# Classification of Regular Embeddings of Hypercubes of Odd Dimension

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**Abstract**

By a regular embedding of a graph into an orientable surface we mean a 2-cell embedding with the automorphism group acting regularly on arcs. In 1997 Nedela and Škoviera [Europ. J. Comb. 18, 807-823] presented a construction giving for each solution of the congruence $e^2 \equiv 1 (\mathrm{mod}\ n)$ a regular embedding $\mathcal{M}_e$ of the hypercube $Q_n$. It was conjectured that all regular embeddings of $Q_n$ can be constructed in this way.

This paper gives a classification of regular embeddings of hypercubes $Q_n$ for $n$ odd. The conjecture is confirmed affirmatively for every odd $n$.

## 1 Introduction

A (topological) *map* is a cellular decomposition of an orientable closed surface. A common way to describe maps is to view them as 2-cell embeddings of graphs. An *automorphism* of a map is an automorphism of the underlying graph which extends to an orientation preserving self-homeomorphism of the supporting surface. It is well known that the automorphism group of a map acts semi-regularly on the set of arcs of the underlying graph and in an extreme case, when the action is regular, the map itself is called *regular*. A *regular embedding* of a graph is a 2-cell embedding of a graph into a surface in a way that the associated map is regular.

An abstract characterization of graphs underlying regular maps was given by Gardiner et al. in [6]. A classification of regular embeddings of complete graphs can

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be found in James and Jones [9]. In [15] regular embeddings of the cocktail-party graphs were classified. In [4] the regular embeddings of complete $n$-multipartite graphs $K_{p,p,\ldots,p}$, where $p$ is a prime and $n$ is a positive integer are classified. A more general result is proved in [5], where regular embeddings of the graphs of order $pq$ for $p$ and $q$ primes are classified. There are two other popular families of graphs which are worth to consider, the complete bipartite graphs and the $n$-dimensional cubes. As concerns the complete bipartite graphs $K_{n,n}$, the classification of all regular embeddings is known for some particular infinite classes of $n$. For instance, a complete list of the maps in the case $n$ is a power of an odd prime is given in [11]. The aim of this paper is to deal with regular embeddings of $n$-dimensional cubes. The $n$-dimensional cube $Q_n$ is a graph whose vertex-set is formed by the vectors of dimension $n$ over the field $\mathbb{F}_2$, two vertices are adjacent if they differ in precisely one coordinate. The existence of at least two regular embeddings of $Q_n$ for every $n$ was known a long time ago. For instance, $Q_3$ has the well-known 4-gonal regular embedding in the sphere, as well as a hexagonal regular embedding in the torus, the map $\{6,3\}_{2,0}$ in the Coxeter-Moser notation [3, page 107]. In [16] for every solution of the congruence $e^2 \equiv 1 (\mathrm{mod}\ n)$ a regular embedding of $Q_n$ is constructed, and it is proved that different solutions give rise to nonisomorphic maps. It is conjectured that there are no other regular embeddings of $Q_n$. The main goal of this paper is to derive a classification of regular embeddings of $Q_n$ for $n$ odd. It transpires that the above conjecture holds true for all odd $n$, while the classification problem for even $n$ remains open. More information on some related results one can find in the survey paper [17].

## 2 Regular maps and regular embeddings of $Q_n$

Let $\mathcal{G} = \mathcal{G}(V, D)$ be a graph with a vertex set $V = V(\mathcal{G})$ and an arc set $D = D(\mathcal{G})$. By $S_V$ and $S_D$ we denote the symmetric groups on $V$ and on $D$, respectively. The involution $L$ in $S_D$ interchanging the two arcs underlying every given edge is called the *arc-reversing involution*. An element $R$ in $S_D$ which cyclically permutes the arcs initiated at $v$ for each vertex $v \in V(\mathcal{G})$ is called a *rotation*. In the theory of maps it is natural to allow the underlying graphs to have loops and multiple edges. If the graph $\mathcal{G}$ is simple (without loops and multiple edges), $\mathrm{Aut}(\mathcal{G})$ is considered as a subgroup of both $S_V$ and $S_D$, and the same notation is used for convenience. In the investigation of maps, it is often useful to replace topological maps on orientable surfaces with their combinatorial counterparts. A map $\mathcal{M}$ with underlying graph $\mathcal{G}$ can be identified with a triple $\mathcal{M} = (D, R, L)$, where $D$ is the set of arcs of $\mathcal{G}$, $R$ is a rotation and $L$ is the arc-reversing involution of $\mathcal{G}$ (see [7, 10]). Given graph $\mathcal{G}$

the involution $L$ is determined, hence we can write $\mathcal{M} = (\mathcal{G}; R)$ for $\mathcal{M}$ as well. By the connectivity of $\mathcal{G}$, the group $\mathrm{Mon}(\mathcal{M}) := \langle R, L\rangle$ is a transitive subgroup of $S_D$. Given two maps $\mathcal{M}_1 = (\mathcal{G}_1;\, R_1) = (D_1, R_1, L_1)$ and $\mathcal{M}_2 = (\mathcal{G}_2;\, R_2) = (D_2, R_2, L_2)$, a homomorphism $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ is a mapping $D_1 \to D_2$ satisfying $R_1\phi = \phi R_2$ and $L_1\phi = \phi L_2$. Since $\mathrm{Mon}(\mathcal{M}_2)$ acts transitively on $D_2$, every map homomorphism is an epimorphism. If $\phi$ is a bijection $D_1 \to D_2$ it is a *map isomorphism.* In particular, if $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$, then $\phi$ is called an *automorphism* of $\mathcal{M}$. The automorphisms of $\mathcal{M}$ form a group $\mathrm{Aut}(\mathcal{M}) \le \mathrm{Aut}(\mathcal{G})$, called the *automorphism group* of the map $\mathcal{M}$. By this definition, $\mathrm{Aut}(\mathcal{M}) \le C_{S_D}(\mathrm{Mon}(\mathcal{M}))$, the centraliser of $\mathrm{Mon}(\mathcal{M})$ in $S_D$. Also $\mathrm{Aut}(\mathcal{M})$ acts semi-regularly on $D$, which follows from the transitivity of $\mathrm{Mon}(\mathcal{M})$ on $D$. If the action is regular, the map $\mathcal{M}$ is called *regular.* Note that in maps with simple underlying graphs both graph and map automorphisms act faithfully on vertices. Hence, whenever it will be convenient we will consider them as permutations of vertices as well.

Let $\phi$ be a map homomorphism $\mathcal{M} \to \bar{\mathcal{M}}$. The preimages over an arc $\bar{x} \in \bar{\mathcal{M}}$ in $\bar{\mathcal{M}}$ form a *fibre* over $\bar{x}$ and the subgroup $N \le \mathrm{Aut}(\mathcal{M})$ of elements fixing a fibre setwise is called the *group of covering transformations.* Similarly, the preimages over a vertex $\bar{v}$ form a (*vertex-*)*fibre* over $\bar{v}$. If both maps are regular then $N$ acts regularly on each fibre over an arc and it is a normal subgroup of $\mathrm{Aut}(\mathcal{M})$. In this case, we can identify the $|N|$ arcs in the fibre over an arc $\bar{x}$ of $\bar{\mathcal{M}}$ with the set $\{\bar{x}\} \times N$. Then the relation between the rotations of the maps $\mathcal{M}$ and $\bar{\mathcal{M}}$ is expressed by the following formula $(\bar{x}, g)^R = (\bar{x}^{\bar{R}}, g)$, where $g \in N$. Moreover, if the covering $\phi$ has no branch points at vertices, i.e., if $N$ intersects a vertex-stabiliser trivially then the above formula means that for each vertex $v$ in the fibre over $\bar{v}$, the (local) rotation $R$ in $v$ copies the (local) rotation $\bar{R}$ in $\bar{v}$. Let us stress that given regular covering of graphs $X \to \bar{X}$, a map $\bar{\mathcal{M}} = (\bar{X}, \bar{R})$ 'lifts' in a unique way onto a map $\mathcal{M} = (X, R)$ with the rotation determined by the above formula.

Conversely, let $\mathcal{M} = (D, R, L)$ and $N \trianglelefteq \mathrm{Aut}(\mathcal{M})$. Set $\mathcal{M}/N = (\bar{D}, \bar{R}, \bar{L})$ where $\bar{D} = \{xN \mid x \in D\}$ is formed by orbits of the action of $N$, $(xN)^{\bar{R}} = x^R N$ and $(xN)^{\bar{L}} = x^L N$. It is easy to see that the natural projection taking an arc $x \mapsto xN$ is a homomorphism $\mathcal{M} \to \mathcal{M}/N$.

The following lemma makes the above correspondence more precise.

**Proposition 2.1** [12, page 452] *Let $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ be a homomorphism between two regular maps. Then there exists a normal subgroup $N \trianglelefteq \mathrm{Aut}(\mathcal{M}_1)$ such that $\mathcal{M}_2 \cong \mathcal{M}_1/N$ and $\phi\psi = \pi$, where $\psi$ is a map isomorphism $\mathcal{M}_2 \to \mathcal{M}_1/N$ and $\pi$ is the natural projection taking an arc $x \mapsto xN$.*

In what follows we define two known families of regular maps and explain a relation between them. By $\mathbb{F}_q$, $\mathbb{Z}_n$, $\mathbb{Z}_n^*$ and $\mathbb{D}_{2n}$, we denote the finite field with $q$

elements, the cyclic group of order $n$, the multiplicative group modulo $n$ and the dihedral group of order $2n$, respectively.

**Regular maps of dipoles:** Regular maps with two vertices are classified in [14] (see [1] and [3] as well). The underlying graph is an $n$-dipole, which consists of $n$ parallel edges joining the two vertices. The edges can be assigned by the elements of $\mathbb{Z}_n$ so that the set of arcs is $\mathbb{Z}_2 \times \mathbb{Z}_n$. Given $n$ and $e$ satisfying $e^2 \equiv 1(\mathrm{mod}\ n)$, a map of an $n$-dipole is defined as follows: $\mathcal{D}(n,e) = (\mathbb{Z}_2 \times \mathbb{Z}_n, \bar{R}, \bar{L})$, where $(i,j)^{\bar{R}} = (i, j+e^i)$ and $(i,j)^{\bar{L}} = (i+1, j)$.

We have the following result classifying two-vertex regular maps.

**Lemma 2.2** [14] *A two-vertex regular map is isomorphic to $\mathcal{D}(n,e)$ for some $n$ and $e$, $e^2 \equiv 1(\mathrm{mod}\ n)$. Moreover, $\mathcal{D}(n,e) \cong \mathcal{D}(n,f)$ if and only if $e \equiv f(\mathrm{mod}\ n)$.*

**Regular maps $\mathcal{M}(n,e)$ of $Q_n$:** In [16] the following family of regular maps with the underlying graph $Q_n$ is defined. The vertices of $Q_n$ are the $n$-dimensional vectors of $V = V(n,2)$ over $\mathbb{F}_2$, two being adjacent if they differ only in a $j$-th component for some $j \in \{0, 1, \dots, n-1\}$. Let $\{\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_{n-1}\}$ denote the standard basis for $V$. Then, an arc of $Q_n$ is an ordered pair of vertices $(\mathbf{v}, \mathbf{v}+\mathbf{e}_j)$ for some $\mathbf{v} \in V$ and for some $j$. The arc-reversing involution is given by $(\mathbf{v}, \mathbf{v}+\mathbf{e}_j)^L = (\mathbf{v}+\mathbf{e}_j, \mathbf{v})$. Let $e$ be an element of $\mathbb{Z}_n$ such that $e^2 \equiv 1(\mathrm{mod}\ n)$ . Set $(\mathbf{v}, \mathbf{v}+\mathbf{e}_j)^R = (\mathbf{v}, \mathbf{v}+\mathbf{e}_{j+e^{||v||}})$, where $||\mathbf{v}||$ is the sum of components of $\mathbf{v}$ modulo 2. Denote by $\mathcal{M}(n,e) = (Q_n; R)$= the above map. It is proved in [16] that the maps $\mathcal{M}(n,e)$ are regular and different $e$ give rise to nonisomorphic maps.
Set

$$W = \left\{ (x_0, x_1, x_2, \dots, x_{n-1}) \in V \mid \sum_{i=0}^{n-1} x_i = 0 \right\}. \tag{2.1}$$

Observe that $W$ determines the bipartition $W$, $V \setminus W$ of the vertex set of $Q_n$. The group of translations $t(V) \in \mathrm{Aut}(Q_n)$ consisting of automorphisms $t(\mathbf{v})$ taking $\mathbf{x} \mapsto \mathbf{x} + \mathbf{v}$ is a normal subgroup of $\mathrm{Aut}(Q_n)$ acting freely on vertices of $Q_n$. Let $t(W)$ denotes the subgroup of index 2 of $t(V)$ given by $W$. The following lemma relates the two families of regular maps defined above.

**Lemma 2.3** *There is a map homomorphism $\mathcal{M}(n,e) \to \mathcal{D}(n,e)$ with the group of covering transformations $N = t(W)$, that is $\mathcal{M}(n,e)/t(W) \cong \mathcal{D}(n,e)$.*

**Proof** Set $\phi(\mathbf{v}, \mathbf{v}+\mathbf{e}_j) = (||\mathbf{v}||, j)$. Clearly, $\phi$ maps the arcs of $\mathcal{M}(n,e) = (D, R, L)$ onto the arcs of $\mathcal{D}(n,e) = (\bar{D}, \bar{R}, \bar{L})$. One can see that

$$(\mathbf{v}, \mathbf{v}+\mathbf{e}_j)^{\phi\bar{R}} = (||\mathbf{v}||, j)^{\bar{R}} = (||\mathbf{v}||, j+e^{||\mathbf{v}||}) = (\mathbf{v}, \mathbf{v}+\mathbf{e}_{j+e^{||\mathbf{v}||}})^{\phi} = (\mathbf{v}, \mathbf{v}+\mathbf{e}_j)^{R\phi},$$

and

$$
\begin{aligned}
(\mathbf{v}, \mathbf{v}+\mathbf{e}_j)^{\phi \bar{L}} &= (||\mathbf{v}||, j)^{\bar{L}} = (||\mathbf{v}||+1, j) \\
&= (\mathbf{v}+\mathbf{e}_j, \mathbf{v}+\mathbf{e}_j+\mathbf{e}_j)^{\phi} = (\mathbf{v}+\mathbf{e}_j, \mathbf{v})^{\phi} = (\mathbf{v}, \mathbf{v}+\mathbf{e}_j)^{L\phi}.
\end{aligned}
$$

Hence $\phi$ is a map homomorphism.

By the regularity of the action, there is a unique subgroup $N \trianglelefteq \mathrm{Aut}(\mathcal{M}(n,e))$ acting regularly on the fibre $F = \{(\mathbf{v}, \mathbf{v}+\mathbf{e}_0) \mid \mathbf{v} \in W\}$ over $(0,0)$. Since $t(W)$ acts regularly on $F$, $N = t(W)$ provided $t(W) \leq \mathrm{Aut}(\mathcal{M}(n,e))$. For any $\mathbf{w} \in W$, we have

$$
\begin{aligned}
(\mathbf{v}, \mathbf{v}+\mathbf{e}_j)^{t(\mathbf{w})R} &= (\mathbf{v}+\mathbf{w}, \mathbf{v}+\mathbf{w}+\mathbf{e}_j)^R = (\mathbf{v}+\mathbf{w}, \mathbf{v}+\mathbf{w}+\mathbf{e}_{j+e^{||v+w||}}) \\
&= (\mathbf{v}, \mathbf{v}+\mathbf{e}_{j+e^{||v||}})^{t(\mathbf{w})} = (\mathbf{v}, \mathbf{v}+\mathbf{e}_j)^{Rt(\mathbf{w})}.
\end{aligned}
$$

Since $t(W)$ is a group of graph automorphisms we have $t(\mathbf{w})L = Lt(\mathbf{w})$ as well. Thus $t(W) \leq \mathrm{Aut}(\mathcal{M}(n,e))$. It follows that $N = t(W)$. □

The following conjecture for the maps $\mathcal{M}(n,e)$ is stated.

**Conjecture** [16, page 821] Every regular map $\mathcal{M}$ with the underlying graph $Q_n$ is isomorphic to one of the maps $\mathcal{M}(n,e)$ for some $e$ satisfying $e^2 \equiv 1 (\mathrm{mod}\ n)$.

As was already mentioned the conjecture will be confirmed for odd $n$.

The following result will be used in next section

**Proposition 2.4** [8, VI. Hauptsatz 4.3] *Let $G = N_1 N_2 \cdots N_k$, where $N_i$ is a nilpotent subgroup of $G$ for all $i \in \{1, 2, \ldots, k\}$, and $N_i N_j = N_j N_i$ for any $i$, $j$. Then $G$ is solvable.*

# 3 Classification Theorem

Let us first introduce some notation. For two positive integers $m$ and $n$, by $(m,n)$ we denote the greatest common divisor of $m$ and $n$. For a group $G$ and a subgroup $H$, we use $C_G(H)$ and $N_G(H)$ to denote the centraliser and normaliser of $H$ in $G$, respectively. A semidirect product of the group $N$ by the group $H$ is denoted by $N{:}H$. By $V = V(n,q)$, $\mathrm{AG}(V)$, $\mathrm{GL}(n,q)$, and $\mathrm{AGL}(n,q)$, respectively, we denote the $n$-dimensional row linear space, affine geometry corresponding to $V$, general linear group and affine transformation group over the field $\mathbb{F}_q$. Recall that for any $\mathbf{v} \in V$, we denote by $t(\mathbf{v})$ the translation corresponding to $\mathbf{v}$ in $\mathrm{AG}(V)$ and by $t(V)$ the translation subgroup of $\mathrm{AGL}(n,q)$. For group theoretical notions not defined here the reader is referred to [8].

The following result will be crucial for our investigation.

**Lemma 3.1** *Let $G = \langle r, \ell \rangle$ be a permutation group of odd degree $n$ on a set $\Omega$ such that $|\ell| = 2$, $\langle r \rangle$ acts regularly on $\Omega$ and let the point stabilisers of $G$ be 2-groups. Then $\langle r \rangle \lhd G$ and $G$ is isomorphic to one of the following groups:*

$$L(n,e) = \langle r, \ell \mid r^n = \ell^2 = 1, r^\ell = r^e \rangle, \tag{3.1}$$

*where $e^2 \equiv 1 (\mathrm{mod}\ n)$. Moreover, if $e_1 \not\equiv e_2 (\mathrm{mod}\ n)$, then $L(n,e_1) \not\cong L(n,e_2)$.*

**Proof** Let $G$ be a group satisfying the assumptions. Denote $R = \langle r \rangle$. Since $G = RG_v = G_vR$, by Proposition 2.4 $G$ is solvable. Observe that it is sufficient to prove $R \lhd G$ is a normal subgroup.

Since $|\Omega|$ is odd, $G$ has no normal 2-subgroups. Denote by $N = O_{2'}(G)$ the largest normal subgroup of odd order of $G$. Then $1 < N \leq R$ and $G/N$ has no normal subgroups of odd order. Set $K/N = O_2(G/N)$, the largest normal 2-subgroup of $G/N$.

Assume $N$ is intransitive on $\Omega$. Then $N < R$ and $K < G$. Now $K/N$ is a normal 2-subgroup of $G/N$. Suppose $N < C_K(N)$. Since $C_K(N) \lhd G$, we know that $C_K(N)/N$ is a 2-subgroup, which forces that $C_K(N) = N \times H$ for some nontrivial 2-subgroup $H$ of $G$ . This implies $G$ contains a normal 2-subgroup $H$, a contradiction. Therefore $C_K(N) = N$. Set $C = C_G(N)$. Then $R \leq C \lhd G$ and $C \cap K = N$, which implies that $C/N \cap K/N = 1$. Let $D/N$ be the minimal normal subgroup of $G/N$ contained in $C/N$. Then $D/N$ is a 2-subgroup, which in turn forces $D/N \leq O_2(G/N) = K/N$, contradicting to $C/N \cap K/N = 1$.

Now $N$ is transitive on $\Omega$ and equivalently, $R = N \lhd G$ as required.

One can easily check that $L(n,e_1) \cong L(n,e_2)$ implies $e_1 \equiv e_2 (\mathrm{mod}\ n)$. □

Let $n$ be an odd integer and let $V = V(n,2)$. Let $Q_n$ be the $n$-dimensional cube with the vertex set $V$. Let $\mathbb{S}_n$ denote the symmetric group of degree $n$. Then we define an action of $\mathbb{S}_n$ on $V$ by $\mathbf{x}^h = (x_{h^{-1}(0)}, x_{h^{-1}(1)}, \ldots, x_{h^{-1}(n-1)})$ for any $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1}) \in V$ and $h \in \mathbb{S}_n$. It induces a monomorphism from $\mathbb{S}_n$ into $\mathrm{GL}(n,2)$. We shall use the same notation for $\mathbb{S}_n$ as well as for its isomorphic image in $\mathrm{GL}(n,2)$. Clearly, the above action of $\mathbb{S}_n$ gives rise to a subgroup $t(V) : \mathbb{S}_n \leq \mathrm{Aut}(Q_n)$, which is a subgroup of $\mathrm{AGL}(n,2)$, noting that for any $\mathbf{x} \in V$ and $h \in \mathbb{S}_n$ we have $t(\mathbf{x})^h = t(\mathbf{x}^h)$. It is not difficult to prove that a stabiliser of a vertex $u$ in the action of $\mathrm{Aut}(Q_n)$ acts faithfully on the neighbours of $u$ proving the equality $t(V) : \mathbb{S}_n = \mathrm{Aut}(Q_n)$, (see [2, page 261]).

We shall need the following lemma.

**Lemma 3.2** *Let $\pi : X \to \bar{X}$ be a covering between connected bipartite graphs. Let the bipartition of $X$ be $A \cup B$ and let the bipartition of $\bar{X}$ be $C \cup D$. Then either $C = \pi(A)$ and $D = \pi(B)$, or $C = \pi(B)$ and $D = \pi(A)$.*

**Proof** Since $\pi$ is a covering, $\pi^{-1}(C) \cup \pi^{-1}(D)$ is a bipartition of $X$. Then there is no edge joining $F = (\pi^{-1}(C) \cap A) \cup (\pi^{-1}(D) \cap B)$ to its complement. By the connectivity of $X$ we have $F = A \cup B$ or $F = \emptyset$ proving the lemma. □

Now we are ready to prove the main result of the paper.

**Theorem 3.3** *Let $Q_n$ be the $n$-dimensional cube, where $n$ is odd. Then any regular map $\mathcal{M}$ with the underlying graph $Q_n$ is isomorphic to $\mathcal{M}(n, e)$ for some integer $e$ satisfying $e^2 \equiv 1 (\text{mod } n)$. Moreover, different solutions of the congruence $e^2 \equiv 1 (\text{mod } n)$ give rise to non-isomorphic maps.*

**Proof** Let $G \leq \text{Aut}(Q_n) = t(V) : \mathbb{S}_n$ be the automorphism group of a regular map $\mathcal{M}$ with the underlying graph $Q_n$. Then $|G| = |\text{Aut}(\mathcal{M})| = n2^n$ and $G = \langle r, \ell \rangle$, where $|\ell| = 2$, $|r| = n$ and $|G_v| = |\langle r \rangle| = n$ for some $v \in V(Q_n)$. Let $S$ be any Sylow 2-subgroup of $G$. Since $S_v \leq G_v \cong \mathbb{Z}_n$ and $n \neq 2$, it follows that $S_v = 1$, that is, $S$ acts regularly on $V(Q_n)$. Now $\bar{G} := G/(t(V) \cap G) \cong Gt(V)/t(V)$ is a subgroup of $\text{Aut}(Q_n)/t(V) \cong \mathbb{S}_n$. Let $\bar{r}$ and $\bar{\ell}$ denote the respective images of the generators $r$, $\ell$ in the quotient group $\bar{G}$.

*Case 1* If $\bar{\ell} = \bar{1}$, then $\bar{G} = \langle \bar{r} \rangle \cong \mathbb{Z}_n$ and $\bar{\mathcal{M}} = \mathcal{M}/(G \cap t(V))$ is a one-vertex map with the automorphism group $\mathbb{Z}_n$. In this case $t(V) \cap G = t(V)$.

*Case 2* Let $\bar{\ell} \neq \bar{1}$ so that $\langle \bar{\ell} \rangle \cong \mathbb{Z}_2$. Then, $\bar{G} = \langle \bar{r}, \bar{\ell} \rangle$ is isomorphic to a permutation group of odd degree $n$ such that $|\bar{\ell}| = 2$, $\bar{G}$ contains a cyclic regular subgroup $\langle \bar{r} \rangle \cong \mathbb{Z}_n$ and point stabilisers are 2-groups. By Lemma 3.1, $G/(t(V) \cap G) = \bar{G} = \langle \bar{r} \rangle : \langle \bar{\ell} \rangle \cong \mathbb{Z}_n : \mathbb{Z}_2 \cong L(n, e)$ for some $e$, and the quotient $\bar{\mathcal{M}} = \mathcal{M}/(G \cap t(V))$ is a regular map with the automorphism group $\langle \bar{r}, \bar{\ell} \rangle \cong L(n, e)$. Since $|L(n, e)| = 2n$, $\bar{\mathcal{M}}$ is a 2-vertex regular map and $t(V) \cap G$ is a subgroup of index 2 of $t(V)$ acting freely on vertices of $Q_n$. Hence, it determines a regular covering between the underlying graphs. Since both graphs are bipartite and connected, by Lemma 3.2 $G \cap t(V)$ fixes a bipartition set in $Q_n$ setwise, consequently $t(W) = G \cap t(V)$ (see (2,1)).

In both cases, $t(W) \leq t(V) \cap G$ is a normal subgroup of $G$ and the quotient $\mathcal{M}/t(W)$ is a 2-vertex regular map. Hence, $\mathcal{M}/t(W) \cong \mathcal{D}(n, e)$ for some $e$, $e^2 \equiv 1 (\text{mod } n)$ by Lemma 2.2 and the map $\mathcal{M}$ is uniquely determined. By Lemma 2.3, $\mathcal{M} \cong \mathcal{M}(n, e)$. □

Note that by [13, p.84] given odd integer $n$ the number of solutions of the congruence $e^2 \equiv 1 (\bmod\ n)$ is $2^k$ where $k$ is the number of different primes dividing $n$. By Theorem 3.3 there are exactly $2^k$ non-isomorphic regular embeddings of $Q_n$.

**Concluding remarks**

(1) The face-size of $\mathcal{M} = \mathcal{M}(n, e)$ is 4 if $e = -1$, otherwise it is $\frac{2n}{(e+1,n)}$. Consequently, the genus of the supporting surface is $g(\mathcal{M}) = 2^{n-3}(n-4)+1$ if $e = -1$, and $g(\mathcal{M}) = 2^{n-2}(n-2-(e+1,n))+1$ otherwise.

(2) The group $L(n, e) = \langle r, \ell \rangle$ defined in (3.1) acts as a map automorphism group on the map $\mathcal{D}(n, e)$. One can prove that $\mathrm{Aut}(\mathcal{M}(n, e)) \cong \langle r, t(\mathbf{e}_0)b \rangle = t(W) : \langle r, t(\mathbf{y})b \rangle$, where $\mathbf{y} = (1, 1, \ldots, 1)$ and $r, b \in \mathbb{S}_n$ are such that $r = (0, 1, \ldots, n-1)$ and $b$ takes $i \mapsto ei$.

(3) As was already noted any Sylow 2-group $H$ of $\mathrm{Aut}(\mathcal{M}(n, e))$ acts regularly on the vertices of $Q_n$. One may choose $H = t(W) : \langle t(\mathbf{e}_0)b \rangle$, for instance. Consequently, $Q_n$ can be expressed as a Cayley graph based on $H$ and the map $\mathcal{M}(n, e)$ can be alternatively described as a Cayley map with respect to $H$. (see [18] for definitions).

(4) As was already noted, the problem to classify regular embeddings of $Q_n$ for even $n \geq 8$ remains open.

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