CIRCULANTS AND THE CHROMATIC INDEX OF STEINER TRIPLE SYSTEMS

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Abstract. We complete the determination of the chromatic number of 6-valent circulants of the form $C(n; a, b, a+b)$ and show how this can be applied to improving the upper bound on the chromatic index of cyclic Steiner triple systems.

Introduction

A *Steiner triple system* $STS(v)$ is a pair (V, B), where V is a set of v points and B is a collection of sets of cardinality 3, called triples or blocks, satisfying the following condition: each pair x, y of points is contained in exactly one triple. It is well known that a Steiner triple system on v points exists if and only if $v \equiv 1$ or 3 mod 6.

A *block-color class* is a system of pairwise disjoint triples. An m-block-coloring is a partitioning of the set B into m color classes. The *chromatic index* $\chi'(S)$ of a Steiner triple system S is the least m for which an m-block-coloring exists. A *block intersection graph* of a Steiner triple system $S = (V, B)$ is a graph with the vertex set B; the vertices are adjacent if and only if the respective triples intersect. Since the degree v of a block intersection graph equals $3(v-3)/2$, Brooks' theorem (see, e.g., [10]) gives an upper bound $\chi'(S) \leq 3(v-3)/2$ for $v > 7$. An obvious lower bound is $\chi'(S) \geq (v-1)/2$ if $v \equiv 3$ mod 6, and $(v+1)/2$ for $v \equiv 1$ mod 6. Hence we have $(v-1)/2 \leq \chi'(S) \leq 3(v-3)/2$ if $v \equiv 3$ mod 6 and $(v+1)/2 \leq \chi'(S) \leq 3(v-3)/2$ if $v \equiv 1$ mod 6. The lower bound $\chi'(S) = (v-1)/2$ is reached if and only if the Steiner triple system is resolvable.

The upper bound $\chi'(S) \leq 3(v-3)/2$ seems to be weak in general. In fact, using probabilistic methods Pippenger and Spencer in [8] proved that

$\chi'(STS(v))$ is asymptotic to $v/2$. Also no examples of STS with $v > 7$ exceeding the above lower bounds by more than two are known. For more information on the chromatic index of Steiner triple systems the reader is referred to Chapter 18 of [4].

For some classes of STS the upper bound was improved. In particular, Colbourn and Colbourn [3] improved it for cyclic $STS(v)$ by proving $\chi'(STS(v)) \leq v$. A Steiner triple system $STS(v)$ is called *cyclic* if it is isomorphic to one whose points are $0, 1, \ldots, v-1$ and the mapping $i \mapsto i+1$ mod v is an automorphism. The result in [3] is based on the following idea. Let $S = STS(v)$ be a cyclic Steiner triple system. The block intersection graph has $v(v-1)/6$ vertices and it admits an induced action of the cyclic group of order v. Then the orbits of the induced action decompose the intersection graph into $(v-1)/6$ six-valent circulants of order v if $v \equiv 1$ mod 6, and into $(v-3)/6$ six-valent circulants of order v and one short orbit if $v \equiv 3$ mod 6. By Brooks' theorem each of the circulants (with one exception) can be colored by 6 colors. Taking different sets of colors for different orbits one gets a v-block coloring of any cyclic STS. Using some known results on the chromatic number of toroidal triangulations we derive that with two exceptions the 6-valent circulants coming from a cyclic $STS(v)$ can be 5-colored. Consequently, the upper bound for the chromatic index can be improved. Furthermore, we discuss colorability of 6-valent circulants, in which one of the three generators is the sum of the other two. The results are applied to get better upper bounds on the chromatic index for some classes of cyclic $STS(v)$.

Coloring 6-valent circulants with connection set $\{a, b, a+b\}$

It is not difficult to see that the induced subgraph of an orbit of length v in a Steiner triple system $STS(v)$ in the block intersection graph is a circulant $G = C(n; a, b, c)$, where one of the generators, say c, is the sum of the others (up to replacing c by $n - c$). In what follows we assume w.l.o.g. $c = a + b$. We may think of each edge as being colored by one of the generators. A triangle of G is *3-edge-colored* if its edges are colored by the three colors a, b and c. Each edge lies in exactly two 3-edge-colored triangles. The six 3-edge-colored triangles based at a vertex j induce a cyclic rotation of the neighbours; up to replacing some of a, b, c by $-a, -b, -c$, it has a form $(j+a, j+b, j+c, j-a, j-b, j-c)$ for each vertex $j \in Z_n$. Clearly the 3-edge-colored cycles as well as the local rotation of colors $(a, b, c, -a, -b, -c)$ are preserved by the cyclic automorphism $j \mapsto j+1$. By Euler-Poincare theorem a component of G is a one-skeleton of a vertex-transitive triangulation of the torus. Note that there are $n/gcd(a,b,c)$ components of G, and they are all isomorphic to $C = (n'; a', b', c')$, where $n' = n/d$, $a' = a/d$, $b' = b/d$, $c' = c/d$ for $d = \gcd(a,b,c)$. Many papers were devoted to the problem to determine the chromatic number of triangulations of surfaces satisfying certain conditions. Here we mention only few of those which are closely related to our problem.

Note that the complete graph K_7 can be expressed as $K_7 \cong C(7; 1,2,3)$, and let T_{11} be the circulant $C(11; 1,2,3)$. It is important to note that the chromatic number of K_7 is seven and the chromatic number of T_{11} is 6. Albertson and Hutchinson [1] (see [9] as well) proved that these two graphs are the only 6-valent graphs on the torus which cannot be 5-colored.

Theorem 1. *[Theorem 4.4, [1]] Let G be a toroidal 6-valent graph other than K_7 and T_{11}. Then G is 5-colorable.*

Corollary 2. *Let $S = STS(v)$ be a cyclic Steiner triple system. Then*

(a) $\chi'(S) \leq \frac{5v+13}{6}$ if $v \equiv 1$ mod 6,

(b) $\chi'(S) \leq \frac{5v+9}{6}$ if $v \equiv 3$ mod 6.

Proof. As already mentioned the vertices of the block intersection graph decompose into $(v-1)/6$ orbits of length v if $v \equiv 1$ mod 6, and into $(v-3)/6$ orbits of length v and one short orbit if $v \equiv 3$ mod 6. The subgraph induced on a full orbit is a six-valent circulant $C(v; a, b, c)$ with $c = a+b$ for some a, b. A connectivity component of such a circulant triangulates the torus. By Theorem 1 each such circulant can be 5-colored except when it is a disjoint union of K_7's, or a disjoint union of T_{11}'s. This can happen only if $7|v$ or $11|v$, respectively. Moreover, at most one orbit can be a union of K_7's and at most one orbit can be a union of T_{11}'s. We color the vertices of the block intersection graph by using distinct sets of five colors for each full orbit which is neither a union of K_7's, nor a union of T_{11}'s. For the latter two orbits, if they appear, we use 7 and 6 colors, respectively. Finally, in case $v \equiv 3$ mod 6 we use one color to color the short orbit. By adding the number of colors used we get the upper bounds. □

The bound in Corollary 2(b) cannot be improved in general by the method used in the proof of Corollary 2. To see this, let $v = 3p$, p a prime, such that

(*) $p \equiv 3$ $(mod\ 4)$, the order of 2 mod $3p$ is $p-1$, and the order of 2 mod p is $\frac{p-1}{2}$.

Now, for each prime p satisfying (*) and $v = 3p$, there exist cyclic Steiner triple systems of order v such that for each full orbit of triples the corresponding 6-valent circulant is 5-chromatic, so that the bound of Corollary 2 becomes $\chi'(S) \leq \frac{5v-9}{9}$. Indeed, the set $\{\{2^{2i}, 2.2^{2i}, 3.2^{2i}\} : i = 0, 1, \ldots, \frac{v-3}{6}$ is a solution to the 2nd Heffter's difference problem (cf. [4]); note the crucial fact that $\frac{p-1}{2}$ is odd. Clearly, all circulants corresponding to the full orbits of any cyclic STS(v) whose difference triples are as above are multiplier-isomorphic to $C(v; 1, 2, 3)$.

The first few values of v satisfying condition (*) are $v = 21, 69, 141, 213, 237, 309, 501$... It is not known whether there are infinitely many primes p satisfying (*) although this is likely. Let us note that there exist cyclic STSs of other orders $\equiv 3$ $(mod\ 6)$ (e.g. $v = 33$), or even of orders $\equiv 1$ $(mod\ 6)$ (e.g. $v = 37$) for which the circulants corresponding to any full orbit are 5-chromatic.

On the other hand, for many cyclic STSs our method gives a much better upper bound. For example, for the projective STS of order 31 (whose blocks are the lines of PG(4,2)), the bound of Corollary 2 yields the value of 20. We found an 18-block-colouring of PG(4,2) which implies that the chromatic index of PG(4,2) equals 18 ($\chi' \geq 18$ follows from Wilson's argument, see [4], p.345) The spectrum of values for bounds obtained by adding the chromatic numbers of circulants corresponding to the individual orbits of STSs of small orders is given in the table below.

Table

Order	Spectrum
7	7
9	-
13	10
15	9
19	12,15
21	13,14,16
25	18,20
27	18
31	20,21,22,23
33	21,22,23,25,26
37	24,25,26,27,28,30

39	23,24,25,26,27,28,29
43	28,29,30,31,32,33,34
45	26,27,28,29,30,31

Let $min_v = \frac{v-1}{2}$ if $v \equiv 3 \ (mod\ 6)$ and $min_v = \frac{v+1}{2}$ if $v \equiv 1 \ (mod\ 6)$. We checked on a computer that for all *cyclic* Steiner triple systems of order $v \leq 43$ (note that the number of cyclic Steiner triple systems of order 43 is known to be 9508, not a small number [4]) , we have for the chromatic index the bounds $min_v \leq \chi' \leq min_v + 2$. This suggests that the following could be true:

Conjecture. The chromatic index of every cyclic STS(v), $v > 7$, takes on one of three values: min_v, $min_v + 1$, $min_v + 2$.

It would be tempting to conjecture that the same holds for *any* Steiner triple system but admittedly the evidence for such a claim is scarce.

Let us note that the problem of determining the chromatic number of general circulants is NP-hard [2]. However, we are able to say more in the special case that we consider, namely for the 6-valent circulants $C(n; a, b, a+b)$.

Some recent results proved by Collins and Hutchinson [6] and in more general setting by Hutchinson, Richter and Seymour [7] suggest that most of the considered circulants are even 4-colorable. Indeed, Yeh and Zhu [Theorem 5, [11]] proved the following statement conjectured by Collins, Fisher and Hutchinson in [5]:

Theorem 3. *(Yeh,Zhu [11]) Let $G = C(n; 1, r, r+1)$, $n \geq 2(r+1) \geq 6$ be a circulant (possibly with multiple edges for $n = 2(r+1)$). Then G is 4-colorable except when one of the following cases occurs:*

(1) $n = 2r+2$, $n = 2r+3$, $n = 3r+1$ or $3r+2$ and n is not divisible by 4,

(2) $r = 2$ and 4 does not divide n,

(3) (r, n) is one of $(3, 13)$, $(3, 17)$, $(3, 18)$, $(3, 25)$, $(4, 17)$, $(6, 17)$, $(6, 25)$, $(6, 33)$, $(7, 19)$, $(7, 25)$, $(7, 26)$, $(9, 25)$, $(10, 25)$, $(10, 26)$, $(10, 37)$, $(14, 33)$.

Using the above theorem one can classify k-chromatic 6-valent circulants for all admissible integers k, $3 \leq k \leq 7$ as follows.

Theorem 4. *Let $G = C(n; a, b, c)$ be a connected 6-valent circulant, where $n \geq 7$, $c = a + b$ or $n - c = a + b$ are pairwise distinct positive integers different from $n/2$. Let $\chi(G)$ be the chromatic number of G. Then*

(1) $\chi(G) = 7$ if and only if $G \cong K_7 \cong C(7; 1, 2, 3)$,

(2) $\chi(G) = 6$ if and only if $G \cong T_{11} \cong C(11; 1, 2, 3)$,

(3) $\chi(G) = 5$ if and only if $G \cong C(n; 1, 2, 3)$ and $n \neq 7, 11$ is not divisible by 4, or G is isomorphic to one of the following circulants: $C(13; 1, 3, 4)$, $C(17; 1, 3, 4)$, $C(18; 1, 3, 4)$, $C(19; 1, 7, 8)$, $C(25; 1, 3, 4)$, $C(26; 1, 7, 8)$, $C(33; 1, 6, 7)$, $C(37; 1, 10, 11)$.

(4) $\chi(G) = 3$ if and only if $3|n$ and none of a, b, c is divisible by 3.

(5) $\chi(G) = 4$ in all the remaining cases.

Proof. (1) follows from Brooks' theorem. (2) comes from Albertson and Hutchinson [1]. It is easy to see that none of the circulants listed in (3) is 4-colorable. Assume G is not 4-colorable. If $\gcd(n, a) = 1$, $\gcd(n, b) = 1$ or $\gcd(n, c) = 1$ then G is (multiplier-) isomorphic to $C(n; 1, r, r + 1)$. Applying Theorem 3 and reducing the number of exceptions by taking representatives up to isomorphism we get the list of 5-chromatic six-valent circulants. Assume the greatest common divisors $\gcd(n, a)$, $\gcd(n, b)$ and $\gcd(n, c)$ to be all > 1. We show that in this case G can be 4-colored. Let, say, $\gcd(n, a) = d > 1$. Since each of the pairs (a, b), (b, c), (a, c) generates Z_n, it follows that at most one of $\gcd(a, n)$, $\gcd(b, n)$ and $\gcd(c, n)$ is even. Hence we may assume that both $\gcd(a, n) = d$ and $\gcd(b, n) = k$ are odd. Clearly $db \in \langle a \rangle$, hence $db = xa$ for some x, $0 \leq x \leq m - 1$, where $m = n/d$. If $\gcd(x, m) = 1$ then $k = \gcd(n, b) = 1$, a contradiction. Moreover, $\gcd(m, x) \neq 2$ because $\gcd(m, x) = 2$ would imply $|b| = n/2 = n/\gcd(b, n)$, forcing $\gcd(b, n) = 2$ contradicting the assumptions. Hence, $\gcd(m, x) > 2$. Let $x = 0$. Then the circulant G forms a so-called right diagonal $m \times d$ grid, $m, d \geq 3$ which by Theorem 3.3 in [6] is 4-colorable. If $x \neq 0$ then consider the factor of G by $\langle xa \rangle$. It is a 6-valent circulant $\bar{G} = C(\bar{n}; \bar{a}, \bar{b}, \bar{c})$ such that one of the generators is the sum of the other two, $\langle \bar{a} \rangle$ is a subgroup of index d and of cardinality $\bar{m} = \bar{n}/d \geq 3$. Moreover, $d\bar{b} = 0$, and consequently, the order of $\bar{b}$ is d. As above the 4-colorability of $\bar{G}$ follows from Theorem 3.3 in [6]. The 4-coloring of the original circulant G can be constructed by lifting the 4-coloring of $\bar{G}$ along the covering $G \to \bar{G}$.

If $3|n$ and none of the generators is divisible by 3 then the assigment $j \mapsto \bar{j}$, where $\bar{j}$ is the residue class of j mod 3 defines a 3-coloring of the circulant G.

Assume there is a 3-coloring of G. If one of the $\gcd(n,a)$, $\gcd(n,b)$ or $\gcd(n,c)$ equals one, then the circulant is isomorphic to $C(n;1,r,r+1)$ for some $1 < r < r+1 < n/2$. The vertices 0, r, $r+1$ are colored by three different colors, say A,B and C, respectively. It follows that the vertex 1 is colored by B, vertex $r+2$ by $A, \ldots$. It is easy to see that this coloring extends to a coloring of G if and only if $3|n$ and $r \equiv 1$ mod 3. Hence, all the generators are not $\equiv 0$ mod 3, and this property remains valid when multiplying by integers coprime to n as well as when replacing generators by their inverses. Assume now that none of $\gcd(n,a)$, $\gcd(n,b)$, $\gcd(n,c)$ equals one. If, say, $\gcd(n,a) = d > 1$ then considering 3-colorability of the subgraph H induced by $\langle a\rangle \cup (b + \langle a\rangle)$ we see that H can be 3-colored only if $3|n/d$. Hence $3|n$. Moreover, the 3-coloring is unique up to a permutation of colors. Since a is a generator of a cyclic subgroup of order divisible by 3, it is coprime to 3. The same argument applies to b and c, proving that all a, b and c are coprime to 3. $\square$

Let us call circulants of type (1), (2) or (3) from preceding Theorem *exceptional.*

Corollary 5. Let $S = STS(v)$ be a cyclic Steiner triple system of order v, and assume that no circulant induced on an orbit of the block intersection graph contains as a component an exceptional circulant. Then $\chi'(S) \leq \frac{2}{3}(v-1)$ if $v \equiv 1$ $(mod\ 6)$, and $\chi'(S) \leq \frac{2}{3}v - 1$ if $v \equiv 3$ $(mod\ 6)$.

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