Non-existence of nonorientable regular embeddings of n-dimensional cubes

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Abstract

By a regular embedding of a graph K into a surface we mean a 2-cell embedding of K into a compact connected surface with the automorphism group acting regularly on flags. Regular embeddings of the n-dimensional cubes Q_n into orientable surfaces exist for any positive integer n. In contrast to this, we prove the non-existence of nonorientable regular embeddings of Q_n for $n > 2$.

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1 Introduction

In what follows by a graph we mean a simple connected graph of minimum valency ≥ 3. A *map* is a 2-cell embedding of a graph into a compact, connected surface. An automorphism of a map is an automorphism of a graph which extends to a self-homeomorphism of the underlying surface. Automorphisms of a map $\mathcal{M}$ act semi-regularly on its *flags* - mutually incident triples of the form vertex-edge-face. Hence the size of the automorphism group $\mathrm{Aut}(\mathcal{M})$ of $\mathcal{M}$ is bounded by $4|E|$, where $|E|$ is the number of edges. If the equality $\mathrm{Aut}(\mathcal{M}) = 4|E|$ holds then the action of $\mathrm{Aut}(\mathcal{M})$ is regular and the map $\mathcal{M}$ itself is called *regular*. Equivalently, $\mathcal{M}$ is regular if there are three pairwise distinct (nontrivial) involutions λ, ρ and τ in $\mathrm{Aut}(\mathcal{M})$ each fixing two

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elements of $\{v, e, f\}$ for some flag (v, e, f). It follows that $G = \mathrm{Aut}\,(\mathcal{M}) = \langle \lambda, \rho, \tau \rangle$. In what follows we shall assume that λ and τ are in the stabiliser of the edge e, and that τ fixes v, as well. By regularity of the action, $\tau\lambda = \lambda\tau$. Thus the stabiliser $G_e \cong Z_2 \times Z_2$ is dihedral of order 4. Similarly, the stabiliser G_v of the vertex v is generated by the involutions ρ and τ, and consequently, it is dihedral of order $2m$, where m is the valency. It follows that with each regular map $\mathcal{M}$ we can associate a triple of involutions λ, ρ, τ acting on the set of flags such that $\mathrm{Aut}\,(\mathcal{M}) = \langle \lambda, \rho, \tau \rangle$ and $\tau\lambda = \lambda\tau$. In such case we write $\mathcal{M} = \mathcal{M}(\lambda, \rho, \tau)$. Two regular maps $\mathcal{M}(\lambda, \rho, \tau)$ and $\mathcal{M}'(\lambda', \rho', \tau')$ with the same underlying graph K are *isomorphic* if there is a graph automorphism ψ of K such that $\psi^{-1}\lambda\psi = \lambda'$, $\psi^{-1}\rho\psi = \rho'$ and $\psi^{-1}\tau\psi = \tau'$. Detailed explanation of the above representation of regular maps can be found in [4, Theorem 3]. It is proved there that map automorphisms (acting by definition on flags) can be interpreted as graph automorphisms. Consequently, if the underlying graph of a map is simple of valency at least 3 then each map automorphism can be viewed as a permutation of the set of vertices of the considered map.

Basics of the theory of regular maps as well as some more information can be found in papers [7, 8, 11, 14, 15].

If a regular map is determined by an embedding $i : K \mapsto S$ of a graph K into a surface S we say that i is a *regular embedding* of K. The surface underlying a map $\mathcal{M}$ is nonorientable if and only if there is a cycle C whose regular neighbourhood forms a subspace homeomorphic to a Möbius band. Such a cycle will be called *orientation reversing*. If $\mathcal{M}$ is regular with $\mathrm{Aut}\,\mathcal{M} = \langle \lambda, \rho, \tau \rangle$ then the nonorientability of S is reflected by the fact that the even word subgroup $\mathrm{Aut}^{\,+}(\mathcal{M}) = \langle \rho\tau, \tau\lambda \rangle = \langle R, L \rangle = \mathrm{Aut}\,(\mathcal{M})$, where $R = \rho\tau$ and $L = \tau\lambda$. In particular, if there exists an orientation reversing cycle C of length n in $\mathcal{M}$ then there is an associated identity of the form $LR^{m_1}LR^{m_2}\ldots LR^{m_n} = \tau$ in $\mathrm{Aut}\,(\mathcal{M})$. Vice-versa, an existence of such an identity in $\mathrm{Aut}\,(\mathcal{M})$ forces $\mathrm{Aut}\,(\mathcal{M}^+) = \mathrm{Aut}\,(\mathcal{M})$, so $\mathcal{M}$ is nonorientable.

Only for few families of graphs a complete classification of their regular embeddings is known. General, Sabidussi-like, characterisation of graphs admitting a regular embedding can be found in [4]. Regular embeddings of complete graphs K_n were classified by James [5] and Wilson [16]. They exist only if n is 2,3,4 or 6. In contrast to this, regular embeddings of the complete bipartite graphs $K_{n,n}$ (see [1, page 134]) exist for any n. For an odd n, it is shown that (up to isomorphism) there exists exactly one orientable regular embedding of $K_{n,n}$ [10].

Regular embeddings of the n-dimensional cubes Q_n in orientable surfaces are known to exist for any n, see [12, page 821]. In particular, let n be odd and let k be the number of distinct prime divisors of n. Then the construction in [12] gives 2^k nonisomorphic orientable regular embeddings of Q_n. On the other hand, it follows from the classification result proved in [3] that for an odd n, orientable regular embeddings of Q_n are exactly those constructed in [12]. For an even n, the classification of the nonisomorphic orientable regular embeddings of Q_n is not yet complete. It is shown in [9] that there exist at least $2^{\lfloor \frac{n}{4} \rfloor + 1}$ nonisomorphic orientable regular embeddings of Q_n. However, except $n = 2$, no nonorientable regular embeddings are known until now. For $n = 2$, Q_2 is just the 4-cycle and there is a regular embedding of the 4-cycle in the projective plane.

In this paper, we prove that there are no nonorientable regular embeddings of Q_n for any n ($n > 2$). This completes the classification of regular embeddings of Q_n for n odd. In particular, we have the following theorem.

Theorem 1.1 *Let $n > 1$ be an odd integer and let k be the number of different prime divisors of n. Then the number of nonisomorphic regular embeddings of the n-dimensional cube is 2^k.*

The problem to classify regular embeddings of Q_n for n even remains open.

2 Non-existence of nonorientable regular embeddings of Q_n

Let Δ denote the set $\{0, 1, \ldots, n-1\}$. Throughout this paper, we denote the vertex set of Q_n by

$$\mathbf{V} = \mathbb{Z}_2^n = \{\mathbf{a} = (a_0, a_1, \ldots, a_{n-1}) \mid a_i \in \mathbb{Z}_2 = \{0, 1\}\ \},$$

and its edge set by

$$E = \{\{\mathbf{a}, \mathbf{a} + \mathbf{e}_i\} \mid \mathbf{a} \in \mathbf{V}, i \in \Delta\},$$

where $\mathbf{e}_i$ is the element of $\mathbb{Z}_2^n$ whose all components are zero except **the** $(i+1)$-**th** component. Denote by $\tilde{\mathbf{V}}$ the set of translations, i.e. the action of $\tilde{\mathbf{b}} \in \tilde{\mathbf{V}}$ on $\mathbf{V}$ is given by $\mathbf{a}^{\tilde{\mathbf{b}}} = \mathbf{a} + \mathbf{b}$. Further, denote by S_Δ the symmetric group on Δ. Let $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ be a vector in $\mathbf{V}$. Inspecting the induced action of $\mathrm{Aut}\,(Q_n)$ on the set of all 4-cycles of Q_n one can easily see that the action of a vertex-stabiliser $(\mathrm{Aut}\,(Q_n))_v$ is faithful on the set of neighbours of v. Since $(\mathrm{Aut}\,(Q_n))_v$ acts faithfully on the set of neighbours of v, the (full) automorphism group of Q_n is isomorphic to the semidirect product $\tilde{\mathbf{V}} \ltimes S_\Delta$, where S_Δ acts on $\tilde{\mathbf{V}}$ by permuting its components as follows: $\tilde{\mathbf{a}}^\alpha = \tilde{\mathbf{b}}$ with

$$\mathbf{b} = \mathbf{a}^\alpha = (a_{\alpha^{-1}(0)}, a_{\alpha^{-1}(1)}, \ldots, a_{\alpha^{-1}(n-1)}),$$

see [2, page 261]. Note that we have the following commuting rules:

$$\tilde{\mathbf{a}}\alpha = \alpha\tilde{\mathbf{a}}^{\alpha}, \quad \alpha\tilde{\mathbf{a}} = \tilde{\mathbf{a}}^{\alpha^{-1}}\alpha.$$

It follows that each element of $\mathrm{Aut}\,(Q_n)$ can be uniquely expressed as a product $\tilde{\mathbf{a}}\alpha$, where $\tilde{\mathbf{a}} \in \tilde{\mathbf{V}}$ and $\alpha \in S_\Delta$.

Denote by ρ and τ the involutory permutations acting on Δ defined as follows: $\rho(k) = -k-1 \pmod n$, $\tau(k) = -k \pmod n$. The faithfulness of the action of a vertex stabiliser of $\mathrm{Aut}\,(Q_n)$ allows us to identify the involutions y, z defining a regular map $\mathcal{M}(x,y,z)$ of Q_n with some involutions in the symmetric group S acting on Δ. However, since $\mathrm{Aut}\,(Q_n) = \tilde{\mathbf{V}} \ltimes S_\Delta$, the elements of S_Δ are naturally embedded in the automorphism group of Q_n. It depends on the point of view whether the above product is considered to be external or internal.

Lemma 2.1 *Let $\mathcal{M} = \mathcal{M}(x,y,z)$ be a regular map whose underlying graph is Q_n. Then $\mathcal{M}(x,y,z) \cong \mathcal{M}(\tilde{\mathbf{e}}_\mathbf{0}\sigma, \rho, \tau)$, for some $\sigma \in S_\Delta$ satisfying the conditions $\sigma(0) = 0$, $\sigma^2 = id$ and $\sigma(-k) = -\sigma(k)$ for all $k \in \Delta$.*

Proof: By the transitivity of the action of $\mathrm{Aut}\,(M)$ on flags we may choose a representation $\mathcal{M}(x,y,z)$ with respect to the flag (v,e,f), where the vertex v is $\mathbf{0}$, the edge e is $\{\mathbf{0}, \mathbf{e_0}\}$ and f is one of the two faces incident to both v and e. Then the product yz cyclically permutes the neighbours of $\mathbf{0}$ as follows: $(\mathbf{e_0}, \mathbf{e_{i_1}}, \mathbf{e_{i_1}}, \ldots, \mathbf{e_{i_{n-1}}})$. Since y and z are map automorphisms fixing the vertex $\mathbf{0}$, y is locally an axial symmetry interchanging the edges $\{\mathbf{0}, \mathbf{e_0}\}$, $\{\mathbf{0}, \mathbf{e_{i_{n-1}}}\}$, while z is locally an axial symmetry fixing the edge $\{\mathbf{0}, \mathbf{e_0}\}$ and transposing the edges $\{\mathbf{0}, \mathbf{e_{i_{n-1}}}\}$, $\{\mathbf{0}, \mathbf{e_{i_1}}\}$. Recall, that the stabiliser of $\mathbf{0}$ in $Aut(Q_n)$ acts faithfully on the neighbours of $\mathbf{0}$. It follows that y and z viewed as members of S_Δ are defined by

$$y = (0, i_{n-1})(i_1, i_{n-2})(i_2, i_{n-3})\cdots \quad \text{and} \quad z = (0)(i_1, i_{n-1})(i_2, i_{n-2})\cdots.$$

Moreover, since $Aut(Q_n) = \tilde{\mathbf{V}} \ltimes S_\Delta$ and x inverts the edge $\{\mathbf{0}, \mathbf{e_0}\}$ the involutory automorphism x can be expressed as $x = \tilde{\mathbf{e}}_\mathbf{0}\beta$, for some $\beta \in S_\Delta$. Let $\phi \in S_\Delta$ be the permutation fixing 0 and taking $i_j \mapsto j$ for any $j = 1, \ldots, n-1$.

Then,

$$\begin{aligned} x^\phi &= \tilde{\mathbf{e}}_\mathbf{0}^\phi \beta^\phi = \tilde{\mathbf{e}}_\mathbf{0}\sigma, \quad y^\phi = (0, n-1)(1, n-2)(2, n-3)\cdots = \rho \ \text{ and} \\ z^\phi &= (0)(1, n-1)(2, n-2)\cdots = \tau, \end{aligned}$$

for some $\sigma = \beta^\phi \in S_\Delta$. The above three equalities establish $\mathcal{M}(x,y,z) \cong \mathcal{M}(\tilde{\mathbf{e}}_\mathbf{0}\sigma, \rho, \tau)$ via the map isomorphism $\phi \in S_\Delta \leq Aut(Q_n) = \tilde{\mathbf{V}} \ltimes S_\Delta$.

Since $id = (\tilde{\mathbf{e}}_\mathbf{0}\sigma)^2 = \tilde{\mathbf{e}}_\mathbf{0}\tilde{\mathbf{e}}_{\sigma(0)}\sigma^2$, it follows that $\sigma(0) = 0$ and $\sigma^2 = id$. Moreover, the identity $\tilde{\mathbf{e}}_\mathbf{0}\sigma\tau = \tau\tilde{\mathbf{e}}_\mathbf{0}\sigma$ implies $-\sigma(k) = \sigma(-k)$ for any $k \in \Delta$. □

Let us call the above triple $(\tilde{\mathbf{e}}_\mathbf{0}\sigma, \rho, \tau)$ of generators a *canonical triple* of $\mathcal{M}$. Given $n \geq 3$, the permutations ρ and τ are fixed as well as the translation $\tilde{\mathbf{e}}_\mathbf{0}$. It follows that a regular embedding of Q_n is determined by an involutory permutation σ in S_Δ.

Recall that subgraphs of a graph G whose connected components are Eulerian graphs form a vector space over Z_2 called a *cycle space* $\mathcal{C}(G)$ of G. Here a subgraph F is interpreted as a vector over Z_2 identifying the edge-set of F. In what follows we shall call elements of the cycle space *cycles*. Alternatively, the cycle space is called a *circuit space* in [13, page 539] while its elements are called *circs* there. More information on cycle (circuit) spaces can be found in [13].

Let G be embedded into a nonorientable surface. A (simple) cycle is orientation preserving, or positive, if its regular neighbourhood is homeomorphic to a cylinder, otherwise it is called orientation reversing (or negative). Hence we have an assignment ω taking values in the 2-element multiplicative group $\{1, -1\}$ and associating to every (simple) cycle one of the two numbers. It is easy to see that ω extends to all elements of the cycle space such that we have a morphism $\omega : \mathcal{C}(G) \to \{1, -1\}$. The kernel of ω formed by positive "cycles" is a subspace of index 2 in $\mathcal{C}(G)$.

The following lemma turns out to be useful.

Lemma 2.2 *The set of 4-cycles is a generating set for $\mathcal{C}(Q_n)$. In particular, any cycle C in Q_n can be expressed as a sum of 4-cycles in Q_n.*

Proof: Given vector $\mathbf{v}$ denote by $||\mathbf{v}||$ the sum of its coordinates and for $C \in \mathcal{C}(Q_n)$ denote by $||C||$ maximum of $||\mathbf{v}|| - ||\mathbf{u}||$ where the maximum is taken through all pairs of vertices $\mathbf{v}$, $\mathbf{u}$ incident to edges of C. If C contains no edges we set $||C|| = 0$. We proceed by induction with respect to $||C||$. The statement is clearly true for trivial C with $||C|| = 0$. Assume the statement is true for all C with $||C|| < d$. Let $||C|| = d$. Each connected component of C is an Eulerian graph so we have a sequence of its vertices determined by an Eulerian trail. Let us fix such a trail for each connected component. Let us consider the set of all subpaths $\mathbf{wvu}$ of length two of the above trails with the property that the central vertex $\mathbf{v}$ has maximal $||\mathbf{v}||$. The central vertices of the above subpaths of length two form a subsequence in each connectivity component of C (ordered by an order induced by the Eulerian trial C). Concatenating these subsequences we get a sequence $\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_s}$ of the vertices incident to edges of C with the maximal sum of coordinates $||\mathbf{v}||$. Note that each such a vertex v appears in the sequence $p/2$ times where p is the degree of v in C. By definition of the adjacency relation no two of these vertices are adjacent in Q_n. For any $r = 1, 2, \ldots, s$, there is a unique path of length 2 of the form $\mathbf{a_r} + \mathbf{e_{i_r}}, \mathbf{a_r}, \mathbf{a_r} + \mathbf{e_{j_r}}$ determined by the respective Eulerian trail. Denote by $C(r)$ the 4-cycle $C(r) = (\mathbf{a_r} + \mathbf{e_{i_r}}, \mathbf{a_r}, \mathbf{a_r} + \mathbf{e_{j_r}}, \mathbf{a_r} + \mathbf{e_{i_r}} + \mathbf{e_{j_r}}, \mathbf{a_r} + \mathbf{e_{i_r}})$. Then, the cycle-sum $C + \sum_{r=1}^{s} C(r)$ is a cycle C' such that $||C'|| < d$. By induction hypothesis, C' is a cycle-sum of 4-cycles in Q_n. Hence, C can be expressed as a cycle-sum of 4-cycles in Q_n as well. □

We have the following corollary.

Corollary 2.3 *For any embedding $i : Q_n \hookrightarrow S$ of Q_n into some nonorientable surface S, there exists a 4-cycle in Q_n which is orientation reversing with respect to i.*

Proof: Assume that the statement is not true. By Lemma 2.2, any cycle C is a cycle-sum of 4-cycles in Q_n. Consequently, every cycle C of Q_n is orientation preserving, a contradiction with the nonorientability of the embedding. □

Now we are ready to prove the nonexistence of nonorientable regular embeddings of Q_n.

Theorem 2.4 *For any $n > 2$ there is no nonorientable regular embedding of Q_n.*

Proof: Suppose that $\mathcal{M}$ is a map determined by a nonorientable regular embedding of Q_n. By Lemma 2.1 $\mathcal{M}$ can be represented by a canonical triple of involutory automorphisms $(\tilde{\mathbf{e}}_\mathbf{0}\sigma, \rho, \tau)$, where $\rho = (0, n-1)(1, n-2)(2, n-3)\cdots$ and $\tau = (0)(1, n-1)(2, n-2)\cdots$ of Q_n. Note that $\rho\tau = (0, 1, \ldots, n-1)$. For the convenience, let $R = \rho\tau = (0, 1, \ldots, n-1)$ and $L = \tilde{\mathbf{e}}_\mathbf{0}\sigma\tau = \tilde{\mathbf{e}}_\mathbf{0}\alpha$, where $\alpha = \sigma\tau$. In the following computations the elements of Δ are considered as elements of the additive group Z_n and the operators "+" and "−" are considered modulo n. It is easily checked that $\alpha(0) = 0$, $\alpha^2 = 1$ and $\alpha(-k) = -\alpha(k)$ for all $k \in \Delta$.

Since $\mathcal{M}$ is nonorientable, $\langle L, R\rangle = \langle \tilde{\mathbf{e}}_\mathbf{0}\sigma\tau, \rho\tau\rangle = \langle \tilde{\mathbf{e}}_\mathbf{0}\sigma, \rho, \tau\rangle$. By Corollary 2.3, there exists an orientation reversing 4-cycle C in Q_n. By the regularity of the action on the set of flags there is an associated identity in Aut$(\mathcal{M})$ of the form

$$\tilde{\mathbf{e}}_\mathbf{0}\alpha R^a \tilde{\mathbf{e}}_\mathbf{0}\alpha R^b \tilde{\mathbf{e}}_\mathbf{0}\alpha R^c \tilde{\mathbf{e}}_\mathbf{0}\alpha R^d = \tau,$$

for some $a, b, c, d \in \Delta - \{0\}$. Using the commuting rules we rewrite the left hand side as follows:

$$\tilde{\mathbf{e}}_\mathbf{0}\alpha R^a \tilde{\mathbf{e}}_\mathbf{0}\alpha R^b \tilde{\mathbf{e}}_\mathbf{0}\alpha R^c \tilde{\mathbf{e}}_\mathbf{0}\alpha R^d = \tilde{\mathbf{e}}_\mathbf{0}\tilde{\mathbf{e}}_{\alpha(a)}\tilde{\mathbf{e}}_{\alpha(\alpha(b)+a)}\tilde{\mathbf{e}}_{\alpha(\alpha(\alpha(c)+b)+a)}\alpha R^a \alpha R^b \alpha R^c \alpha R^d.$$

Since the translation part of τ is trivial, we have $\tilde{\mathbf{e}}_{\mathbf{0}}\tilde{\mathbf{e}}_{\alpha(a)}\tilde{\mathbf{e}}_{\alpha(\alpha(b)+a)}\tilde{\mathbf{e}}_{\alpha(\alpha(\alpha(c)+b)+a)} = id$. So, we get $\alpha(\alpha(b)+a) = 0$ and $\alpha(\alpha(\alpha(c)+b)+a) = a$ because none of a, b and c is 0. Hence, $b = -\alpha(a)$ and $c = -\alpha(b) = a$. It follows that

$$\alpha R^a \alpha R^{-\alpha(a)} \alpha R^a \alpha R^d = (0)(1, n-1)(2, n-2)\cdots = \tau.$$

The image of 0 in the left permutation is $0^{\alpha R^a \alpha R^{-\alpha(a)} \alpha R^a \alpha R^d} = \alpha(a) + d$ and it is equal to $0^\tau = 0$. Hence, $d = -\alpha(a)$ and we have

$$\alpha R^a \alpha R^{-\alpha(a)} \alpha R^a \alpha R^{-\alpha(a)} = (\alpha R^a \alpha R^{-\alpha(a)})^2 = (0)(1, n-1)(2, n-2)\cdots = \tau.$$

Direct calculation gives us $(-\alpha(a))^\tau = (-\alpha(a))^{(\alpha R^a \alpha R^{-\alpha(a)})^2} = -\alpha(a)$. Since $a \neq 0$, n is even and $\alpha(a) = \frac{n}{2}$. The fact that $\alpha(\frac{n}{2}) = \alpha(-\frac{n}{2}) = -\alpha(\frac{n}{2})$ implies that $a = \alpha(\frac{n}{2}) = \frac{n}{2}$. It follows that $(\alpha R^{\frac{n}{2}})^4 = \tau$. Moreover, for any $f \neq \frac{n}{2}$, $(\alpha R^f \alpha R^{-\alpha(f)})^2$ is the identity because $(\alpha R^f \alpha R^{-\alpha(f)})^2 \in \langle \tau, \rho \rangle$ fixes 0 and $-\alpha(f)$.

Take $f = 1$. Then, the image of $\frac{n}{2}$ under the identity $(\alpha R \alpha R^{-\alpha(1)})^2$ is

$$\begin{aligned}
\left(\frac{n}{2}\right)^{(\alpha R \alpha R^{-\alpha(1)})^2} &= \left(\alpha\left(\frac{n}{2}+1\right) - \alpha(1)\right)^{\alpha R \alpha R^{-\alpha(1)}} \\
&= \alpha\left(\alpha\left(\alpha\left(\frac{n}{2}+1\right) - \alpha(1)\right) + 1\right) - \alpha(1) = \frac{n}{2}.
\end{aligned}$$

The first and the second equalities are extracted from the facts that $\alpha(\frac{n}{2}) = \frac{n}{2}$ and for any $k, t \in \Delta$, $k^\alpha = \alpha(k)$ and $k^{R^t} = k + t$.

Since $\alpha^2 = 1$ and $\alpha(-k) = -\alpha(k)$ for any $k \in \Delta$, the above equation implies that

$$\alpha\left(\alpha(1) + \frac{n}{2}\right) + \alpha\left(\alpha(1) - \alpha\left(1 + \frac{n}{2}\right)\right) = 1 \tag{1}$$

Take $f = 1 + \frac{n}{2}$. The image of $\frac{n}{2}$ under the identity $\left(\alpha R^{1+\frac{n}{2}} \alpha R^{-\alpha(1+\frac{n}{2})}\right)^2$ is

$$\begin{aligned}
\left(\frac{n}{2}\right)^{(\alpha R^{1+\frac{n}{2}} \alpha R^{-\alpha(1+\frac{n}{2})})^2} &= \left(\alpha(1) - \alpha\left(1 + \frac{n}{2}\right)\right)^{\alpha R^{1+\frac{n}{2}} \alpha R^{-\alpha(1+\frac{n}{2})}} \\
&= \alpha\left(\alpha\left(\alpha(1) - \alpha\left(1 + \frac{n}{2}\right)\right) + 1 + \frac{n}{2}\right) - \alpha\left(1 + \frac{n}{2}\right) = \frac{n}{2}.
\end{aligned}$$

This equation means that

$$\alpha\left(\alpha\left(1 + \frac{n}{2}\right) + \frac{n}{2}\right) + \alpha\left(\alpha\left(1 + \frac{n}{2}\right) - \alpha(1)\right) = 1 + \frac{n}{2} \tag{2}$$

On the other hand, the image of $\alpha(1)$ under $(\alpha R^{\frac{n}{2}})^4 = \tau$ is

$$\begin{aligned}
\alpha(1)^{(\alpha R^{\frac{n}{2}})^4} &= \left(1 + \frac{n}{2}\right)^{(\alpha R^{\frac{n}{2}})^3} = \left(\alpha\left(1 + \frac{n}{2}\right) + \frac{n}{2}\right)^{(\alpha R^{\frac{n}{2}})^2} \\
&= \alpha\left(\alpha\left(\alpha\left(1 + \frac{n}{2}\right) + \frac{n}{2}\right) + \frac{n}{2}\right) + \frac{n}{2} = -\alpha(1).
\end{aligned}$$

Hence, the following equation holds:

$$\alpha\left(\alpha\left(1+\frac{n}{2}\right)+\frac{n}{2}\right)+\alpha\left(\alpha(1)+\frac{n}{2}\right)=\frac{n}{2} \tag{3}$$

Using equation (2), we get

$$\alpha\left(\alpha\left(1+\frac{n}{2}\right)+\frac{n}{2}\right)=-\alpha\left(\alpha\left(1+\frac{n}{2}\right)-\alpha(1)\right)+1+\frac{n}{2}.$$

Inserting the above equation into (3), we get the following:

$$-\alpha\left(\alpha\left(1+\frac{n}{2}\right)-\alpha(1)\right)+1+\frac{n}{2}+\alpha\left(\alpha(1)+\frac{n}{2}\right)=\frac{n}{2}$$

which is equivalent to the equation

$$\alpha\left(\alpha\left(1+\frac{n}{2}\right)-\alpha(1)\right)-\alpha\left(\alpha(1)+\frac{n}{2}\right)=1. \tag{4}$$

However, using (1) and (4), we derive

$$\begin{aligned} 1 &= \alpha\left(\alpha(1)+\frac{n}{2}\right)+\alpha\left(\alpha(1)-\alpha\left(1+\frac{n}{2}\right)\right) \\ &= -\left(\alpha\left(\alpha\left(1+\frac{n}{2}\right)-\alpha(1)\right)-\alpha\left(\alpha(1)+\frac{n}{2}\right)\right)=-1 \end{aligned}$$

which is impossible for $n>2$. □

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