and $m(m-1)/2$ edges. Every two distinct vertices of K_m are connected exactly by one edge.

$$n^{r-1}cw(G) = \Theta\left(m^{r+1}\right) \qquad r^2(m-1)^2 << \frac{m^r}{4m^{\frac{r}{2}-1}} = \frac{m^{\frac{r}{2}+1}}{4}$$
$$\frac{n^{2r-1}}{2\pi(G)n^{r-1}} = \Theta(m^{r+1})$$

3.4.6 Product of CCC graphs

The m-dimensional cube-connected cycle, denoted by $CCC(m)$, is constructed from m-dimensional hypercube Q_m by replacing each vertex of Q_m with an undirected cycle of length m.

Note that approximation of cutwidth of complete transposition graph from Table 1 comes from approximation for the cutwidth of the hypercube.

$$n^{r-1}cw(G) = \Theta\left(m^{r-1}2^{mr}\right) \qquad (mr)^2 << \frac{m^r 2^{mr}}{m^2 2^m m^{\frac{r}{2}-1} 2^{m(\frac{r}{2}-1)}}$$
$$\frac{n^{2r-1}}{2\pi(G)n^{r-1}} = \Theta(m^{r-1}2^{mr}) \qquad (mr)^2 << m^{\frac{r}{2}-1}2^{\frac{mr}{2}}$$

Graph	1-AL model	General model
Complete transposition graph product	$\Theta\left((m!)^{\frac{3r}{2}} rm(m-1)\right)$	$\Theta\left((m(m!)^r)^{\frac{3}{2}}\right)$
deBruijn product	$\Theta\left(r\frac{n^{\frac{3r}{2}}}{\log n}\right)$	$\Theta\left(\frac{n^{\frac{3}{2}(r+1)}}{\log^{\frac{3}{2}} n}\right)$
Star graph product	$\Theta\left((m!)^{\frac{3r}{2}} r(m-1)\right)$	$\Theta\left((m!)^{\frac{3r}{2}}\right)$
Butterfly product	$\Theta(m^{\frac{3r}{2}-1} 2^{\frac{3mr}{2}} r)$	$\Theta(m^{\frac{3r}{2}-\frac{3}{2}} 2^{\frac{3mr}{2}})$
Product of complete graphs	$\Theta\left(m^{\frac{3r}{2}+1} r(m-1)\right)$	$\Theta\left(m^{\frac{3r}{2}+\frac{3}{2}}\right)$
CCC graph product	$\Theta\left(m^{\frac{3r}{2}} r 2^{\frac{3mr}{2}}\right)$	$\Theta\left(m^{\frac{3r}{2}-\frac{3}{2}} 2^{\frac{3mr}{2}}\right)$
Linear array product	$\Theta\left(m^{\frac{3r}{2}-1} r\right)$	$\Theta\left(m^{\frac{3r}{2}-\frac{3}{2}}\right)$
Hypercube	$\Theta\left(2^{\frac{3m}{2}} m\right)$[18]	$\Theta\left(2^{\frac{3m}{2}}\right)$[18]

Table 2: Optimal layout volumes of several product graphs

3.4.7 Product of linear arrays

Linear array (path) of length m is the graph whose nodes are all integers from 1 to m and whose edges connect each integer $i(1 \leq i \leq m)$ with $i+1$.

$n^{r-1} cw(G) = \Theta(m^{r-1})$ $\qquad$ $4r^2 << \frac{m^r}{m^2 m^{\frac{r}{2}-1}}$

$\frac{n^{2r}-1}{2\pi(G)n^{r-1}} = \Theta(m^{r-1})$ $\qquad$ $4r^2 << m^{\frac{r}{2}-1}$

4 Results review

Table 2 contains a summary of results gained by substitution of graph parameters from Table 1 into formulas from Theorem 2 and Theorem 3.

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