

# A classification of regular embeddings of graphs of order a product of two primes

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## Abstract

In [5], the classification problem of regular embeddings of a given graph was described in terms of pure group theory. With the philosophy in [5], we shall classify the regular embeddings of simple graphs of order  $pq$  for any two primes  $p$  and  $q$  (not necessarily distinct) in this paper (see Theorem 4.8). The classification is based on the direct analysis of the structure of the arc-regular subgroups with the cyclic stabilizers of the automorphism groups of such graphs. Our analysis is independent of the classification of primitive permutation groups of degree  $p$  or degree  $pq$ . It is also independent of the classification of the arc-transitive graphs of order  $pq$  ( $p \neq q$ ).

## 1 Introduction

A (topological) *map* is a cellular decomposition of an orientable closed surface. A common way to describe maps is to view them as 2-cell embeddings of graphs. An *automorphism* of a map is an automorphism of the underlying graph which extends to an orientation preserving self-homeomorphism of the supporting surface. It is well-known that the automorphism group of a map acts semiregularly on the set of arcs of the underlying graph and in the extreme case, when the action is regular, the map itself is called *regular*. A *regular embedding* of a graph is a 2-cell embedding of a graph into a surface in a way that the associated map is regular. Regular maps were studied in connection with various branches of mathematics

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including Riemann surfaces and algebraic curves. For more information about regular maps and their connections to other fields of mathematics we refer the reader to [11, 12, 13].

One of the research problems for regular maps is to classify the regular embeddings of a given underlying graph. Generally, it is very difficult. Regular embeddings of the complete graph  $K_n$  are classified in [1, 10, 23]. It is known [6] that the automorphism group of a regular map is a two-generator group acting with a cyclic stabilizer, and conversely, any two-generator group with one generator being involution determines a regular map (depending on the choice of the generators one can derive, and one usually derives, more than one regular maps from such a group). In [5], the regular maps (so called *algebraic maps*) with a simple underlying graph are described by the coset graphs, and then their classification problem is transferred into a pure group theoretical problem. With such coset graph description, the authors [5] classified the regular embeddings of complete  $n$ -multipartite graphs  $K_{p,p,\dots,p}$ , where  $p$  is a prime and  $n$  is a positive integer.

When classifying regular embeddings of graphs, the technique often used is to project a given map onto a smaller one, and then to employ information about the quotient. Although underlying graphs of regular maps may have multiple edges, a regular map with multiple edges projects onto another one with a simple underlying graph that has the same set of vertices and the same adjacency relation. Hence, regular maps with multiple edges can be described as some "extensions" of regular embeddings of simple graphs. Therefore, from now on, all the maps considered throughout the paper will be assumed to have simple underlying graphs. Since the regular maps with  $p$  or  $pq$  vertices, where  $p$  and  $q$  are primes, appear as quotients of many other regular maps, they are natural candidates to be dealt with. The classification of regular maps with  $p$  vertices is done in [5]. The classification of such regular maps can be derived from the classification of arc-transitive graphs with prime number of vertices, see [3]. In this paper, we shall give a classification of regular embeddings of graphs of order  $pq$ , where  $p$  and  $q$  are primes (not necessarily distinct).

Since the classification of arc-transitive graphs of order  $pq$  ( $p \neq q$ ) is known (see [15]-[18]), it is possible to use these results for our purpose. However, there are some arguments that do not support this idea. First of all, many different families of graphs included in that classification have to be checked unnecessarily. For some of the families, either there are no regular maps, or the classification is highly non-trivial. This situation can be well demonstrated in the family of complete  $q$ -partite graphs  $K_{p,\dots,p}$  which are obviously arc-transitive of order  $pq$ .

They admit a regular embedding into a surface if and only if either  $q \leq 3$ , or  $p = q$ , see [5]. Perhaps, the most important argument to attack the problem independently of the classification of arc-transitive graphs of order  $pq$  consists in the fact that it depends on the classification of primitive groups of degree  $pq$  (see [14]). Since the classification of primitive groups of degree  $pq$  depends on the classification of finite simple groups, it is worth classifying regular maps with  $pq$  vertices independently of it. Our approach avoids the employment of this classification. In fact, it is independent of the classification of finite simple groups.

The paper is organized as follows. After this introductory section, a brief description of algebraic maps and some preliminary group theoretical results will be given in Sections 2 and 3. Section 4 contains the proof of the classification theorem (see Theorem 4.8). In Section 5 we compute the genera of the maps obtained in Theorem 4.8.

## 2 Algebraic maps

Let  $\mathcal{G} = \mathcal{G}(V, D)$  be a simple graph with the vertex set  $V = V(\mathcal{G})$  and the arc set  $D = D(\mathcal{G})$ . By  $S_V$  and  $S_D$ , we denote the symmetry groups on the vertex set and on the arc set, respectively. The involution  $L$  in  $S_D$  interchanging the two arcs underlying the same edge is called the *arc-reversing involution*. An element  $R$  in  $S_D$  which cyclically permutes the arcs initiated at  $v$  for each vertex  $v \in V(\mathcal{G})$  is called a *rotation*. Since the graph  $\mathcal{G}$  has been assumed to be simple,  $\text{Aut}(\mathcal{G})$  is considered as a subgroup of both  $S_V$  and  $S_D$ , and the same notation is used for convenience. In an investigation of maps, it is often useful to replace topological maps on orientable surfaces with their combinatorial counterparts. It is well-known that graph embeddings into orientable surfaces can be described by means of rotations (see [7, 12]). A map  $\mathcal{M}$  with the underlying graph  $\mathcal{G}$  can be identified with a pair  $\mathcal{M} = \mathcal{M}(\mathcal{G}; R)$ , where  $R$  is a rotation of  $\mathcal{G}$  with the arc-reversing involution  $L$ . By the connectivity of  $\mathcal{G}$ ,  $\text{Mon}(\mathcal{M}) := \langle R, L \rangle$  is a transitive subgroup of  $S_D$ . Let  $\mathcal{G}_1$  and  $\mathcal{G}_2$  be two graphs, and let  $L_1$  and  $L_2$  be the respective arc-reversing involutions. Given two maps  $\mathcal{M}_1 = \mathcal{M}(\mathcal{G}_1; R_1)$  and  $\mathcal{M}_2 = \mathcal{M}(\mathcal{G}_2; R_2)$ , a graph isomorphism  $\phi : \mathcal{G}_1 \rightarrow \mathcal{G}_2$  is called a *map isomorphism* from  $\mathcal{M}_1$  to  $\mathcal{M}_2$  if  $R_1\phi = \phi R_2$ , noting that  $L_1\phi = \phi L_2$ . In particular, if  $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$ , then  $\phi$  is called an *automorphism* of  $\mathcal{M}$ . The automorphisms of  $\mathcal{M}$  form a group  $\text{Aut}(\mathcal{M}) \leq \text{Aut}(\mathcal{G})$ , called the *automorphism group* of the map  $\mathcal{M}$ . By the definition,  $\text{Aut}(\mathcal{M}) \leq C_{S_D}(\text{Mon}(\mathcal{M}))$ , the centralizer of  $\text{Mon}(\mathcal{M})$  in  $S_D$ . Therefore,  $\text{Aut}(\mathcal{M})$  acts semi-regularly on  $D$ , which comes

from the transitivity of  $\text{Mon}(\mathcal{M})$  on  $D$ . If the action is regular, the map  $\mathcal{M}$  is called *regular*. As a consequence of some well-known results in a permutation group theory (see [8, I.Theorem 6.5]), we infer that in a regular map  $\mathcal{M}$ , the two associated permutation groups  $\text{Aut}(\mathcal{M})$  and  $\text{Mon}(\mathcal{M})$  on  $D$  can be viewed as the right and the left regular representations of an abstract group  $G$ , so that  $G \cong \text{Aut}(\mathcal{M}) \cong \text{Mon}(\mathcal{M})$  mutually centralizing each other in  $S_D$  (see [12]).

In [5], a group theoretical method for classifying the regular maps with a given graph was introduced. Here we give a brief description of the main idea.

Let  $G$  be a finite group and  $H$  a proper subgroup of  $G$  with  $\bigcap_{g \in G} H^g = 1$ . Let  $B = B^{-1}$  be a double coset of  $H$  in  $G$ . From now on, we use  $\mathcal{G} = \mathcal{G}(G; H, B)$  to denote the *coset graph* with  $V(\mathcal{G}) = \{Hg \mid g \in G\}$  and  $D(\mathcal{G}) = \{(Hg, Hbg) \mid b \in B, g \in G\}$ . Note that  $G$  acts faithfully and arc-transitively on the coset graph by the right multiplication. In what follows, we often identify the group  $G$  with the corresponding group of right multiplications.

**Definition 2.1** *Let  $G = \langle r, \ell \rangle$  be a finite two-generator group with  $\ell^2 = 1$  and  $\langle r \rangle \cap \langle r \rangle^\ell = 1$ . By an algebraic map  $\mathcal{M}(G; r, \ell) = (\mathcal{G}; R)$ , we mean the map whose underlying graph is the coset graph  $\mathcal{G} = \mathcal{G}(G; \langle r \rangle, \langle r \rangle \ell \langle r \rangle)$  and the rotation  $R$  is determined by  $e^R = e^{g_1^{-1} r g_1} = (\langle r \rangle g_1, \langle r \rangle g_2 g_1^{-1} r g_1)$  for any arc  $e = (\langle r \rangle g_1, \langle r \rangle g_2)$  in  $D(\mathcal{G})$ .*

By the definition,  $G$  acts arc-regularly on  $\mathcal{G}$  (by the right multiplication) and preserves the rotation  $R$  of the map  $\mathcal{M}(G; r, \ell)$ . Therefore, any algebraic map  $\mathcal{M}$  is regular with  $\text{Aut}(\mathcal{M}) \cong \text{Mon}(\mathcal{M}) \cong G$ . It was shown in [5, Propositions 2.2, 2.3] that every regular embedding of a graph can be described by an algebraic map, and that two such algebraic maps  $\mathcal{M}(G; r_1, \ell_1)$  and  $\mathcal{M}(G; r_2, \ell_2)$  are isomorphic if and only if there exists an automorphism  $\sigma \in \text{Aut}(G)$  such that  $r_1^\sigma = r_2$  and  $\ell_1^\sigma = \ell_2$ . Note that isomorphic regular maps have isomorphic automorphism groups. Therefore, one can transfer the classification problem of regular embeddings of a given graph into a purely group theoretical problem. More precisely, one may classify all the regular maps with a given underlying graph  $\mathcal{G}$  of valency  $n$  in the following two steps:

- (1) Find the representatives  $G$  (as abstract groups) of the isomorphism classes of arc-regular subgroups of  $\text{Aut}(\mathcal{G})$  with cyclic point stabilizers.
- (2) For each group  $G$  given in (1), determine all the algebraic regular maps  $\mathcal{M}(G; r, \ell)$  with the underlying graphs isomorphic to  $\mathcal{G}$ , or equivalently, determine the representatives of the orbits of  $\text{Aut}(G)$  on the set of generating pairs  $(r, \ell)$  of  $G$  such that  $|r| = n$ ,  $|\ell| = 2$  and the coset graph  $\mathcal{G}(G; \langle r \rangle, \langle r \rangle \ell \langle r \rangle) \cong \mathcal{G}$ .

### 3 Preliminary results in group theory

In this section, we give some group theoretical results. First, we introduce some notations.

By  $(r, s)$  and  $[r, s]$ , we denote *g.c.d* and *l.c.m.* of two positive integers  $r$  and  $s$ , respectively; and by  $[a, b]$ , (with an abuse of notation) the commutator of two elements  $a$  and  $b$  in a group. By  $\mathbb{Z}_n$  and  $\mathbb{D}_{2n}$ , the cyclic group of order  $n$  and the dihedral group of order  $2n$ , respectively; by  $\mathbb{F}_p$  the finite field of  $p$  elements; and by  $C_G(H)$  and  $N_G(H)$ , the centralizer and normalizer of  $H$  in a group  $G$ , respectively. For a subgroup  $H$  of the group  $G$ ; by  $N : H$  a semiproduct of the group  $N$  by the group  $H$ ; by  $G^{k+}$  the subgroup  $\langle g \mid g \in G, g^k = 1 \rangle$  of a group  $G$  for some positive integer  $k$ ; and by  $S^*$  the multiplication group of a ring  $S$ . By  $V = V(2, p)$ ,  $\text{PG}(V)$ ,  $\text{AG}(V)$ ,  $\text{GL}(2, p)$ ,  $\text{PGL}(2, p)$ , and  $\text{AGL}(2, p)$ , respectively, we denote the 2-dimensional row linear space, projective geometry, affine geometry, general linear group, projective general linear group, and affine transformation group over the field  $\mathbb{F}_p$ .

For any  $\alpha \in V$ , by  $t_\alpha$  we denote the translation corresponding to  $\alpha$  in  $\text{AG}(V)$  and by  $T$ , the translation subgroup of  $\text{AGL}(2, p)$ . Then  $\text{AGL}(2, p) = T : \text{GL}(2, p)$ . We adopt the matrix notation for  $\text{GL}(2, p)$  and, for convenience, denote a matrix  $x = (a_{ij})_{2 \times 2}$  by  $x = \|\|a_{11}, a_{12}; a_{21}, a_{22}\|\|$ . We have  $g^{-1}t_\alpha g = (t_\alpha)^g = t_{\alpha g}$  for any  $t_\alpha \in T \leq \text{AGL}(2, p)$  and any  $g \in \text{GL}(2, p) \leq \text{AGL}(2, p)$ . Hereafter, let us fix  $a = t_{(1,0)}$ ,  $b = t_{(0,1)}$ , so that  $\langle a, b \rangle = T$  is the translation subgroup of  $\text{AGL}(2, p)$ . Let  $G = T : \langle x \rangle$ , where  $x = \|\|e, u; f, w\|\|$  is an element of order  $n$  in  $\text{GL}(2, p)$ . Then, as an abstract group,  $G$  can be presented by

$$G = \langle a, b, x \mid a^p = b^p = x^n = [a, b] = 1, a^x = a^e b^u, b^x = a^f b^w \rangle. \quad (3.1)$$

Conversely, any group  $G$  given by (3.1) can be viewed as a subgroup of  $\text{AGL}(2, p)$  containing  $T$  by identifying  $a, b, x$  with  $t_{(1,0)}, t_{(0,1)}$  and  $\|\|e, f; u, w\|\|$ , respectively. This identification will be frequently used later.

The knowledge on 2-dimensional linear groups as well as on projective groups or on affine groups will be used throughout this paper. The reader is assumed to be familiar with it. To make the proofs more transparent in what follows, we have decided to include some known facts. Proposition 3.1 can be extracted from [22] and [4], and Proposition 3.2 can be obtained from Proposition 3.1 and some results in [8, I.8].

**Proposition 3.1** *Let  $p$  be an odd prime. Then, the maximal subgroups of  $\text{PGL}(2, p)$  are one of the following:*

one conjugacy class of subgroups isomorphic to  $\mathbb{Z}_p : \mathbb{Z}_{p-1}$ ; one class isomorphic to  $\mathbb{D}_{2(p-1)}$ , when  $p \geq 7$ ; one class isomorphic to  $\mathbb{D}_{2(p+1)}$ ; one class isomorphic to  $S_4$ , when either  $p = 5$  or  $p \equiv 3, 13, 27, 37 \pmod{40}$ ; and one subgroup isomorphic to  $\text{PSL}(2, p)$ .

**Proposition 3.2** *For a prime  $p$ , let  $\mathbb{F}_p^* = \langle \theta \rangle$ ,  $G = \text{GL}(2, p)$ ,  $Z = Z(G)$  and  $\overline{G} = \text{PGL}(2, p)$ . Then,*

- (1)  $\text{GL}(2, 2) = \langle x, y \rangle \cong \mathbb{D}_6$ , where  $x = ||1, 1; 1, 0||$  and  $y = ||0, 1; 1, 0||$ .
- (2) For  $p \geq 3$ , all the elements of the form  $||e, f\theta; f, e||$  form a cyclic subgroup  $H$  of  $G$  which is of order  $p^2 - 1$  and  $Z \leq H$ . Moreover,  $N_G(H) \cong H : \langle y \rangle$ , where  $y = ||1, 0; 0, -1||$  is an involution. For any  $n$  satisfying  $n \mid (p^2 - 1)$  but  $n \nmid (p - 1)$ , each cyclic subgroup of order  $n$  is conjugate to a subgroup of  $H$ . Each irreducible subgroup  $L$  of  $\text{GL}(2, p)$  on its action on  $V$  is conjugate to a subgroup of  $H$  and the action  $\overline{L} := LZ/Z$  on  $\text{PG}(V)$  is regular.
- (3) For  $p \geq 3$ , let  $D$  be the diagonal subgroup of  $G$ . Then,  $N_G(D) = D : \langle y \rangle$ , where  $y = ||0, 1; 1, 0||$  is an involution. For any two divisors  $n$  and  $m$  of  $p - 1$ , each subgroup isomorphic to  $\mathbb{Z}_n \times \mathbb{Z}_m$  is conjugate to a subgroup of  $D$ . Moreover, for each element  $h$  in  $D \setminus Z$ ,  $\overline{h} := hZ/Z$  fixes precisely two points in  $\text{PG}(V)$ .
- (4) For  $p \geq 3$ ,  $G$  contains one class of subgroups isomorphic to  $\mathbb{Z}_p : \mathbb{Z}_{p-1}$  with a representative  $H = \langle x, y \rangle$ , where  $x = ||1, 1; 0, 1||$  and  $y = ||1, 0; 0, \theta||$ . Moreover,  $\overline{H} := HZ/Z$  is a point-stabilizer of  $\text{PGL}(2, p)$  in the action on  $\text{PG}(V)$ .

**Lemma 3.3** *Let  $F = T : \langle x \rangle$  and  $F' = T : \langle x' \rangle$  be two subgroups of  $A := \text{AGL}(2, p)$ , where  $T$  is the translation subgroup, and  $x$  and  $x'$  are nontrivial elements in  $G = \text{GL}(2, p)$ . Suppose  $\sigma$  is an isomorphism from  $F$  to  $F'$  mapping  $\langle x \rangle$  to  $\langle x' \rangle$ . Then, there exists an element  $u \in G$  such that  $\sigma = I(u)|_F$ , where  $I(u)$  is the inner automorphism of  $A$  induced by  $u$ . In particular, if  $\langle x \rangle = \langle x' \rangle$  then  $u \in N_G(\langle x \rangle)$ .*

**Proof** Since both  $F$  and  $F'$  have only one subgroup isomorphic to  $T$ , the isomorphism  $\sigma$  fixes  $T$  setwise. With our notations,  $T = \langle a, b \rangle$ , where  $a = t_{(1,0)}$  and  $b = t_{(0,1)}$ . Let  $a^\sigma = a^{u_{11}}b^{u_{12}}$  and  $b^\sigma = a^{u_{21}}b^{u_{22}}$  for some  $u_{ij} \in \mathbb{Z}_p$ . Then, we have  $a^\sigma = t_{(u_{11}, u_{12})}$  and  $b^\sigma = t_{(u_{21}, u_{22})}$ . Moreover, it is easy to check that for any  $\alpha \in V$ ,  $(t_\alpha)^\sigma = t_{\alpha u} = (t_\alpha)^u$  for  $u = (u_{ij}) \in G$ . By  $(x^{-1}t_\alpha x)^\sigma = (t_{\alpha x})^\sigma$  and  $x^\sigma \in G$ , we have  $t_{\alpha x^\sigma} = t_{\alpha x u}$ , which forces that  $x^\sigma = u^{-1}x u$ . Therefore,  $\sigma = I(u)|_F$ . The second part of the lemma is immediate.  $\square$

**Lemma 3.4** *Let  $p$  and  $q$  be two primes with  $p \geq q$ ,  $(t_1, t_2) \in \mathbb{Z}_p^* \times \mathbb{Z}_q^*$ , and let  $h = [|t_1|, |t_2|]$ . Define the group*

$$G(p, q, t_1, t_2) = \langle a, b, x \mid a^p = b^q = x^h = [a, b] = 1, a^x = a^{t_1}, b^x = b^{t_2} \rangle.$$

*Then,  $G(p, q, t_1, t_2) \cong G(p, q, t'_1, t'_2)$  if and only if in  $\mathbb{Z}_p^* \times \mathbb{Z}_q^*$ , we have  $\langle (t_1, t_2) \rangle = \langle (t'_1, t'_2) \rangle$  for  $p \geq q$ , or  $\langle (t_1, t_2) \rangle = \langle (t'_2, t'_1) \rangle$  for  $p = q$ .*

**Proof** Let  $G = G(p, q, t_1, t_2)$  and  $G' = G(p, q, t'_1, t'_2)$ , and let us denote the three generators of  $G'$  by  $a'$ ,  $b'$  and  $x'$ . We distinguish the following two cases.

*Case 1*  $p > q$  : Assume  $\langle (t_1, t_2) \rangle = \langle (t'_1, t'_2) \rangle$ . Then,  $(t_1, t_2) = (t'_1, t'_2)^j$  for some  $j \in \mathbb{Z}_h^*$ , and the assignment  $\tau : a \rightarrow a', b \rightarrow b'$  and  $x \rightarrow x'^j$  extends to an isomorphism  $G \rightarrow G'$ . Conversely, assume that  $\sigma : G \rightarrow G'$  is an isomorphism. Then,  $a^\sigma = a'^i$ ,  $b^\sigma = b'^j$ ,  $x^\sigma = a'^e b'^f x'^k$  for some  $i \in \mathbb{Z}_p^*$ ,  $j \in \mathbb{Z}_q^*$  and  $k \in \mathbb{Z}_h^*$ . Considering the action of  $\sigma$  on the definition relations of  $G$ , we see that  $t_1 \equiv (t'_1)^k \pmod{p}$  and  $t_2 \equiv (t'_2)^k \pmod{q}$ , forcing  $(t_1, t_2) \leq \langle (t'_1, t'_2) \rangle$ . By  $|\langle (t_1, t_2) \rangle| = |\langle (t'_1, t'_2) \rangle|$ , we get  $\langle (t_1, t_2) \rangle = \langle (t'_1, t'_2) \rangle$ .

*Case 2*  $p = q$  : The subcase  $p = 2$  is trivial. So, in what follows, we assume  $p \geq 3$ . The proof of the sufficiency can be proceeded in a similar way as in Case (1), by noting the symmetry of  $a$  and  $b$ . Conversely, assume that  $\sigma : G \rightarrow G'$  is an isomorphism. Without any loss of generality, we may assume that both  $G = \langle a, b, x \rangle$  and  $G' = \langle a', b', x' \rangle$  are subgroups of  $\text{AGL}(2, p)$  containing  $T$ . As above, we set  $a = a' = t_{(1,0)}$ ,  $b = b' = t_{(0,1)}$ ,  $x = [|t_1, 0; 0, t_2|]$ , and  $x' = [|t'_1, 0; 0, t'_2|]$ . Then,  $\sigma$  fixes  $T$  setwise. Since both  $G$  and  $G'$  have only one conjugacy class of subgroups of order  $h$ , we may assume  $\langle x \rangle^\sigma = \langle x' \rangle$ . By Lemma 3.3,  $\sigma = I(u)|_G$  for some inner automorphism  $I(u)$  of  $\text{AGL}(2, p)$ , where  $u \in \text{GL}(2, p)$  and  $u^{-1}xu = x'^j$  for some integer  $j$ . A direct checking shows that  $u$  is of the form either  $||e, 0; 0, f||$  or  $||0, e; f, 0||$ . The desired result follows.  $\square$

The following propositions will be used later.

**Proposition 3.5** [8, I.4.5] *Let  $G$  be a group and  $H$  a subgroup of  $G$ . Then,  $N_G(H)/C_G(H)$  is isomorphic to a subgroup of  $\text{Aut}(G)$ .*

**Proposition 3.6** [9] *Let  $G$  be a finite group and  $P$  a Sylow  $p$ -subgroup of  $G$ . If  $N_G(P) = C_G(P)$ , then  $P$  has a normal  $p$ -complement in  $G$ .*

It is well-known that every group of odd order or of order  $p^a q^b$  for two primes  $p$  and  $q$  is solvable. Using this fact and Proposition 3.6, one can easily obtain the following proposition.

**Proposition 3.7** *Every group containing a cyclic Sylow 2-subgroup is solvable.*

**Proposition 3.8** [24, 11.6, 11.7] *Every permutation group of prime degree  $p$  is either insolvable 2-transitive or isomorphic to  $\mathbb{Z}_p : \mathbb{Z}_s$  for  $s \mid (p - 1)$ . Moreover, such an insolvable 2-transitive group cannot have a cyclic point-stabilizer.*

For an odd prime  $p$  and an even divisor  $s$  of  $p - 1$ , denote by  $\mathcal{G}(p, s)$  the Cayley graph  $\text{Cay}(\mathbb{Z}_p, S)$ , where  $S$  is the subgroup of order  $s$  in the multiplicative group  $\mathbb{Z}_p^*$ .

**Proposition 3.9** [3] *For any odd prime  $p$  and an even divisor  $s$  of  $p - 1$ , there is a unique arc-transitive graph of order  $p$  and of valency  $s$ , up to isomorphism. It is isomorphic to the Cayley graph  $\mathcal{G}(p, s)$ , and its automorphism group is isomorphic to  $\mathbb{Z}_p : \mathbb{Z}_s$  if  $s < p - 1$ , and to the symmetry group  $S_p$  if  $s = p - 1$ .*

We conclude this section by proving a group theoretical result which is crucial for the proof of the main theorem. It enables us to avoid the employment of the classification of primitive groups of degree  $p$  or  $pq$  for two primes  $p$  and  $q$ . Moreover, it derives the classification theorem of regular maps in this paper independently of the classification of finite simple groups.

**Lemma 3.10** *If a group  $G$  contains a cyclic subgroup  $H$  of index  $pq$  for two primes  $p$  and  $q$  ( $p$  may be equal to  $q$ ), then  $G$  is solvable.*

**Proof** Without any loss of generality, we may assume that  $p \leq q$  and the core  $H_G := \bigcap_{g \in G} g^{-1} H g$  is trivial. Let  $|H| = n$ , so that  $|G| = pqn$ . If  $p = q = 2$ , then  $G$  is isomorphic to a subgroup of  $S_4$  and so  $G$  is solvable. If  $p$  or  $n$  is odd, then  $G$  is of odd order or  $G$  contains a cyclic Sylow 2-subgroup, which forces that  $G$  is solvable, by the solvability of the groups of odd order and Proposition 3.7. Hence, in what follows we assume that  $p = 2$ ,  $q$  is odd and  $n$  is even. Noting that the core  $H_G = 1$ , we consider the faithful right multiplication representation of  $G$  on  $\Omega := \{Hg \mid g \in G\}$ , where  $|\Omega| = 2q$ . We split our discussion into two cases.

*Case 1*  $G$  acts imprimitively on  $\Omega$  : Let  $\mathbf{B}$  be a complete block system of  $G$  and let  $K$  be the kernel of  $G$  on  $\mathbf{B}$ . If the blocks in  $\mathbf{B}$  are of size  $q$ , then  $|G/K| = 2$  and  $|K| = nq$ . By Proposition 3.7,  $K$  is solvable and consequently,  $G$  is solvable. Therefore, we assume that the action of  $G$  induces only blocks of size 2. Let us first deal with the case  $K \neq 1$ . Then,  $K$  is a 2-group which is transitive on each block. For any block  $B \in \mathbf{B}$  and any vertex  $u \in B$ , we have that  $\overline{G}_B = G_B K / K = G_u K / K \cong G_u / (G_u \cap K)$ , a cyclic group, where  $G_u \cong \mathbb{Z}_n$ .

Thus,  $\overline{G}$  is a permutation group on  $\mathbf{B}$  of degree  $q$ , with a cyclic point-stabilizer, which by Proposition 3.8 forces  $\overline{G}$  to be solvable. Consequently,  $G$  is solvable as well. Next, let us consider the case  $K = 1$ . Then,  $G$  is insolvable, otherwise, the minimal normal subgroup of  $G$  induces a block system with the nontrivial kernel. (Note that each minimal normal subgroup of a solvable group must be elementary abelian.) Now,  $|G_B| = 2n$  and  $G$  is isomorphic to an insolvable subgroup of  $S_q$ . In particular, each Sylow  $q$ -subgroup of  $G$  has the order  $q$ . Then, by Proposition 3.8,  $\overline{G} = G/K \cong G$  is 2-transitive on  $\mathbf{B}$  implying that  $G_B$  is transitive on  $\mathbf{B}$ . Consequently,  $G_u$  has 1, 2 or 4 orbits on  $\Omega \setminus B$ , which means that  $n = 2(q - 1)$ ,  $q - 1$  or  $\frac{q-1}{2}$ . The first case implies that the involution of  $G_u$  is contained in  $K$ , a contradiction. By Sylow-Theorem, we have  $(qk + 1) \mid 2nq$  for some positive integer  $k$ . However, this condition cannot hold for either  $n = q - 1$  or  $n = \frac{q-1}{2}$ .

*Case 2*  $G$  is primitive on  $\Omega$  : For any group  $L$ , by  $\pi(L)$  we denote the set of all the primes which are divisors of  $|L|$ . The statement will be proved by induction on  $|\pi(G_u) \setminus \{2, q\}|$ . If  $|\pi(G_u) \setminus \{2, q\}| = 0$ , then  $G$  is of order  $2^a q^b$  for some positive integers  $a$  and  $b$ , and consequently,  $G$  is solvable. Assume  $|\pi(G_u) \setminus \{2, q\}| \geq 1$ . Take  $\ell \in \pi(G_u) \setminus \{2, q\}$  and let  $L$  be a Sylow  $\ell$ -subgroup of  $G$  contained in  $G_u$ . Then,  $G_u \leq C_G(L) \leq N_G(L) < G$ . Since  $G$  is primitive,  $G_u$  is a maximal subgroup, which forces that  $G_u = C_G(L) = N_G(L)$ , noting that  $G_u$  is core free. By Proposition 3.6,  $L$  has a normal complement  $M$  in  $G$ . Then,  $M$  is transitive on  $\Omega$ . Note that  $|\pi(M_u) \setminus \{2, q\}| = |\pi(G_u) \setminus \{p, q\}| - 1$  and for any  $u \in \Omega$ ,  $M_u \leq G_v$  is cyclic. If  $M$  is primitive on  $\Omega$ , then by the induction hypothesis we have that  $M$  is solvable. If  $M$  is imprimitive on  $\Omega$ , then by Case 1, one can derive the solvability of  $M$ . Since both  $M$  and  $G/M \cong L$  are solvable,  $G$  is solvable.  $\square$

Let us remark that there is not much room for a possible generalization of Lemma 3.10. In fact, if we replace  $pq$  by  $pqr$ , Lemma 3.10 cannot hold. For instance, the underlying graph of the icosahedron has 12 vertices and the automorphism group of this map is  $A_5$ , the smallest simple nonabelian group.

## 4 Classification Theorem

Throughout this section, let  $p$  and  $q$  be primes, not necessarily distinct. Recall that  $\mathcal{G}(q, s)$  denotes the unique arc-transitive graph of order  $q$  and valency  $s$  for some even divisor  $s$  of  $q - 1$  (see Proposition 3.9). If  $s = 2$ , then  $\mathcal{G}(q, s)$  is a circle  $C_q$  of order  $q$ . For given two graphs  $\mathcal{G}$  and  $\mathcal{H}$ , we use  $\mathcal{G}[\mathcal{H}]$  to denote their lexicographic product, while  $\mathcal{G} \otimes \mathcal{H}$ , denotes their tensor product. In particular,

$K_n[\overline{K}_m]$  is the complete  $n$ -partite graph  $K_{m,\dots,m}$ , and  $K_2 \otimes \mathcal{G}(p, s)$  is the canonical bipartite cover of  $\mathcal{G}(p, s)$ , see [19] for the definition.

In order to prove the main theorem, we first introduce six different families of two generator groups  $G = \langle r, \ell \rangle$  with  $|\ell| = 2$ ,  $|r| = n$ ,  $\langle r \rangle \cap \langle r \rangle^\ell = 1$  and  $|G| = npq$ . We prove in Theorem 4.1 that every arc-regular subgroup of the automorphism group of an arc-transitive graph of order  $pq$  with cyclic vertex-stabiliser belongs to one of these families of groups. And, for each of these groups  $G$ , we shall find all the corresponding algebraic maps in Lemmas 4.2-4.7. Finally, we state Theorem 4.8 establishing the classification.

(I) Let  $p \geq 7$ ,  $h$  any odd divisor of  $p - 1$  with  $h \geq 3$ , and let  $t$  be any fixed element of order  $2h$  in  $\mathbb{Z}_p^*$ . Define a group

$$G_1 = G_1(p, h) = \langle x, y \mid x^p = y^{2h} = 1, x^y = x^t \rangle. \quad (4.1)$$

Note that  $G_1 \cong \mathbb{Z}_p : \mathbb{Z}_{2h}$ .

(II) Let  $p \geq 3$ ,  $h$  any even divisor of  $p^2 - p$  with  $h \geq 2$ , and let  $t$  be any fixed element of order  $h$  in  $\mathbb{Z}_{p^2}^*$ . Define a group

$$G_2 = G_2(p, h) = \langle x, y \mid x^{p^2} = y^h = 1, x^y = x^t \rangle. \quad (4.2)$$

(III) Let  $p \geq q \geq 2$ ,  $pq > 4$  and  $(t_1, t_2) \in \mathbb{Z}_p^* \times \mathbb{Z}_q^*$  such that  $t_1 \neq t_2$  if  $p = q$ , and  $\langle (t_1, t_2) \rangle$  contains  $(-1, 1)$  if  $q = 2$ , and contains  $(-1, -1)$  if  $q \geq 3$ . Let  $h = [ |t_1|, |t_2| ]$ , where  $h \geq 2$  is even. Define a group

$$G_3 = G_3(p, q, t_1, t_2) = \langle a, b, x \mid a^p = b^q = x^h = [a, b] = 1, a^x = a^{t_1}, b^x = b^{t_2} \rangle. \quad (4.3)$$

Note that given  $p$  and  $q$ , a necessary and sufficient condition for two such groups with different parameters  $t_1$  and  $t_2$  being isomorphic has been determined in Lemma 3.4.

(IV) Let  $\mathbb{F}_p^* = \langle \theta \rangle$  and let  $x$  be an element of order  $h$  in  $\text{GL}(2, p)$ , where  $h \geq 3$ , defined as follows:

- (1) If  $p = 2$  and  $h = 3$ , then  $x = ||1, 1; 1, 0||$ ; or
- (2) if  $p \geq 3$ ,  $h \mid (p^2 - 1)$  but  $h \nmid (p - 1)$ , then  $x = ||e, f\theta; f, e||$  for some fixed pair  $(e, f)$  such that  $|x| = h$ .

Let  $T = \langle a, b \rangle$  be the translation subgroup of  $\text{AGL}(2, p)$  as before. Define a group

$$G_4 = G_4(p, h) = T : \langle x \rangle \leq \text{AGL}(2, p). \quad (4.4)$$

(V) Define a group

$$G_5 = G_5(p) = \langle a, b, x \mid a^p = b^p = x^2 = [a, b] = 1, a^x = b. \rangle \quad (4.5)$$

(VI) Let  $\mathbb{F}_p^* = \langle \theta \rangle$  and let  $H = \langle x, y \rangle$  be a subgroup in  $\text{GL}(2, p)$  isomorphic to a Frobenius group  $\mathbb{Z}_q : \mathbb{Z}_h$  with  $h \geq 2$ , and two elements  $x$  and  $y$  are defined as follows:

- (1) If  $p = 2$ ,  $q = 3$  and  $h = 2$ , then set  $x = ||1, 1; 1, 0||$  and  $y = ||0, 1; 1, 0||$ ;
- (2) if  $p > q \geq 3$ ,  $q \mid (p - 1)$  and  $h = 2$ , then  $x = ||t, 0; 0, t^{-1}||$  where  $t = \theta^{\frac{p-1}{q}}$  and  $y = ||0, 1; 1, 0||$ ;
- (3) if  $p > q \geq 3$ ,  $q \mid (p + 1)$  and  $h = 2$ , then  $x = ||e, f\theta; f, e||$  for some fixed pair  $(e, f)$  such that  $|x| = h$ , and  $y = ||1, 0; -1, 0||$ ; or
- (4) if  $p = q \geq 3$  and  $h$  is an even divisor of  $p - 1$ , then  $x = ||1, 1; 0, 1||$  and  $y = ||1, 0; 0, t||$ , where  $t = \theta^{\frac{p-1}{h}}$ .

Define a group

$$G_6 = G_6(p, q, h) = T : H \leq \text{AGL}(2, p). \quad (4.6)$$

**Theorem 4.1** *Let  $\mathcal{G}$  be any connected graph of order  $pq$  and valency  $n$ , whose automorphism group  $\text{Aut}(\mathcal{G})$  contains an arc-regular subgroup  $G$  with a cyclic point-stabilizer. Then,  $G$  is isomorphic to one of the groups  $G_i$  for  $1 \leq i \leq 6$ , defined in (I)-(VI), with parameters satisfying the following conditions:*

- (1)  $G \cong G_1(p, h)$ , where  $p \geq 7$ ,  $q = 2$  and  $n = h \geq 3$ ;
- (2)  $G \cong G_2(p, h)$ , where  $p = q \geq 3$ , and  $n = h \geq 2$ ;
- (3)  $G \cong G_3(p, q, t_1, t_2)$ , where  $p \geq q \geq 2$ ,  $pq > 4$  and  $n = h = [|t_1|, |t_2|] \geq 2$  is even;
- (4)  $G \cong G_4(p, h)$ , where  $p = q \geq 2$ ,  $n = h \geq 3$ ;
- (5)  $G \cong G_5(p)$ , where  $p \geq 2$ ,  $q = 2$  and  $n = p$ ;
- (6)  $G \cong G_6(p, q, h)$ , where either  $p \geq 2$ ,  $q \geq 3$ ,  $q \mid (p \pm 1)$  and  $h = 2$ ; or  $p = q \geq 3$  and  $h$  is an even divisor of  $p - 1$ . In both cases,  $n = hp$ .

Moreover, for each  $i$ , the groups  $G_i$  are uniquely determined by its admissible parameters, and for different  $i$ , the groups  $G_i$  are pairwise nonisomorphic.

**Proof** The last statement of the lemma follows from the definitions of  $G_i$  for  $1 \leq i \leq 6$ . Therefore, we only need to determine the structure of the groups  $G$ , the arc-regular subgroups of  $\text{Aut}(\mathcal{G})$  with the cyclic point-stabilizer. Clearly,  $G = \langle r, \ell \rangle$ , where  $\ell$  is an involution and  $\langle r \rangle = G_v$  for any fixed vertex  $v$  in  $V(\mathcal{G})$ . By Lemma 3.10,  $G$  is a solvable group of order  $npq$  with  $n = |G_v|$ . We divide the proof into two cases according to whether the action of  $G$  on  $V(\mathcal{G})$  is primitive or imprimitive.

*Case 1*  $G$  is primitive on  $V(\mathcal{G})$  :

Let  $N$  be a minimal normal subgroup of  $G$ . Then,  $N$  is transitive on  $V(\mathcal{G})$ , because  $G$  is primitive. Since  $G$  is solvable,  $N$  is elementary abelian, which forces that  $N$  acts semiregularly and so regularly on  $V(\mathcal{G})$ . This implies  $p = q$  and  $N \cong \mathbb{Z}_p \times \mathbb{Z}_p$ . Therefore,  $G = N : G_v$  is isomorphic to a subgroup of order  $np^2$  in  $\text{AGL}(2, p)$ , where  $G_v$  is an irreducible cyclic subgroup of  $\text{GL}(2, p)$  acting on  $V = V(2, p)$ .

Assume  $p = 2$ , so that  $G_v \leq \text{GL}(2, 2) \cong \mathbb{D}_6$ . In this case,  $G_v$  is the unique subgroup of order 3 of  $\text{GL}(2, 2)$ . Therefore,  $G \cong G_4(2, 3)$ , defined in (4.4).

Assume  $p \geq 3$ . Then, by Proposition 3.2 (2),  $n \mid (p^2 - 1)$  but  $n \nmid (p - 1)$ , and  $\text{GL}(2, p)$  has only one conjugacy class of irreducible cyclic subgroups of order  $n$ . Therefore,  $\text{AGL}(2, p)$  has only one conjugacy class of subgroups isomorphic to  $G$  and so  $G \cong G_4(p, h)$ , defined in (4.4), by setting  $h = n \geq 3$ .

*Case 2*  $G$  is imprimitive on  $V(\mathcal{G})$  :

Without any loss of generality, we assume that  $G$  has a complete block system  $\mathbf{B} = \{B_0, B_1, \dots, B_{q-1}\}$ , where  $|B_i| = p$ . By  $\overline{\mathcal{G}}$  we denote the quotient (block) graph of  $\mathcal{G}$  corresponding to  $\mathbf{B}$ , also we assume that  $\overline{\mathcal{G}}$  has the valency  $s$ . Let  $K$  be the kernel of  $G$  acting on  $\mathbf{B}$  and set  $\overline{G} = G/K$ . Since  $G$  is solvable,  $\overline{G}$  is solvable, which implies that  $\overline{G}$  is isomorphic either to  $\mathbb{Z}_2$  if  $q = 2$ , or to  $\mathbb{Z}_q : \mathbb{Z}_s$  if  $q \geq 3$  for an even divisor  $s$  of  $q - 1$ . In particular,  $\overline{G}$  acts arc-regularly on the graph  $\overline{\mathcal{G}}$ . If  $q = 2$ , then  $K \neq 1$ . Suppose that  $q \geq 3$ . Then, we also claim that  $K \neq 1$ . Suppose  $K = 1$ , by the contrary. Then,  $G \cong \overline{G} \cong \mathbb{Z}_q : \mathbb{Z}_s$ , and so  $G$  contains a normal subgroup of order  $q$  which induces an imprimitive block system, say  $\mathbf{B}_1$ . Let  $K_1$  be the kernel of  $G$  on  $\mathbf{B}_1$ . Then,  $q \mid |K_1|$  and  $G/K_1$  is cyclic, contradicting to the arc-transitivity of  $G/K_1$  on the quotient graph corresponding to  $\mathbf{B}_1$ .

First, suppose that  $K$  acts unfaithfully on a block, say  $B_0$ . Then, the kernel  $K_{(B_0)}$  of  $K$  on  $B_0$  is a nontrivial normal subgroup of  $K$ . Since  $\overline{\mathcal{G}}$  is connected, there exist two blocks  $B_i$  and  $B_j$  such that  $\{B_i, B_j\} \in E(\overline{\mathcal{G}})$  and  $K_{(B_0)}$  fixes  $B_i$  pointwise and is transitive on  $B_j$ . It implies that the induced subgraph  $\mathcal{G}(B_i \cup B_j) \cong K_{p,p}$ .

Therefore,  $n = sp$  and  $K \cong \mathbb{Z}_p \times \mathbb{Z}_p$ . Since  $\overline{\mathcal{G}}$  is arc-transitive, it follows that  $\mathcal{G} \cong \mathcal{G}(q, s)[\overline{K}_p]$  is the lexicographic product of the (unique) connected arc-transitive graph  $\mathcal{G}(q, s)$  of order  $q$  and valency  $s$  with the complement of the complete graph of order  $p$ . In this case, by a non-trivial argument, it has been proved in [5] that  $G$  is isomorphic to  $G_2(p, h)$  for  $p \mid h$ , see (4.2) where  $h = n$ , or to  $G_5(p)$  defined by (4.5) where  $n = p$ , or to  $G_6(p, q, h)$  defined by (4.6) where  $n = ph$ .

Next, suppose that  $K$  acts faithfully on each block. Then,  $K_v \cong \mathbb{Z}_{\frac{n}{s}}$  and  $K \cong \mathbb{Z}_p : \mathbb{Z}_{\frac{n}{s}}$ . In what follows, we assume that  $P = \langle x \rangle$  is the Sylow  $p$ -subgroup of  $K$ . Note that  $P$  is also normal in  $G$ , because it is a characteristic subgroup of  $K$ . If  $p = q = 2$ , then  $G$  is primitive on  $V(\Gamma)$ . Therefore, we assume that  $pq \neq 4$  and distinguish the following three cases: (i)  $p > q = 2$ ; (ii)  $q \geq 3$  and  $p \neq q$ ; and (iii)  $q = p \geq 3$ .

*Subcase (i)  $p > q = 2$*  : In this case,  $G$  is an extension of  $K \cong \mathbb{Z}_p : \mathbb{Z}_n$  by  $\mathbb{Z}_2$ , and  $G_v \cong \mathbb{Z}_n$ , where  $n \mid (p - 1)$ . Since  $(p, 2n) = 1$  and  $G$  is solvable,  $P$  has a complement, say  $H$ , in  $G$ .

If  $C_G(P) = P$ , then by Proposition 3.5 we have that  $H \cong G/P = G/C_G(P)$  is isomorphic to a subgroup of  $\text{Aut}(P) \cong \mathbb{Z}_{p-1}$ , forcing  $H \cong \mathbb{Z}_{2n}$ . Thus,  $G \cong \mathbb{Z}_p : \mathbb{Z}_{2n}$ , a Frobenius group. Assume that  $G = \langle x \rangle : \langle y \rangle$ , where  $|x| = p$  and  $|y| = 2n$ . Since  $G$  has only one conjugacy class of cyclic subgroups of order  $2n$ , we may assume that  $r = y^{2i}$  and  $\ell = x^j y^n$  so that  $G = \langle r, \ell \rangle$  and  $\ell^2 = 1$ . If  $n$  is even, then  $\langle r, \ell \rangle \leq \langle x, y^2 \rangle \neq G$ . Hence,  $n$  must be odd. Therefore,  $G \cong G_1(p, h)$ , defined in (4.1), by setting  $h = n \geq 3$ .

If  $C_G(P) \neq P$ , then  $|C_G(P)| = 2p$  as  $C_K(P) = P$ . Moreover,  $C_G(P) = P \times \langle b \rangle$  for some involution  $b$ , which implies that  $\langle b \rangle$  is normal in  $G$ , and so it is contained in the center of  $G$ . Setting  $h = n \geq 2$ , we get  $G \cong G_3(p, 2, t_1, 1)$  defined in (4.3). Since  $\ell \in G \setminus \langle b \rangle$ , it follows that  $n$  must be even.

*Subcase (ii)  $q \geq 3$  and  $p \neq q$*  : Since  $K \cap C_G(P) = P$ ,  $C_G(P) \neq G$ . If  $C_G(P) = P$ , then by using Proposition 3.5 we obtain that  $G/P = G/C_G(P)$  is isomorphic to a subgroup of  $\text{Aut}(P) \cong \mathbb{Z}_{p-1}$ , which is cyclic. However, from  $(G/P)/(K/P) \cong G/K \cong \mathbb{Z}_q : \mathbb{Z}_s$  where  $s \geq 2$ , we deduce that  $G/P$  cannot be cyclic. Therefore,  $C_G(P) \neq P$ . Since  $C_G(P)K/K \cong C_G(P)/(C_G(P) \cap K) \cong C_G(P)/P$ , it follows that  $C_G(P)K/K$  is a nontrivial normal subgroup of  $G/K$  and so  $C_G(P)K/K \cong \mathbb{Z}_q$ . Therefore,  $C_G(P)/P \cong \mathbb{Z}_q$  implying  $|C_G(P)| = pq$ . Hence,  $C_G(P) = P \times Q$  for a subgroup  $Q$  of  $G$ . Since  $Q$  is a characteristic subgroup in  $C_G(P)$ , it is normal in  $G$ . Since  $C_G(P)$  is transitive on  $V(\mathcal{G})$ ,  $P \times Q$  acts regularly on  $V(\mathcal{G})$ . Therefore,  $G = (P \times Q) : G_v \leq (\mathbb{Z}_p \times \mathbb{Z}_q) : \mathbb{Z}_n$ , where  $\mathbb{Z}_n$  is isomorphic to a subgroup of  $\text{Aut}(\mathbb{Z}_p \times \mathbb{Z}_q) \cong \mathbb{Z}_p^* \times \mathbb{Z}_q^*$ . Taking into account the symmetry of

$P$  and  $Q$  in this case, we may assume  $p > q \geq 3$ , noting that if  $q = 2$  then we are back to Subcase (i). Now, let  $P = \langle a \rangle$ ,  $Q = \langle b \rangle$  and  $G_v = \langle x \rangle$  and suppose that  $a^x = a^{t_1}$  and  $b^x = b^{t_2}$ , where  $(t_1, t_2) \in \mathbb{Z}_p^* \times \mathbb{Z}_q^*$ . Then,  $n = |x| = [|t_1|, |t_2|]$ . Suppose  $G = \langle r, \ell \rangle$  for an involution  $\ell$ , where  $r \in \langle x \rangle$ . Then,  $\ell = a^i b^j x^{\frac{n}{2}}$  for some  $i \in \mathbb{Z}_p^*$  and  $j \in \mathbb{Z}_q^*$ . From  $\ell^2 = 1$ , we have that  $t_1^{\frac{n}{2}} = -1$  in  $\mathbb{Z}_p$  and  $t_2^{\frac{n}{2}} = -1$  in  $\mathbb{Z}_q$ , which means  $(-1, -1) \leq \langle (t_1, t_2) \rangle$ . Therefore, for  $i=1$  and  $2$ ,  $|t_i|$  is even and  $|t_i|/(|t_1|, |t_2|)$  is odd. Thus, setting  $h = n \geq 2$ , we end with  $G \cong G_3(p, q, t_1, t_2)$  for  $p > q \geq 3$ , defined in (4.3).

*Subcase (iii)  $q = p \geq 3$  :* In this case, the Sylow  $p$ -subgroup  $P$  of  $G$  is a normal subgroup of  $G$  and its order is  $p^2$ . Hence,  $G = P : G_v$ .

If  $P \cong \mathbb{Z}_{p^2}$ , then  $G \cong \mathbb{Z}_{p^2} : \mathbb{Z}_n \leq \mathbb{Z}_{p^2} : \text{Aut}(\mathbb{Z}_{p^2})$ , where  $n \mid (p-1)$  as  $p \nmid n$ . Since  $\text{Aut}(\mathbb{Z}_{p^2}) \cong \mathbb{Z}_{p^2-p}$ , a cyclic group, and all the complements of  $P$  in  $G$  are conjugate, we derive that  $G \cong G_2(p, h)$  for  $h \mid (p-1)$  defined in (4.2), by setting  $h = n \geq 2$ .

If  $P \cong \mathbb{Z}_p^2$ , then  $G = P : G_v \cong \mathbb{Z}_p^2 : \mathbb{Z}_n \leq \text{AGL}(2, p)$ , and by Lemma 3.2(3)  $G_v = \langle x \rangle$  can be chosen to be a diagonal subgroup not contained in  $Z$ . Let  $x = [|t_1, 0; 0, t_2|]$ . Let  $\ell = a^i b^j x^{\frac{n}{2}}$  be an involution for some  $i, j \in \mathbb{Z}_p^*$ . Then, from  $\ell^2 = 1$ , we have  $x^{\frac{n}{2}} = z$ , the center involution of  $\text{GL}(2, p)$ , forcing that  $|t_i|/(|t_1|, |t_2|)$  is odd for  $i=1, 2$ . Therefore,  $G \cong G_3(p, p, t_1, t_2)$ , by setting  $h = n > 2$  (see (4.3)).

□

Our next task is to determine all the nonisomorphic algebraic maps coming from the groups  $G = G_i$  for  $1 \leq i \leq 6$ . By the results of Section 2, we only need to determine all the representatives of the orbits of in the action of  $\text{Aut}(G)$  on the set of the generating pairs  $(r, \ell)$  of  $G$  satisfying  $|\ell| = 2$ ,  $|r| = n$ ,  $|G| = npq$  and  $\langle r \rangle \cap \langle r \rangle^\ell = 1$ . It will be carried out case by case throughout Lemmas 4.2-4.7. Recall that  $\phi(h)$  denotes the Euler function, i.e., the number of positive integers less than  $h$  coprime to  $h$ .

**Lemma 4.2** *Let  $G_1 = G_1(p, h)$  be a group defined in (4.1). Then, any regular map  $\mathcal{M}(G_1; r, \ell)$  with the underlying graph  $\mathcal{G}$  of order  $2p$  is isomorphic to one of the following  $\phi(h)$  regular maps:*

$$\mathcal{M}_1 = \mathcal{M}_1(p, h, i) = \mathcal{M}(G_1; y^{2i}, xy^h), \quad (4.7)$$

where  $p \geq 7$ ,  $h \geq 3$ ,  $h$  is an odd divisor of  $p-1$  and  $i \in \mathbb{Z}_h^*$ . Moreover,  $\mathcal{G}$  is a bipartite graph of valency  $h$ .

**Proof** Recall that

$$G_1 = G_1(p, h) = \langle x, y \mid x^p = y^{2h} = 1, x^y = x^t \rangle.$$

In this case,  $q = 2$  and  $n = |r| = h \geq 3$ . In what follows, we find all the representatives of the orbits of  $\text{Aut}(G_1)$  on the set of admissible generating pairs  $(r, \ell)$  of  $G_1$ . Since  $G_1$  has only one conjugacy class of involutions as well as subgroups of order  $h$ , we may assume  $r = y^{2^i}$  for any  $i \in \mathbb{Z}_h^*$ . Moreover, each involution in  $G_1 \setminus \langle y \rangle$  has the form  $x^j y^h$  for  $j \in \mathbb{Z}_p^*$  and the automorphism  $\tau$  of  $G$  defined by  $x^\tau = x^j$  and  $y^\tau = y$  fixes  $r$  and sends  $xy^h$  to  $x^j y^h$ . Therefore, we may fix  $\ell = xy^h$ .

Now, suppose that  $\sigma \in \text{Aut}(G_1)$  fixes  $\langle y^2 \rangle$  setwise and  $\ell$  pointwise. Then,  $x^\sigma = x$  and  $y^\sigma = y^j$  for some  $j \in \mathbb{Z}_{2h}^*$ . Since  $\sigma$  preserves the relation  $x^y = x^t$ , we have  $y^{-j} x y^j = x^t$  and so  $t^j \equiv t \pmod{p}$ , which implies that  $j = 1$  in  $\mathbb{Z}_{2h}^*$  and consequently,  $\sigma = 1$ . Therefore, we have the  $\phi(h)$  choices for  $(r, \ell)$  and so obtain  $\phi(h)$  nonisomorphic regular maps as claimed in (4.7).  $\square$

**Lemma 4.3** *Let  $G_2 = G_2(p, h)$  be a group defined in (4.2). Then, any regular map  $\mathcal{M}(G_2; r, \ell)$  with the underlying graph  $\mathcal{G}$  of order  $p^2$  is isomorphic to one of the following  $\phi(h)$  regular maps:*

$$\mathcal{M}_2 = \mathcal{M}_2(p, h, i) = \mathcal{M}(G_2; y^i, xy^{\frac{h}{2}}), \quad (4.8)$$

where  $p \geq 3$ ,  $h \geq 2$ ,  $h$  is an even divisor of  $p^2 - p$  and  $i \in \mathbb{Z}_h^*$ . Moreover, if  $p \mid h$  then  $\mathcal{G} \cong \mathcal{G}(p, \frac{h}{p})[\overline{K}_p]$ ; and if  $(p, h) = 1$  then  $h \mid (p - 1)$  and  $\mathcal{G}$  is a  $p$ -fold cover of the graph  $\mathcal{G}(p, h)$ .

**Proof** Recall that

$$G_2 = G_2(p, h) = \langle x, y \mid x^{p^2} = y^h = 1, x^y = x^t \rangle.$$

In this case,  $p = q$  and  $n = h$ . Applying a similar argument as in Lemma 4.2, one can show that all the admissible pairs  $(r, \ell)$  are of the form  $r = y^i$  and  $\ell = xy^{\frac{h}{2}}$  for  $i \in \mathbb{Z}_h^*$ . Hence, there are exactly  $\phi(h)$  nonisomorphic regular maps, as shown in (4.8). Moreover, the structure of the underlying graph  $\mathcal{G}$  can be seen from the corresponding coset graph.  $\square$

**Lemma 4.4** *Let  $G_3 = G_3(p, q, t_1, t_2)$  be a group defined in (4.3). Then, any regular map  $\mathcal{M}(G_3; r, \ell)$  with the underlying graph  $\mathcal{G}$  of order  $pq$  is isomorphic to one of the following regular maps:*

$$\mathcal{M}_3 = \mathcal{M}_3(p, q, t_1, t_2, i) = \mathcal{M}(G_3; x^i, abx^{\frac{h}{2}}), \quad (4.9)$$

where  $p \geq q \geq 2$ ,  $pq > 4$  and  $(t_1, t_2) \in \mathbb{Z}_p \times \mathbb{Z}_q$  such that  $t_1 \neq t_2$  if  $p = q$ , and  $\langle (t_1, t_2) \rangle$  contains  $(-1, 1)$  if  $q = 2$ , and contains  $(-1, -1)$  if  $q \geq 3$ ; and  $i \in \mathbb{Z}_h^*/(\mathbb{Z}_h^*)^{2+}$  if  $p = q$  and  $h = |t_1| = |t_2|$ , and  $i \in \mathbb{Z}_h^*$  otherwise.

Hence, we have  $\phi(h)/|(\mathbb{Z}_h^*)^{2+}|$  or  $\phi(h)$  nonisomorphic regular maps according to whether  $p = q$  and  $|t_1| = |t_2|$  or not.

**Proof** Recall that

$$G_3 = G_3(p, q, t_1, t_2) = \langle a, b, x \mid a^p = b^q = x^h = [a, b] = 1, a^x = a^{t_1}, b^x = b^{t_2} \rangle.$$

The proof is divided into three cases to be discussed separately.

*Case 1*  $p > q = 2$ : Now  $|t_2| = 1$  and  $h = |t_1|$ . In this case,  $b \in Z(G_3)$ , so  $G_3$  has only one class of cyclic subgroups of order  $h$ . Hence, we may assume  $r = x^i$  for  $i \in \mathbb{Z}_h^*$  and  $\ell = a^j b x^{\frac{h}{2}}$  for  $j \in \mathbb{Z}_p^*$ . Since the automorphism  $\tau$  of  $G$  defined by  $a^\tau = a^j$ ,  $b^\tau = b$  and  $x^\tau = x$  fixes  $r$  and sends  $abx^{\frac{h}{2}}$  to  $a^j b x^{\frac{h}{2}}$ , we may fix  $\ell = abx^{\frac{h}{2}}$ . Similarly, one can show that if a  $\sigma$  in  $\text{Aut}(G_3)$  fixes  $\ell$  pointwise and  $\langle r \rangle$  setwise, then it must be the identity. Therefore, we obtain  $\phi(h)$  nonisomorphic regular maps  $\mathcal{M}(p, 2, t_1, 1, i)$  as claimed in (4.9).

*Case 2*  $p > q \geq 3$ : Now both  $|t_1|$  and  $|t_2|$  are even and  $|t_i|^{\frac{h}{2}} = -1$ . Let  $P = \langle a \rangle$  and  $Q = \langle b \rangle$ . We consider  $G_3$  as a subgroup of  $A := (P \times Q) : \text{Aut}(P \times Q) \cong (\mathbb{Z}_p \times \mathbb{Z}_q) : (\mathbb{Z}_p^* \times \mathbb{Z}_q^*)$ .

Note that each cyclic subgroup  $H$  of order  $h$  is a complement of  $P \times Q$  in  $G_3$  and  $H$  induces a faithful automorphism action on it. Hence, we may assume that  $r \in \langle x \rangle$ . Then,  $\ell$  has to be of the form  $a^i b^j x^{\frac{h}{2}}$  for some  $i \in \mathbb{Z}_p^*$  and  $j \in \mathbb{Z}_q^*$ . Since the action of the subgroup of inner automorphisms  $I(g)$  for  $g \in \text{Aut}(P \times Q)$  of  $A$  fixes  $x$  and maps  $ab$  to  $a^i b^j$ , we may fix  $\ell = abx^{\frac{h}{2}}$ . Suppose  $\sigma \in \text{Aut}(G_3)$  fixes  $\ell$  pointwise and  $\langle x \rangle$  setwise. Then,  $\sigma$  satisfies:  $a^\sigma = a$ ,  $b^\sigma = b$ ,  $x^\sigma = x^k$  for some  $k \in \mathbb{Z}_h^*$ . Since  $\sigma$  preserves the definition relations  $a^x = a^{t_1}$  and  $b^x = b^{t_2}$ , we get that  $t_1^{k-1} \equiv 1 \pmod{p}$  and  $t_2^{k-1} \equiv 1 \pmod{q}$ , which implies that  $|t_1| \mid (k-1)$  and  $|t_2| \mid (k-1)$  and so  $h \mid (k-1)$ , implying  $k = 1$  in  $\mathbb{Z}_h^*$ , or equivalently,  $\sigma = 1$ . Hence, we obtain the  $\phi(h)$  nonisomorphic regular maps as required.

*Case 3*  $p = q \geq 3$ : Let  $G = \text{GL}(2, p)$  and let  $D$  be the diagonal subgroup of  $G$ . In this case, by identifying  $a$  with  $t_{(1,0)}$ ,  $b$  with  $t_{(0,1)}$ , and  $x$  with  $\|t_1, 0; 0, t_2\| \in D \setminus Z(G)$ , we may consider  $G_3$  as a subgroup of  $A := \text{AGL}(2, p)$  satisfying  $G_3 = T : \langle x \rangle$ . Set  $z = x^{\frac{h}{2}}$ , which is the center involution of  $G$ . Since  $G_3$  has only one conjugacy class of subgroups of order  $h$  and of order 2, respectively, we may assume that  $r \in \langle x \rangle$ . Therefore,  $\ell$  is of the form  $a^i b^j z$  for some  $i, j \in \mathbb{Z}_p$ , because  $x$  fixes  $(1, 0)$  and  $(0, 1)$ . Since the action of the subgroup of inner automorphisms  $I(g)$  of  $A$  for  $g \in D$  fixes  $r$  and is transitive on the set  $\{t_{(i,j)} \mid i, j \in \mathbb{Z}_p^*\}$ , we may fix  $\ell = abz$ . Suppose that  $1 \neq \sigma \in \text{Aut}(G_3)$  fixes  $\langle x \rangle$  setwise and  $\ell$  pointwise. Then,  $\sigma$  fixes  $ab$ . By Lemma 3.3, we have that  $\sigma = I(u)|_{G_3} \in \text{Inn}(A)$  for some  $u \in N_G(\langle x \rangle)$ . By Proposition 3.2(2),  $C_G(\langle x \rangle) = D$  and  $N_G(\langle x \rangle) = D : \langle y \rangle$ , where  $y = \|[0, 1; 1, 0]\|$ . Hence, we have either  $u = \|[r, 0; 0, s]\|$  or  $u = \|[0, r; s, 0]\|$  for some  $r, s$  in  $\mathbb{Z}_p^*$ . By  $t_{(1,1)u} = (ab)^u = (ab)^\sigma = ab = t_{(1,1)}$ , we have  $r = s = 1$  and so

$u = y$ , noting that  $\sigma \neq 1$ . Therefore,  $x^\sigma = x^y = ||t_2, 0; t_1, 0||$ , which forces that  $|t_1| = |t_2|$  and so  $h = |t_i|$ . Assume  $x^\sigma = x^j$ . Then,  $t_2 = t_1^j = t_2^{j^2}$  and so  $j^2 = 1$  in  $\mathbb{Z}_h^*$ . Therefore, in the case  $p = q$  and  $h = |t_i|$ , we may assume that  $r = x^i$  for  $i \in \mathbb{Z}_h^*/(\mathbb{Z}_h^*)^{2+}$ . In the other cases,  $r = x^i$  for  $i \in \mathbb{Z}_h^*$ .  $\square$

**Lemma 4.5** *Let  $G_4(p, n)$  be a group defined in (4.4). Then, any regular map  $\mathcal{M}(G_4; r, \ell)$  with the underlying graph  $\mathcal{G}$  of order  $p^2$  is isomorphic to one of the following  $\phi(h)/|(\mathbb{Z}_h^*)^{2+}|$  nonisomorphic regular maps*

$$\mathcal{M}_4 = \mathcal{M}_4(p, h, i) = \mathcal{M}(G_4, x^i, az), \quad (4.10)$$

where  $h \mid (p^2 - 1)$  but  $h \nmid (p - 1)$  and  $h \geq 3$ ; either  $z = 1$  for  $p = 2$  or  $z = x^{\frac{h}{2}}$  for  $p \geq 3$ ; and  $i \in \mathbb{Z}_h^*/(\mathbb{Z}_h^*)^{2+}$ .

**Proof** If  $p = 2$  and  $h = 3$ , then  $G_4 \cong A_4$ . In this case, we have only one regular map  $\mathcal{M}_4(2, 3, 1)$  as expected.

Assume that  $p \geq 3$ . In a similar way as in the proof of Lemma 4.4 Case 3, we let  $G = \text{GL}(2, p)$ , and identifying respectively  $a$  with  $t_{(1,0)}$ ,  $b$  with  $t_{(0,1)}$ , and  $x$  with  $||e, f\theta; f, e||$  for some fixed pair  $(e, f)$  such that  $ef \neq 0$  and  $x$  is of order  $h$ , we consider  $G_4$  as a subgroup of  $A := \text{AGL}(2, p)$  of the form  $G_4 = T : \langle x \rangle$ . Then, the central involution  $z = x^{\frac{h}{2}} \neq 1$ . Since  $G_4$  has only one conjugacy class of subgroups of order  $h$  and of order 2, respectively, we may assume that  $r \in \langle x \rangle$ . As well, we may assume that  $\ell = a^i b^j z$  for  $i, j \in \mathbb{Z}_p$  with  $(i, j) \neq (0, 0)$ . By Lemma 3.2 (2), all the elements of the form  $||e', f'\theta; f', e'||$  form a cyclic subgroup  $H$  of  $G$ , which is regular on  $V \setminus \{0\}$ . Therefore, the subgroup of  $A$  consisting of the inner automorphisms  $I(g)$  for  $g \in H$  is transitive on  $\{a^i b^j z \mid i, j \in \mathbb{Z}_p, (i, j) \neq (0, 0)\}$ . Therefore, we may fix  $\ell = az$ .

Suppose that  $1 \neq \sigma \in \text{Aut}(G_4)$  fixes  $\langle x \rangle$  setwise and  $\ell$  pointwise. Then,  $\sigma$  fixes  $a$ . By Lemma 3.3,  $\sigma = I(u)|_{G_4} \in \text{Inn}(G_4)$  for some  $u \in N_G(\langle x \rangle)$ . By Proposition 3.2(2),  $C_G(\langle x \rangle) = H$  and  $N_G(\langle x \rangle) = H : \langle y \rangle$ , where  $y = ||1, 0; 0, -1||$ . Hence,  $u = ||e, f\theta; f, e||$  or  $||e, -f\theta; f, -e||$  for the above fixed pair  $(e, f)$ . Computing  $a^\sigma = a$ , we get  $f = 0$  and thus  $u = y$ . Therefore,  $x^\sigma = x^y = ||e, -f\theta; -f, e||$ . In particular,  $|\sigma| = 2$ . Assume  $x^\sigma = x^j$ . Then,  $j^2 = 1$  in  $\mathbb{Z}_h^*$ . Consequently, we may assume that  $r = x^i$  for  $i \in \mathbb{Z}_h^*/(\mathbb{Z}_h^*)^{2+}$ . Hence, we obtain  $\phi(h)/|(\mathbb{Z}_h^*)^{2+}|$  nonisomorphic regular maps as required in (4.10).  $\square$

The following two lemmas dealing with the families of the groups  $G_5(p)$  and  $G_6(p, q, h)$  are proved in [5].

**Lemma 4.6** *Let  $G_5(p)$  be a group defined in (4.5), where  $p \geq 2$ . Then, any regular map  $\mathcal{M}(G_5; r, \ell)$  with the underlying graph of order  $2p$  is isomorphic to the map*

$$\mathcal{M}_5(p) = \mathcal{M}(G_5; a, x). \quad (4.11)$$

*Its underlying graph is  $K_{p,p}$ .*

**Lemma 4.7** *Let  $G_6 = G_6(p, q, h)$  be a group defined in (4.6). Then, any regular map  $\mathcal{M}(G_6; r, \ell)$  with the underlying graph  $\mathcal{G}$  of order  $pq$  is isomorphic to one of following  $\frac{q-1}{h}\phi(h)$  nonisomorphic regular maps*

$$\mathcal{M}_6(p, q, h, i, j) := \mathcal{M}(G_6; a'y^j, x^i y^{\frac{h}{2}}), \quad (4.12)$$

*where either  $p \geq 2$ ,  $q \geq 3$ ,  $q \mid (p \pm 1)$  and  $h = 2$ , or  $p = q \geq 3$  and  $h$  is an even divisor of  $q - 1$ . In both cases,  $i \in \mathbb{Z}_q^*/(\mathbb{Z}_q^*)^{h+}$  and  $j \in \mathbb{Z}_h^*$ ; and*

$$a' = \begin{cases} t_{(1,0)} & \text{if } p = 2, q \mid (p+1) \text{ or } p = q, \\ t_{(1,1)} & \text{if } q \mid (p-1). \end{cases}$$

*Moreover,  $\mathcal{G} \cong \mathcal{G}(q, h)[\overline{K}_p]$ .*

Combining Lemmas 4.1-4.7, one can obtain the following classification theorem.

**Theorem 4.8** *Let  $\mathcal{M}$  be a regular map with a simple underlying graph  $\mathcal{G}$  of order  $pq$  for any two primes  $p$  and  $q$ ,  $p \geq q$ . Then,  $\mathcal{M}$  is isomorphic to one of the following regular maps uniquely determined by the given integer parameters:*

(1)  $p = q = 2$ ,

$$\mathcal{M} \cong \mathcal{M}_5(2) \text{ and } \mathcal{G} \cong C_4.$$

$$\mathcal{M} \cong \mathcal{M}_4(2, 3, 1) \text{ and } \mathcal{G} \cong K_4.$$

(2)  $p = q \geq 3$ ,

$\mathcal{M} \cong \mathcal{M}_2(p, h, i)$ , where  $h \geq 2$ ,  $h$  is an even divisor of  $p(p-1)$  and  $i \in \mathbb{Z}_h^*$ . The graph  $\mathcal{G}$  is isomorphic to  $\mathcal{G}(p, h/p)[\overline{K}_p]$  if  $p \mid h$ , or to a  $p$ -fold regular cover of  $\mathcal{G}(p, h)$  if  $p \nmid h$ .

$\mathcal{M} \cong \mathcal{M}_3(p, p, t_1, t_2, i)$ , where  $t_1 \neq t_2$ ,  $\langle (t_1, t_2) \rangle$  is a subgroup in  $\mathbb{Z}_p^* \times \mathbb{Z}_p^*$  containing  $(-1, -1)$  and  $h = [|t_1|, |t_2|] > 2$ ; and  $i \in \mathbb{Z}_h^*/(\mathbb{Z}_h^*)^{2+}$  if  $h = |t_1| = |t_2|$ , and  $i \in \mathbb{Z}_h^*$  otherwise. The graph  $\mathcal{G}$  is of valency  $h$  and  $\text{Aut}(\mathcal{M})$  acts imprimitively on  $V(\mathcal{G})$ .

$\mathcal{M} \cong \mathcal{M}_4(p, h, i)$ , where  $h \geq 3$ ,  $h \mid (p^2 - 1)$  but  $h \nmid (p - 1)$ ,  $i \in \mathbb{Z}_h^*/(\mathbb{Z}_h^*)^{+2}$ .  
The graph  $\mathcal{G}$  is of valency  $h$  and  $\text{Aut}(\mathcal{M})$  acts primitively on  $V(\mathcal{G})$ .

$\mathcal{M} \cong \mathcal{M}_6(p, p, h, i, j)$ , where  $h$  is an even divisor of  $p - 1$ ,  $i \in \mathbb{Z}_p^*/(\mathbb{Z}_p^*)^{h+}$ ,  
and  $j \in \mathbb{Z}_h^*$ . Moreover,  $\mathcal{G} \cong \mathcal{G}(p, h)[\overline{K}_p]$ .

(3)  $p \geq 3$  and  $q = 2$ ,

$\mathcal{M} \cong \mathcal{M}_6(2, 3, 2, 1, 1)$ , where  $p = 3$ ; and  $\mathcal{G} \cong \mathcal{G}(3, 2)[\overline{K}_2]$ .

$\mathcal{M} \cong \mathcal{M}_1(p, h, i)$ , where  $p \geq 7$ ,  $h \geq 3$ ,  $h$  is an odd divisor of  $p - 1$  and  
 $i \in \mathbb{Z}_h^*$ . The graph  $\mathcal{G}$  is bipartite of valency  $h$ .

$\mathcal{M} \cong \mathcal{M}_3(p, 2, t_1, 1, i)$ , where  $p \geq 3$ ,  $t_1 \in \mathbb{Z}_p^*$ ,  $i \in \mathbb{Z}_h^*$  for  $h = |t_1|$ , an even  
divisor of  $p - 1$ ; and  $\mathcal{G} \cong K_2 \otimes \mathcal{G}(p, h)$  is a bipartite graph of valency  $h$ .

$\mathcal{M} \cong \mathcal{M}_5(p)$ , where  $p \geq 3$  and  $\mathcal{G} \cong K_{p,p}$ .

(4)  $p > q \geq 3$ ,

$\mathcal{M} \cong \mathcal{M}_3(p, q, t_1, t_2, i)$ , where  $\langle (t_1, t_2) \rangle$  is a subgroup in  $\mathbb{Z}_p^* \times \mathbb{Z}_q^*$  containing  
 $(-1, -1)$  and  $i \in \mathbb{Z}_h^*$  for  $h = [|t_1|, |t_2|]$ . The graph  $\mathcal{G}$  is of valency of  $h$ .

$\mathcal{M}_6(p, q, 2, i, 1)$ , where  $q \mid (p \pm 1)$  and  $i \in \mathbb{Z}_q^*/(\mathbb{Z}_q^*)^{2+}$ . Moreover,  $\mathcal{G} \cong C_q[\overline{K}_p]$ .

It follows that in the (general) case (4) of the theorem,  $p > q \geq 3$ , only two  
sorts of regular maps are admissible. One is formed by regular embeddings of  
the lexicographic products of cycles  $C_q$  with  $p$  isolated vertices. The maps in  
the other family, denoted as  $\mathcal{M}_3(p, q, t_1, t_2, i)$ , are the least regular maps covering  
both a regular embedding of  $\mathcal{G}(p, t_1)$  and of  $\mathcal{G}(q, t_2)$ .

For  $p \geq q \geq 3$ , let  $S_h(p, q)$  be the set of the cyclic subgroups  $H$  of order  $h$  of  
 $\mathbb{Z}_p^* \times \mathbb{Z}_q^*$  that contain  $(-1, -1)$ , and let  $S'_h(p)$  be the set of the cyclic subgroups  
 $H = \langle (t, t^j) \rangle$  of  $\mathbb{Z}_p^* \times \mathbb{Z}_p^*$  for  $j \in \mathbb{Z}_h^*/(\mathbb{Z}_h^*)^{2+}$ , where  $h = |t|$ . Then,  $S_h(p, q)$  and  
 $S'_h(p)$  can be easily determined for any given  $p$  and  $q$  and  $h$ . Finally, denote by  
 $S(p, q) = \cup_h S_h(p, q)$  and similarly by  $S'(p) = \cup_h S'_h(p)$ . Using these notations we  
obtain the following enumeration of nonisomorphic regular embeddings with the  
underlying graphs of order  $pq$ .

**Corollary 4.9** *Let  $p$  and  $q$  be any two primes with  $p \geq q$ . Let  $N(p, q)$  be the  
number of all regular embeddings with the underlying graphs of order  $pq$ . Then,*

$$(1) \quad N(2, 2) = 2, \quad N(3, 2) = 3, \quad N(p, 2) = 1 + \sum_{h \mid (p-1), h \geq 2} \phi(h), \quad \text{if } p \geq 5;$$

$$(2) \quad N(p, p) = \sum_{h|(p-1), 2|h} \phi(h) \left(p + \frac{p-1}{h}\right) + \sum_{h|(p^2-1), h \nmid (p-1), 2|h} \phi(h) / |(Z_h^*)^{2+}| \\ + \sum_{H \in S(p, p) \setminus S'(p)} \phi(|H|) + \sum_{H \in S'(p)} \phi(|H|) / |(Z_{|H|}^*)^{2+}|, \text{ if } p \geq 3;$$

$$(3) \quad N(p, q) = \sum_{H \in S(p, q)} \phi(|H|), \text{ if } p > q \geq 3, q \nmid (p+1) \text{ and } q \nmid (p-1);$$

$$(4) \quad N(p, q) = \frac{q-1}{2} + \sum_{H \in S(p, q)} \phi(|H|), \text{ if } p > q \geq 3, \text{ and either } q \mid (p-1) \text{ or } \\ q \mid (p+1).$$

## 5 Genera of regular maps with $pq$ vertices

An orientable map  $\mathcal{M} = \mathcal{M}(\mathcal{G}, R)$  of *genus*  $g$  is an embedding of the graph  $\mathcal{G}$  into the orientable surface with  $g$  handles. By  $|V|$ ,  $|E|$  and  $|F|$ , we denote the number of vertices, edges and faces of the given map (embedding). Then,  $|V| - |E| + |F| = 2 - 2g$  by the Euler equation. For any regular map, let  $k$  be the face-size and  $n$  the valency of the underlying graph. Then, one can easily see that  $n|V| = k|F| = 2|E|$ . Consequently,

$$g = 1 + \frac{1}{2}(|E| - |V| - |F|) = 1 + \frac{|V|}{2} \left( \frac{n}{2} - \frac{n}{k} - 1 \right).$$

Let  $\mathcal{M} = \mathcal{M}(\mathcal{G}, R)$  be any regular map with underlying graph  $\mathcal{G}$  of order  $pq$  and of valency  $n$ . Then, we have

$$g = 1 + \frac{pq}{2} \left( \frac{n}{2} - \frac{n}{k} - 1 \right). \quad (5.1)$$

Let  $\text{Mon}(\mathcal{M}) = \langle R, L \rangle$  and  $\text{Aut}(\mathcal{M}) = \langle r, \ell \rangle$ . Then, by the definition of a regular orientable map, we have  $k = |RL| = |r\ell|$ . Therefore, in order to compute the genus of every regular map in Theorem 4.8, we need to determine  $k = |r\ell|$  for each group  $G_i$  for  $1 \leq i \leq 6$ .

**Proposition 5.1** *Let  $\mathcal{M}$  be a regular map with a simple underlying graph  $\mathcal{G}$  of order  $pq$  for any two primes  $p$  and  $q$  (see Theorem 4.8) and let  $g$  be the genus of  $\mathcal{M}$ . Then,*

$$(1) \text{ If } \mathcal{M} \cong \mathcal{M}_1(p, h, i), \text{ then } g = 1 + \frac{p(h-3)}{2}.$$

$$(2) \text{ If } \mathcal{M} \cong \mathcal{M}_2(p, h, i), \text{ then } g = 1 + \frac{p(h-4)}{2} \text{ if } 4 \mid h, \text{ and } g = 1 + \frac{p(h-6)}{2} \text{ if } 4 \nmid h.$$

- (3) If  $\mathcal{M} \cong \mathcal{M}_3(p, q, t_1, t_2, i)$ , then  $g = 1 + \frac{pq(h-4)}{4}$  if  $4 \mid h$ , and  $g = 1 + \frac{pq(h-6)}{4}$  if  $4 \nmid h$ .
- (4) If  $\mathcal{M} \cong \mathcal{M}_4(p, h, i)$ , then  $g = 1 + \frac{p^2(h-4)}{4}$  if  $4 \mid h$ , and  $g = 1 + \frac{pq(h-6)}{4}$  if  $h = 3$  and  $4 \nmid h$ .
- (5) If  $\mathcal{M} \cong \mathcal{M}_5(p)$ , then  $g = 1 + \frac{p(p-3)}{2}$ .
- (6) If  $\mathcal{M} \cong \mathcal{M}_6(p, q, h, i, j)$ , then  $g = 1 + \frac{p}{2}(pq - 2p - q)$  if  $p \neq q$ ,  $g = 1 + \frac{p^2}{4}(hp - 4)$  if  $p = q$  and  $4 \mid h$ , and  $g = 1 + \frac{p^2}{4}(hp - 6)$  if  $p = q$  but  $4 \nmid h$ .

**Proof** The proof is divided into six cases according to the six families  $\mathcal{M}_i$ ,  $1 \leq i \leq 6$ , of the maps in Theorem 4.8.

*Case 1*  $\mathcal{M} \cong \mathcal{M}_1(p, h, i)$  : Let  $G_1 = G_1(p, h)$  with the admissible parameters in (4.1). Then,  $G_1 = \langle x \rangle : \langle y \rangle \cong \mathbb{Z}_p : \mathbb{Z}_{2h}$ , where  $h$  is odd. It follows from (4.7) that  $q = 2$ , the valency  $n = h$ ,  $r = y^{2i}$  where  $i \in \mathbb{Z}_h^*$  and  $\ell = xy^h$ . Then,  $r\ell = y^{2i}xy^h = y^{2i+h}x^{-1}$ . By  $\langle x \rangle \triangleleft G_1$  it follows that  $|y^{2i+h}| \mid |r\ell|$ . Noting that  $(2i + h, 2h) = 1$ , we have  $|y^{2i+h}| = 2h \mid k = |r\ell|$ . Since every subgroup of order  $2h$  of  $G_1$  is maximal, we have  $k = 2h$ . Inserting the values of  $q$ ,  $n$ , and  $k$  in (5.1), we get the result given in (1).

*Case 2*  $\mathcal{M} \cong \mathcal{M}_2(p, h, i)$  : Let  $G_2 = G_2(p, h)$  with the admissible parameters in (4.2). Then,  $G_2 = \langle x \rangle : \langle y \rangle \cong \mathbb{Z}_{p^2} : \mathbb{Z}_h$ , where  $h$  is even. It follows from (4.8) that  $q = 2$ , the valency  $n = h$ ,  $r = y^i$  where  $i \in \mathbb{Z}_h^*$  and  $\ell = xy^{\frac{h}{2}}$ . Then,  $r\ell = y^i xy^{\frac{h}{2}} = y^{i+\frac{h}{2}}x^{-1}$ . Note that  $(i + \frac{h}{2}, h)$  is equal to 1 or 2, according to  $4 \mid h$  or  $4 \nmid h$ , respectively. For a reason similar to Case 1, we know that  $k = |r\ell| = h$  or  $\frac{h}{2}$ , if  $4 \mid h$  and  $4 \nmid h$ , respectively. Then, the result in (2) follows from (5.1).

*Case 3*  $\mathcal{M} \cong \mathcal{M}_3(p, q, t_1, t_2, i)$  : Similar to Case 2.

*Case 4*  $\mathcal{M} \cong \mathcal{M}_4(p, h, i)$  : Similar to Case 2.

*Case 5*  $\mathcal{M} \cong \mathcal{M}_5(p, h, i)$  : It is easy to see  $k = |r\ell| = 2p$ , as the desired.

*Case 6*  $\mathcal{M} \cong \mathcal{M}_6(p, h, i)$  : Let  $G_6 = G_6(p, q, h)$  with the admissible parameters in (4.6). Then,  $G_6 = \langle x \rangle : \langle y \rangle \cong \mathbb{Z}_{p^2} : (\mathbb{Z}_q : \mathbb{Z}_h)$ .

First, assume  $p \neq q$ , so that  $h = 2$ . In this case, it follows from (4.12) that  $r = a'y$  and  $\ell = x^i y$ , where  $i \in \mathbb{Z}_q^*/(\mathbb{Z}_q^*)^{2+}$ , and  $a' = t_{(1,0)}$  if  $q \mid (p+1)$ , and  $a' = t_{(1,1)}$  if  $q \mid (p-1)$ . Then,  $r\ell = a'yx^i y = a'x^{-i}$ . Let  $T$  be the translation subgroup of  $\text{AGL}(2, p)$ . Since  $a' \in T \triangleleft \text{AGL}(2, p)$  and  $|x| = q$ , it follows that  $q \mid |r\ell|$ . Since  $x$  has no fixed points in its action on  $V \setminus \{0\}$ , the subgroup  $T : \langle x \rangle$  cannot

contain any element of order  $pq$ . Therefore,  $|r\ell|$  cannot be  $pq$ , and consequently,  $k = |r\ell| = q$ . Noting that  $n = 2p$ , we have the desired result in (6) by (5.1).

Next, assume  $p = q$ . In this case, it follows from (4.12) that  $r = a'y^j$  and  $\ell = x^i y^{\frac{h}{2}}$ , where  $a' = t_{(1,0)}$ ,  $j \in \mathbb{Z}_h^*$ , and  $i \in \mathbb{Z}_q^*/(\mathbb{Z}^*)_q^{h+}$ . Then,  $r\ell = a'y^j x^i y^{\frac{h}{2}} = a'(y^{j+\frac{h}{2}} x^{-i})$ . Denote  $y^{j+\frac{h}{2}} x^{-i}$  by  $y_1$ . Then, from (4.6) we know that  $y_1 = ||1, -i; 0, t^{j-1}||$ . Therefore,  $|y_1| = h$  if  $4 \mid h$ , and  $|y_1| = \frac{h}{2}$  if  $4 \nmid h$ . For a reason similar to the previous one, we have that  $|y_1| \mid |r\ell|$ . A direct computation shows that  $1 \neq (r\ell)^{|y_1|} \in T$ . Therefore,  $k = p|y_1|$ . Noting that  $n = ph$ , we obtain the desired result by inserting the values of  $n$  and  $k$  in (5.1).  $\square$

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