The chirality group and the chirality index of the Coxeter chiral maps.

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Abstract

In this paper we compute the chirality group, the chirality index and the smallest regular coverings of the chiral toroidal maps. We also compute the greatest regular maps covered by chiral toroidal maps.

1 Introduction

Chirality in chemistry is unquestionable an old theme. It terms the handedness of molecules, or the non-existence of plane of symmetry in molecular structures. These so called chiral molecules come in pairs named right- and left-handed enantiomers. In mathematics the combinatorial chirality phenomenon is associate to quasi regular surface structures (maps) having no reflection (orientation preserving automorphisms)[4, 5, 7, 6]. Measuring chirality is certainly new. In [3] it was introduced two chirality measures involving orientably regular hypermaps (this can be extended by means of the generic hypermap to non-orientably regular hypermaps as well): a qualitative measure called *chirality group* and a quantitative measure called *chirality index*. The later measuring being a combinatorial output is not related to the mathematical continuous measuring methods proposed by Zabrodsky, Peleg and Avnir [10] for chemistry. Roughly speaking the chirality group gives a qualitative measuring of the regularity deviation of an orientably regular hypermap $\mathcal{H}$ by expressing this regularity deviation through a quotient group - the covering group - which translate how the smallest regular hypermap covering $\mathcal{H}$ covers $\mathcal{H}$ (or equivalently, how $\mathcal{H}$ covers the biggest regular hypermap covered by $\mathcal{H}$). The chirality index being the size of the chirality group gives a quantitative measuring. This paper follows the paper [2] which classifies the chiral hypermaps up to genus 4. Henceforth we carry the same notations onto this paper and dispense lengthy introduction on the subject. For further inside on the theme we also mention the paper [1] which classifies the chiral hypermaps with one, two, three and four hyperfaces.

1.1 Hypermaps

An oriented hypermap $\mathcal{H}$ is a cellular imbedding of a connected hypergraph $\mathcal{G}$ in a compact orientable surface $\mathcal{S}$ without boundary. The hypergraph $\mathcal{G}$ can be viewed as a bipartite graph with one monochromatic coloured vertices(say vertices coloured 0, or black) representing the hypervertices and the other (say vertices coloured 1, or white) the hyperedges. So the Walsh view of hypermap, often called the *Walsh hypermap*, is a cellular imbedding of a bipartite graph [9]. Algebraically it is realised as a triple $(D; R, L)$ where D is a non-empty finite set (the set of *darts*) and R and L are permutations of D generating the *monodromy group* $Mon(\mathcal{H})$ of $\mathcal{H}$ that acts transitively on D. Topologically we take for *darts* the edges of the underlying bipartite graph $\mathcal{G}$ with an arrow pointing to the hypervertex (white vertex). Fixing a orientation of $\mathcal{S}$, for example the counterclockwise orientation, then R is the permutation of the darts that permutes the darts counterclockwise around their adjacent hypervertices and L is the permutation that permutes the darts counterclockwise around their adjacent hyperedges.

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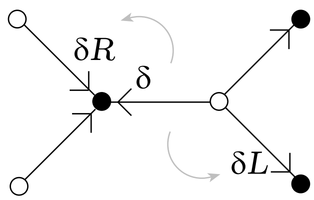

Figure 1

For commodity we draw a dart as an edge of the underlying bipartite graph $\mathcal{G}$ but with a "blade" on one side of the edge acting as two arrows, one pointing to the hypervertex and the other to the counterclockwise orientation (around the hypervertex).

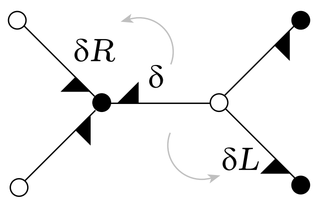

Figure 2

The orbits of the subgroups $\langle R\rangle$, $\langle L\rangle$ and $\langle RL\rangle$ describe three distinct types of cells composing the connected components of $\mathcal{S}\backslash\mathcal{G}$, the *hypervertices*, the *hyperedges* and the *hyperfaces* of $\mathcal{H}$ respectively. The *type* of $\mathcal{H}$ is the triple (l, m, n) composed of orders of R, L and RL respectively; these correspond to the least common multiples of "valencies" of hypervertices, hyperedges and hyperfaces. A *map* is just a hypermap of type $(l, 2, n)$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hypermaps described algebraically by $(D_1; R_1, L_1)$ and $(D_2; R_2, L_2)$ respectively. A *covering* $\mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a function $\phi : D_1 \longrightarrow D_2$ such that $R_1\phi = \phi R_2$ and $L_1\phi = \phi L_2$. Any covering is necessarily onto due to the connectivity of $\mathcal{G}$. If ϕ is injective the covering is said an *isomorphism.* An *automorphism* of $\mathcal{H} = (D; R, L)$ (also called a *symmetry*) is an isomorphism of $\mathcal{H}$ to itself; in other words, an automorphism of $\mathcal{H}$ is a permutation of the dart set D of $\mathcal{H}$ that commutes with R and L. The group of automorphism of $\mathcal{H}$ acts semiregular on D while the monodromy group acts transitively. Hence we have

$$|Aut(\mathcal{H})| \leq |D| \leq |Mon(\mathcal{H})|$$

If one of the equalities hold, that is, if $Aut(\mathcal{H})$ acts transitively on D or $Mon(\mathcal{H})$ acts regularly on D, then the other equality also holds and $\mathcal{H}$ is said *orientably regular.* If in addition $\mathcal{H}$ has orientation inverting automorphisms then $\mathcal{H}$ is said *regular.* However if $\mathcal{H}$ is orientably regular but not regular then $\mathcal{H}$ is said *chiral.*

1.2 The toroidal maps

As classified and described by Coxeter and Moser in [4] the toroidal maps are the regular maps on the Torus $\{4, 4\}_{b,c}$, $\{3, 6\}_{b,c}$ and $\{6, 3\}_{b,c}$, where b and c are non-negative integers. The map $\{4, 4\}_{b,c}$ has $n = b^2 + c^2$ vertices, $2n$ edges, n faces and $4n$ darts. It arises by identifying orientability the opposite sides of a large square with vertices $(0; 0)$, $(b; c)$, $(b - c; b + c)$ and $(-c; b)$ of a square grid with perpendicular unit vectors. Taking $x = (RL)^{-1}$ and $y = L$ then the monodromy group G has presentation

$$G = \langle x, y \mid x^4 = y^2 = (xy)^4 = (xyx)^b(x^2y)^c = 1\rangle \, .$$

The map $\{3, 6\}_{b,c}$ with $t = b^2 + bc + c^2$ vertices, $3t$ edges, $2t$ triangular faces and $6t$ darts arises by identifying a square with vertices $(0; 0)$, $(b; c)$, $(b - c; b + 2c)$ and $(-c; b + c)$ on a triangular grid with unit vectors making an angle of 60 degrees. Writing $x = (RL)^{-1}$ and $y = L$ its monodromy group G has presentation

$$G = \langle x, y \mid x^3 = y^2 = (xy)^6 = (x^{-1}yxy)^b(xyx^{-1}y)^c = 1\rangle \, .$$

The last map $\{6, 3\}_{b,c}$ is just the dual of $\{3, 6\}_{b,c}$.

All these maps, $\{4, 4\}_{b,c}$, $\{3, 6\}_{b,c}$ and $\{6, 3\}_{b,c}$, are reflexible if and only if $bc(b - c) = 0$.

2 Chirality Group

The objective of this section is to show how to compute the chirality group of a chiral hypermap from a presentation of its monodromy group. Given an orientably regular hypermap $\mathcal{H}$ with hypermap subgroup $H \triangleleft \Delta^+$ then the chirality group of $\mathcal{H}$ is the factor group $X(\mathcal{H}) = H/H_{\Delta}$. Before we continue with Coxeter maps we need to establish the following theorem.

Theorem 1 *Let $\mathcal{H}$ be an orientably regular hypermap with Monodromy group $G = Mon(\mathcal{H})$. If G has presentation $\langle x, y \mid R(x, y)\rangle$ then the chirality group $X(\mathcal{H})$ is the normal closure of $\langle R(x^{-1}, y^{-1})\rangle$ in G.*

Proof:
The chirality group is the quotient group $X(\mathcal{H}) = H/H_{\Delta} \cong H^{\Delta}/H$. Since $G = \Delta^+/H$ and $G/X(\mathcal{H}) \cong \Delta^+/H^{\Delta} = \langle x, y \mid R(x, y), R(x^{-1}, y^{-1})\rangle$ then, by von Dyck theorem [8], $X(\mathcal{H}) = \langle R(x^{-1}, y^{-1})\rangle^G$. □

Corollary 2 *If $\mathcal{H}$ is an orientably regular hypermap with Monodromy group $Mon(\mathcal{H}) = \langle x, y \mid x^l = y^m = (xy)^n = R(x, y) = 1\rangle$ then the chirality group $X(\mathcal{H})$ is the normal closure of $\langle R(x^{-1}, y^{-1})\rangle$ in G.*

Proof:
In fact, $x^{-l} = y^{-m} = (yx)^{-n} = 1$ in $Mon(\mathcal{H})$ so we have $X(\mathcal{H}) = \langle x^{-l}, y^{-m}, (yx)^{-n}, R(x^{-1}, y^{-1})\rangle^{Mon(\mathcal{H})} = \langle R(x^{-1}, y^{-1})\rangle^{Mon(\mathcal{H})}$. □

3 Chirality group and index of the toroidal chiral maps

The toroidal chiral maps are the orientably regular maps $\{4, 4\}_{b,c}$, $\{3, 6\}_{b,c}$, $\{6, 3\}_{b,c}$ together with their mirror images $\{4, 4\}_{c,b}$, $\{3, 6\}_{c,b}$, $\{6, 3\}_{c,b}$, where $b > c > 0$. Since $\{6, 3\}_{b,c}$ is the dual of $\{3, 6\}_{b,c}$ both maps have the same chirality group.

3.1 The map $\{4, 4\}_{\mathbf{b,c}}$

The map $\{4, 4\}_{b,c}$ has monodromy group

$$G = \langle x, y \mid x^4 = y^2 = (xy)^4 = (xyx)^b(x^2y)^c = 1\rangle$$

so by corollary 2,

$$X(\{4, 4\}_{b,c}) = \langle (x^{-1}y^{-1}x^{-1})^b(x^{-2}y^{-1})^c\rangle^G = \langle (xyx)^{-b}(yx^2)^{-c}\rangle^G = \langle (yx^2)^c(xyx)^b\rangle^G .$$

Theorem 3 *The chirality group of $\{4, 4\}_{b,c}$ is cyclic generated by some vertical translation $(xyx)^{2d}$, where $d = (b, c)$ is the greatest common divisor of b and c. The chirality index is given by*

$$\kappa = \frac{n}{(n, 2d^2)}$$

where $n = b^2 + c^2$.

Proof:
Inside G we have $(yx^2)^c = (xyx)^b$ and conjugating by x^{-1}, $(xyx)^c = (x^2y)^b$. Then $X(\{4, 4\}_{b,c}) = \langle (xyx)^{2b}\rangle^G = \langle u^{2b}\rangle^G$, by setting $u = xyx$. Let $v = u^x = yx^2$. Then $v^c = u^b$, $u^c = v^{-b}$, and we have the following table:

θ	x	x^2	x^{-1}	y
u^θ	v	u^{-1}	v^{-1}	u^{-1}
v^θ	u^{-1}	v^{-1}	u	v^{-1}

This table says that the set $\{u, u^{-1}, v, v^{-1}\}$ is a complete system of conjugates (i.e. it is closed under conjugation). Then

$$\begin{aligned} X(\{4,4\}_{b,c}) &= \langle u^{2b} \rangle^G \\ &= \langle (u^\theta)^{2b}, (v^\theta)^{2b} \rangle_{\theta=1,x,x^2,x^{-1},y} \\ &= \langle u^{2b}, v^{2b} \rangle \\ &= \langle u^{2b}, u^{2c} \rangle \\ &= \langle u^{2(b,c)} \rangle \\ &= \langle (xyx)^{2(b,c)} \rangle \end{aligned}$$

So $X(\{4,4\}_{b,c})$ is a cyclic group of order

$$|(xyx)^{2(b,c)}| = \frac{|xyx|}{(|xyx|, 2(b,c))}.$$

The word xyx is a translation one step along a vertical line. Take one step as unit of measure and consider the system of coordinates XOY whose axles XX and YY make an angle $\frac{\pi}{2}$.

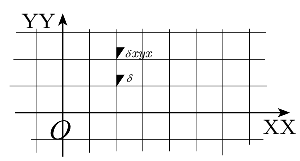

Figure 3

Here we consider two square grides, one, the smallest or finer gride, will be referred simply as the *square gride*; the second, the larger square gride where the shaded square belongs, will be referred as the *large square gride* (Figure 2).

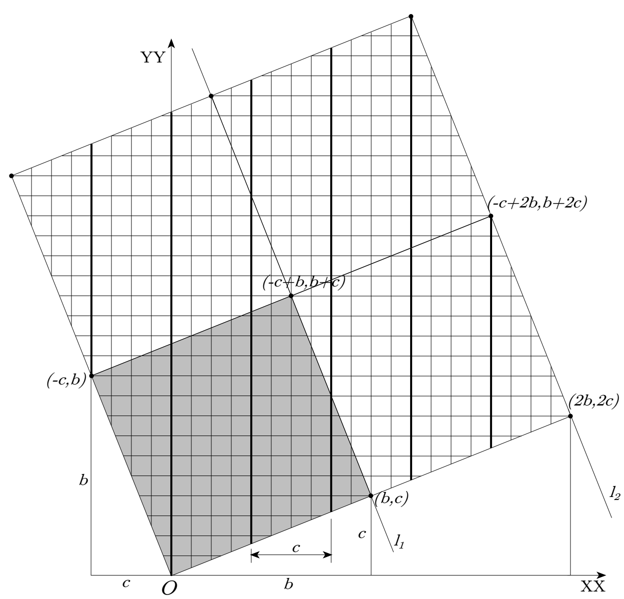

Figure 4

Let l_n be the line passing through the points (nb, nc) and $(-c+nb, b+nc)$, whose equation is $Y - nc = (X - nb)\frac{b}{-c}$. Choose n to be the smallest positive integer such that l_n intersects the YY axle in a vertex of the large square gride. We note that for such n the positive integer Y is the order of xyx. Let V_n be the set of vertices of the large square gride that belong to l_n, that is,

$$V_n = \{(-mc + nb\,;\, mb + nc) \mid m \in \mathbb{N}\}.$$

Then n is the smallest positive integer such that

$$\begin{aligned} (0;Y) \in V_n \quad &\Leftrightarrow \quad nc + n\tfrac{b^2}{c} = mb + nc \\ &\Leftrightarrow \quad n\tfrac{b}{c} = m \in \mathbb{N}\,, \end{aligned}$$

that is, n is the smallest positive integer such that $n\frac{b}{c}$ is an integer. Hence

$$n = \frac{c}{(b,c)}$$

where (,) stands for the greatest common divisor. Then

$$|xyx| = Y = \frac{b^2+c^2}{(b,c)}$$

and so

$$|(xyx)^{2(b,c)}| = \frac{b^2+c^2}{(b^2+c^2\,,\,2(b,c)^2)}$$

is the chirality index of $\{4,4\}_{b,c}$. □

3.2 The map $\{3,6\}_{b,c}$

The map $\{3,6\}_{b,c}$ has monodromy group

$$G = \langle x, y \mid x^3 = y^2 = (xy)^6 = (x^{-1}yxy)^b(xyx^{-1}y)^c = 1\rangle$$

so by corollary 2,

$$X(\{3,6\}_{b,c}) = \langle (xyx^{-1}y)^b(x^{-1}yxy)^c \rangle^G.$$

Theorem 4 *The chirality group of $\{3,6\}_{b,c}$ is cyclic generated by some "horizontal" translation $(xyxyx)^{b-c}$. The chirality index is given by*

$$\kappa = \frac{n}{(n,(b-c)d)}$$

where $n = b^2+bc+c^2$ and $d = (b,c)$.

Proof:
Putting $u = xyx^{-1}y$, $v = x^{-1}yxy$ and $w = xyxyx$, we can write

$$X(\{3,6\}_{b,c}) = \langle u^b v^c \rangle^G.$$

In G we have $(x^{-1}yxy)^b = (yxyx^{-1})^c \Leftrightarrow v^b = u^{-c}$. The following table gives the conjugates of u, v and w by x, x^{-1} and y.

θ	x	x^{-1}	y
u^θ	v^{-1}	w^{-1}	u^{-1}
v^θ	w	u^{-1}	v^{-1}
w^θ	u^{-1}	v	w^{-1}

This shows that $\{u, u^{-1}, v, v^{-1}, w, w^{-1}\}$ is a complete system of conjugates. Conjugating $v^b = u^{-c}$ by x and x^{-1} we have $w^b = v^c$ and $u^b = w^{-c}$. Then

$$\begin{aligned} X(\{3,6\}_{b,c}) &= \langle u^b v^c \rangle^G \\ &= \langle w^{b-c} \rangle^G \\ &= \langle w^{b-c}, (w^x)^{b-c}, (w^{x^{-1}})^{b-c}, (w^y)^{b-c} \rangle \\ &= \langle w^{b-c}, u^{b-c}, v^{b-c} \rangle \\ &= \langle w^{b-c}, v^{b-c} \rangle \end{aligned}$$

since $wv = u = vw$, the chirality group $X(\{3,6\}_{b,c})$ is abelian. So we have $u^{b-c} = v^{b-c}w^{b-c}$.

Consider the system of coordinates XOY with axles making an angle $\frac{\pi}{6}$ and with unit vectors e_x and e_y such that the elements w, u and v act as translations one unit along $< e_y >$, $< e_x >$ and $< e_x - e_y >$.

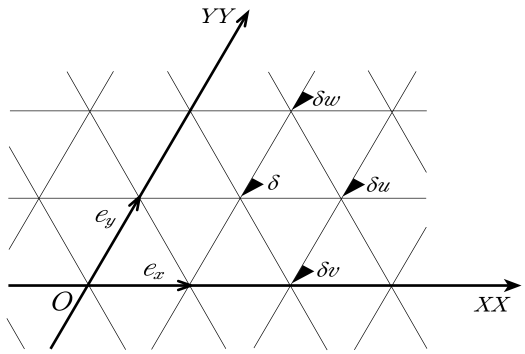

Figure 5

Let R be the fundamental region for $\langle w^{b-c}, v^{b-c}\rangle$ (the shaded region in the figure 6). As w, u and v are conjugate to each other, they all have the same order in $\{3,6\}_{b,c}$.

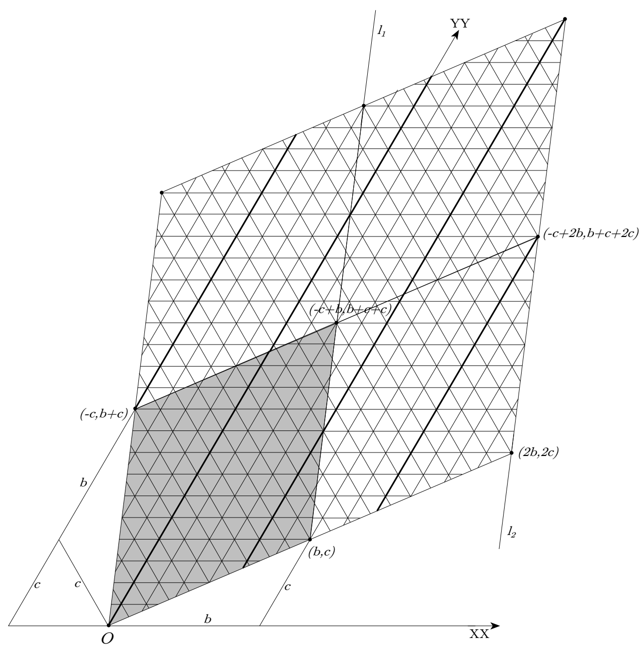

Figure 6

The generators w^{b-c} and v^{b-c} are translations $b-c$ steps along $< e_y >$ and $< e_x - e_y >$. The line l_n passing through the points with coordinates $(nb; nc)$ and $p(-c+bn; b+c+nc)$ has equation $Y - nc = (X - nb)\frac{b+c}{-c}$. Let n be the smallest positive integer such that l_n intersects the line $X = 0$ in a vertex of the tessellation, that is, n is the smallest positive integer such that $Y = n\frac{b^2+bc+c^2}{c}$ is an integer. This integer Y is the order of w. The positive integer n must then be

$$n = \frac{c}{(b^2+bc+c^2, c)} = \frac{c}{(b^2, c)}$$

and so the order of w (hence of v) is

$$|w| = \frac{b^2+bc+c^2}{(b^2, c)}$$

and so,

$$|w^{b-c}| = |v^{b-c}| = \frac{b^2+bc+c^2}{(b^2+bc+c^2,(b-c)(b^2,c))}.$$

The word w^{b-c} moves a fixed flag α along a line l which in the fundamental region R is represented by several line segments. The integer (b,c) reflects the minimal distance between two these consecutive line segments (Figure 7). Since (b,c) divides both b and c, w^b and w^c are elements of $\langle w^{(b,c)}\rangle$. On the other side, $v^c = w^b$ so $v^c \in \langle w^{(b,c)}\rangle$. Powering both sides of $wv = u$ by b we get $v^b = u^b w^{-b} = w^{-(b+c)} \in \langle w^{(b,c)}\rangle$. Hence the chirality group $X(\{3,6\}_{b,c}) = \langle w^{b-c}, v^{b-c}\rangle$ is a subgroup of the cyclic group $\langle w^{(b,c)}\rangle$, so a cyclic group itself. Since a cyclic group C contains only one subgroup of order k for each divisor k of $|C|$ then $\langle w^{b-c}\rangle = \langle v^{b-c}\rangle$ and hence

$$X(\{3,6\}_{b,c}) = \langle w^{b-c}\rangle\,.$$

Then the chirality index of $\{3,6\}_{b,c}$ is given by

$$|X(\{3,6\}_{b,c})| = |w^{b-c}| = \frac{b^2+bc+c^2}{(b^2+bc+c^2,(b-c)(b^2,c))}.$$

The proof now follows from the following lemma. □

Lemma 5 *Let a, b be integer numbers and let $n = b^2+bc+c^2$. Then $(n,(b-c)(b^2,c)) = (n,(b-c)(b,c))$.*

Proof:
This is trivially true if $b = c$, or $b = 0$ or $c = 0$. Due to the symmetry of the polynomial n we may assume that $b > c$. Let $d_1 = (n,(b-c)(b^2,c))$ and $d_2 = (n,(b-c)(b,c))$. Since $(b,c) \mid (b^2,c)$ then $d_2 \mid d_1$. Note that, every prime p dividing d_1 also divides d_2.

Any prime p such that $p \mid d_1$ divides either $(b-c)$ or (b^2,c). Suppose that $p \nmid (b-c)$, then $p \mid (b^2,c)$ and so, $p \mid b$ and $p \mid c$. Hence $p \mid (b-c)$ which is a contradiction. Consequently, any prime $p \mid (b-c)(b^2,c)$ divides necessarily $b-c$.

Let $m = b-c$, then $n = m^2+3cm+3c^2$. Now, $(b^2,c) = (m^2+c^2+2mc,c) = (m^2,c)$, $d_1 = (n,m(m^2,c))$ and $d_2 = (n,m(m,c))$.

Assume that $d_1 \neq d_2$. Then necessarily $(m^2,c) \neq (m,c)$ and $d_1 \nmid d_2$, since $d_2 \mid d_1$. Let p^s (p, prime) be the greatest power of p dividing (m,c). From $(m^2,c) \neq (m,c)$ we conclude that p^s is the greatest power of p that divides m, and p^{s+1} divides c. Since $d_2 \mid d_1$, there exist a power of p dividing d_1 but not d_2. Let p^k be the least power of p in these conditions, that is, $p^k \mid d_1$, $p^k \nmid d_2$ but $p^{k-1} \mid d_2$. As p^s is the greatest power of p that divides m, p^{2s} is the greatest power of p that divides $n = m^2+3cm+3c^2$. As p^{k-1} is the greatest power of p dividing d_2 and p^{2s} is the greatest power of p that divides n then $k-1 = 2s$ and the greatest power of p dividing d_1 is limited by the greatest power of p dividing n, since $p^{2s+1} \mid m(m^2,c)$. Hence $p^{2s} = p^{k-1}$ is the greatest power of p that divides d_1. This means that $p^k \nmid d_1$, which is a contradiction. Hence $d_1 = d_2$. □

4 Regular coverings

We have just computed the chirality group and the chirality index of the chiral coxeter maps $\mathcal{H}$. Yet we have said nothing about their smallest regular coverings $\mathcal{H}_\vartriangle$ and the biggest regular maps $\mathcal{H}^\vartriangle$ covered by them. We know from [3] that if $\mathcal{H}$ is chiral with chirality index κ then $\mathcal{H}_\vartriangle$ with hypermap subgroup the core $H_\vartriangle$ of H in Δ, is the smallest regular hypermap that covers $\mathcal{H}$ and the covering $\mathcal{H}_\vartriangle \longrightarrow \mathcal{H}$ is smooth. On the other side the regular hypermap $\mathcal{H}^\vartriangle$ with hypermap subgroup $H^\vartriangle$, the normal closure of H in Δ, is the greatest regular hypermap covered by $\mathcal{H}$, being the covering $\mathcal{H} \longrightarrow H^\vartriangle$ not necessarily smooth. However, in the case of $\mathcal{H}$ being the Coxeter chiral maps $\mathcal{H} = \{4,4\}_{b,c}$, or $\{3,6\}_{b,c}$, the chirality group $X(\mathcal{H})$ of $\mathcal{H}$ is cyclic generated by some word ω. This word ω is a translation along a vertical (first case) or horizontal (second case) line so factoring by the chirality group the vertex and face valency (i.e. the type) of $\mathcal{H}$ remains unaltered; in other words, the covering $\mathcal{H} \longrightarrow H^\vartriangle$ in these two cases are also smooth. Being both coverings smooth this implies that both $\mathcal{H}_\vartriangle$ and $\mathcal{H}^\vartriangle$ must lie on the torus; in fact, the characteristic χ of $\mathcal{H}_\vartriangle$ and $\mathcal{H}^\vartriangle$ are, respectively, a multiple and a factor of $0 = \chi(\mathcal{H})$.

4.1 The map $\{4,4\}_{\mathbf{b},\mathbf{c}}$

Let $\mathcal{H} = \{4,4\}_{b,c}$, where $b > c > 0$. This has size $4n$ where $n = b^2 + c^2$. As said above, $\mathcal{H}_\Delta$ and $\mathcal{H}^\Delta$ must lie on the torus. As they are regular then they must be one of $\{4,4\}_{x,0}$ of size $4x^2$ or $\{4,4\}_{y,y}$ of size $8y^2$. Notice that the equation $4x^2 = 8y^2 \Leftrightarrow x^2 = 2y^2$ has no integer solution. This means that if a hypermap $\mathcal{H}$ has size $4x^2$ then $\mathcal{H}$ cannot be $\{4,4\}_{y,y}$ for no y, and vice-versa.

Theorem 6 *Let $\mathcal{H} = \{4,4\}_{b,c}$ and $d = (b,c)$. If $\frac{n}{d^2}$ is odd then $\mathcal{H}_\Delta = \{4,4\}_{d\kappa,0}$ and $\mathcal{H}^\Delta = \{4,4\}_{d,0}$. If $\frac{n}{d^2}$ is even then $\mathcal{H}_\Delta = \{4,4\}_{d\kappa,d\kappa}$ and $\mathcal{H}^\Delta = \{4,4\}_{d,d}$.*

Proof:
Notice that d^2 divides n, say $n = d^2x$, and the chirality index of $\mathcal{H}$ is given by

$$\kappa = \frac{n}{(n, 2d^2)} = \frac{x}{(x,2)} \, .$$

If x is odd then $x = \kappa$ and so $|\mathcal{H}_\Delta| = |\mathcal{H}|\kappa = 4xd^2\kappa = 4(d\kappa)^2$ and $|\mathcal{H}^\Delta| = \frac{|\mathcal{H}|}{\kappa} = \frac{4xd^2}{\kappa} = 4d^2$, and hence $\mathcal{H}_\Delta = \{4,4\}_{d\kappa,0}$ and $\mathcal{H}^\Delta = \{4,4\}_{d,0}$. If x is even then $x = 2\kappa$ and so $|\mathcal{H}_\Delta| = 4xd^2\kappa = 8(d\kappa)^2$ and $|\mathcal{H}^\Delta| = \frac{4xd^2}{\kappa} = 8d^2$, and hence $\mathcal{H}_\Delta = \{4,4\}_{d\kappa,d\kappa}$ and $\mathcal{H}^\Delta = \{4,4\}_{d,d}$. □

4.2 The map $\{3,6\}_{\mathbf{b},\mathbf{c}}$

Let $\mathcal{H} = \{3,6\}_{b,c}$, where $b > c > 0$. This has size $6n$ where $n = b^2 + bc + c^2$ and chirality index

$$\kappa = \frac{n}{(n,(b-c)d)} \, ,$$

where $d = (b,c)$. As above, $\mathcal{H}_\Delta$ and $\mathcal{H}^\Delta$ must lie on the torus and as they are regular they must be one of $\{3,6\}_{x,0}$ of size $6x^2$ or $\{3,6\}_{y,y}$ of size $18y^2$. Similarly as above if $\mathcal{H}$ has size $6x^2$ then $\mathcal{H}$ cannot be $\{3,6\}_{y,y}$ for no y, and vice-versa, since the equation $6x^2 = 18y^2 \Leftrightarrow x^2 = 3y^2$ has no integer solution.

Theorem 7 *Let $\mathcal{H} = \{3,6\}_{b,c}$ and $d = (b,c)$. If $(3, \frac{b-c}{d}) = 1$ then $\mathcal{H}_\Delta = \{3,6\}_{d\kappa,0}$ and $\mathcal{H}^\Delta = \{3,6\}_{d,0}$. If $(3, \frac{b-c}{d}) = 3$ then $\mathcal{H}_\Delta = \{3,6\}_{d\kappa,d\kappa}$ and $\mathcal{H}^\Delta = \{3,6\}_{d,d}$.*

Proof:
Let $m = b - c$. Then $n = m^2 + 3cm + 3c^2$ and $d = (b,c) = (m+c,c) = (m,c)$. As $d^2 \mid n$, let $t = \frac{n}{d^2}$. Then

$$\kappa = \frac{t}{(t, \frac{m}{d})} \, .$$

Let $\mu = \frac{m}{d}$ and $\gamma = \frac{c}{d}$. Then $(\mu,\gamma) = 1$ (hence $(\mu^2,\gamma) = 1$ as well), $t = \mu^2 + 3\gamma\mu + 3\gamma^2$ and $(t,\mu) = (3\gamma^2,\mu) = (3,\mu)$. Thus

$$\kappa = \frac{t}{(3,\mu)}$$

(i) $(3,\mu) = 1$. Then $t = \kappa$ and we get $|\mathcal{H}_\Delta| = |\mathcal{H}|\kappa = 6td^2\kappa = 6(d\kappa)^2$ and $|\mathcal{H}^\Delta| = \frac{|\mathcal{H}|}{\kappa} = \frac{6td^2}{\kappa} = 6d^2$. Hence $\mathcal{H}_\Delta = \{3,6\}_{d\kappa,0}$ and $\mathcal{H}^\Delta = \{3,6\}_{d,0}$.

(ii) $(3,\mu) = 3$. Then $t = 3\kappa$ and we have $|\mathcal{H}_\Delta| = 6td^2\kappa = 18(d\kappa)^2$ and $|\mathcal{H}^\Delta| = \frac{6td^2}{\kappa} = 18d^2$. Hence $\mathcal{H}_\Delta = \{3,6\}_{d\kappa,d\kappa}$ and $\mathcal{H}^\Delta = \{3,6\}_{d,d}$. □

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