3-manifolds with Heegaard genus at most two represented by crystallisations with at most 42 vertices

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March 22, 2005

Abstract

It is known that every closed compact orientable 3-manifold $\mathcal{M}$ can be represented by a 4-edge-coloured 4-valent graph called a crystallisation of $\mathcal{M}$. Casali and Grasselli proved that 3-manifolds of Heegaard genus g can be represented by crystallisations with a very simple structure which can be described by a $2(g+1)$-tuple of non-negative integers. The sum of first $g+1$ integers is called complexity of the admissible $2(g+1)$-tuple. If c is the complexity then the number of vertices of the associated graph is $2c$.

In the present paper we describe all prime 3-manifolds of Heegaard genus 2 described by 6-tuples of complexity at most 21.

1 Closed 3-manifolds

In this preliminary section we recall some definitions and known facts about orientable 3-manifolds without boundary.

We denote the *n-dimensional Euclidean space* by E^n, the *unit ball* by $\{x \in E^n : \|x\| \leq 1\}$ by B^n, and the *unit sphere* $\{x \in E^n : \|x\| = 1\}$ by S^{n-1}. We will call a space homeomorphic to B^n (S^{n-1}) a *n-cell* (($n-1$)*-sphere*). Finally, by $D_n = B_n - S^{n-1}$ we denote the *open ball*.

Definition 1 (3-manifold without boundary) *[12] A topological* 3*-manifold is a separable metric space each of whose points has an open neighbourhood homeomorphic to* E^3*. In what follows all the considered* 3*-manifolds will, unless otherwise stated, assumed to be compact, connected and orientable.*

Each 3-manifold $\mathcal{M}$ gives rise to a *fundamental group* $\pi_1(\mathcal{M})$. The fundamental group of a 3-manifold is finitely generated.

Homology group. A *commutator* $[a, b]$ of elements a, b of the group G is the element $[a, b] = aba^{-1}b^{-1}$. The *derived subgroup* $G' \trianglelefteq G$ generated by all commutators is known to be normal. The factor G/G' is called an *abelianisation* of the group G. Abelianisation of the fundamental group $\pi_1(\mathcal{M})$ of a 3-manifold $\mathcal{M}$ is called the *homology group* $H_1(\mathcal{M})$.

Connected sum. The *connected sum* of two 3-manifolds $\mathcal{M}$ and $\mathcal{N}$, denoted $\mathcal{M}\#\mathcal{N}$, is formed by cutting open balls $D_1 \subset \mathcal{M}$ and $D_2 \subset \mathcal{N}$ from both manifolds, with boundaries ∂D_1, ∂D_2 homeomorphic to S^2, and gluing them along $\partial D_1 \subset \mathcal{M}$ and $\partial D_2 \subset \mathcal{N}$ together, following a homeomorphism $\partial D_1 \to \partial D_2$ [17].

Connected sum is a well-defined associative and commutative operation in the category of oriented 3-manifolds and orientation preserving homeomorphisms [12, p. 24]. Let G, H be groups. Their *free product* will be denoted by $G * H$.

Theorem 1.1 (Consequence of Van Kampen's Theorem) *[17, p. 91] [12, p. 25] Let $\mathcal{M}$,$\mathcal{N}$ be 3-manifolds. Let $\mathcal{X} \cong \mathcal{M} \cup \mathcal{N}$. If $\mathcal{M} \cap \mathcal{N}$ is simply connected, then $\pi_1(\mathcal{X})$ is the free product of the groups $\pi_1(\mathcal{M})$ and $\pi_1(\mathcal{N})$ with respect to homomorphisms $\psi_1 : \pi_1(\mathcal{M}) \to \pi_1(\mathcal{X})$ and $\psi_2 : \pi_1(\mathcal{N}) \to \pi_1(\mathcal{X})$ induced by inclusions*

The above theorem deals with a particular case of a more general statement establishing that the fundamental group of a connected sum of two topological spaces is a free product of fundamental groups of the factors. Generally the reverse implication is not true. However, in the particular case of compact, connected 3-manifolds we have the following theorem.

Theorem 1.2 *[12, p. 66] Let $\mathcal{M}$ be a compact, connected 3-manifold. If $\pi_1(\mathcal{M}) \cong G_1 * G_2$ then $\mathcal{M} = \mathcal{M}_1 \# \mathcal{M}_2$ where $\pi_1(\mathcal{M}_i) \cong G_i$, $i = 1, 2$.*

Definition 2 (Prime 3-manifolds) *[12, p. 27] A 3-manifold $\mathcal{M}$ is a prime if $\mathcal{M} = \mathcal{M}_1 \# \mathcal{M}_2$ implies one of $\mathcal{M}_1, \mathcal{M}_2$ to be the S^3.*

Definition 3 (Irreducible 3-manifolds) *[12, p. 28] A 3-manifold is irreducible if each S^2 in $\mathcal{M}$ bounds a 3-cell in $\mathcal{M}$.*

Irreducible 3-manifolds are prime. As a partial converse we have the following lemma.

Lemma 1.3 [12, p. 28] The only prime reducible 3-manifold is $S^1 \times S^2$.

Theorem 1.4 (Existence and uniqueness of decomposition) *[12, p. 31,p. 35] Each compact, connected and orientable 3-manifold $\mathcal{M}$ can be expressed as a connected sum $\mathcal{M} = \mathcal{M}_1 \# \mathcal{M}_2 \# \ldots \# \mathcal{M}_n$ of finite number of prime factors. The decomposition is unique up to reordering of factors in the category of oriented 3-manifolds.*

One of the difficulties in study of the structure of prime 3-manifolds is a possibility of existence of non-trivial simply-connected prime manifolds. Poincaré asserted that a manifold with trivial homology (group) is simply connected and therefore a sphere: "... est simplement c'est-a dire homeómorphe à l'hypersphere" [21]. Shortly thereafter he discovered an example of a nonsimply connected homology 3-sphere. However the following is still unsettled*.

*Recent work of G. Perelman in the theory of Ricci flows suggests that the Poincaré Conjecture could be settled in the affirmative.

Conjecture 1.5 (Poincaré Conjecture) *[12, p. 26] Each closed, connected, simply connected* 3*-manifold is homeomorphic to* S^3*.*

Using the term "homotopy n-sphere" for an n-manifold homotopy equivalent to S^n, we have

Theorem 1.6 *[12, p. 26] A* 3*-manifold* $\mathcal{M}$ *is a homotopy* 3*-sphere if and only if* $\mathcal{M}$ *is closed, connected, simply connected* 3*-manifold.*

A 3-manifold $\mathcal{M}$ with boundary which contains a collection $\{D_1, D_2, \ldots, D_n\}$ of pairwise disjoint, properly embedded 2-cells such that the result of cutting $\mathcal{M}$ along $\bigcup D_i$ is a 3-cell is called a *cube with n-handles* [12, p. 15], or alternatively, *handlebody.* By Van Kampen's theorem $\pi_1(\mathcal{M})$ is a free group of rank n.

A *Heegaard splitting* of a closed, connected and orientable 3-manifold $\mathcal{M}$ is a pair (V_1, V_2) where V_i is a cube with handles (i=1,2), $M = V_1 \cup V_2$, and $V_1 \cap V_2 = \partial V_1 = \partial V_2$.

We note that the boundary of a cube with n-handles, V, is compact, connected surface of Euler characteristic $2 - 2n$ which is orientable if and only if V is orientable. Thus for a Heegard splitting (V_1, V_2) of a 3-manifold $\mathcal{M}$, V_1 and V_2 have the same number of handles and both are orientable since $\mathcal{M}$ is orientable. [12, p. 17]

Theorem 1.7 *[12, p. 17] Each* 3*-manifold* $\mathcal{M}$ *has a Heegaard splitting.*

Definition 4 (Heegaard genus of $\mathcal{M}$) *Heegaard genus of* $\mathcal{M}$ *is minimum of genera of Heegaard splittings of* $\mathcal{M}$*, where the minimum is taken through all Heegaard splittings of* $\mathcal{M}$*.*

Proposition 1.8 (Genus zero splittings) *[5] If a* 3*-manifold has a splitting of genus zero, it is homeomorphic to* S^3*.*

A *lens space* $\mathcal{L}(p,q)$ is a 3-manifold formed as a quotient of S^3 (considered as the unit sphere of $\mathbb{C}^2$) by the cyclic group $\mathbb{Z}_p$ of isometries generated by $(z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi iq/p} z_2)$ for coprime integers p and q. The splitting of genus one has Heegaard surface $|z_1|^2 = |z_2|^2 = \frac{1}{2}$ and divides $\mathcal{L}(p,q)$ into two solid tori given by $|z_1|^2 \leq \frac{1}{2}$ and $|z_1|^2 \geq \frac{1}{2}$ [5]. Let us remark that lens spaces are irreducible 3-manifolds [12, p. 28].

The genus one splitting of $S^1 \times S^2$ is even easier to see – take two solid tori and glue them by the identity map along their boundaries reversing the orientation of one of the handlebodies. Analysis of all possible torus gluing maps shows that the manifolds listed are the only ones with genus one splittings [5][12, pp. 20 – 23].

Proposition 1.9 (Genus one splittings) *[5][12, pp. 20 – 23] If* 3*-manifold* $\mathcal{M}$ *has a splitting of genus one, it is either homeomorphic to a lens space or to* $S^1 \times S^2$*.*

Generally, to understand the structure of 3-manifolds the following approach was used; using certain set of operations each 3-manifold is decomposed into prime 3-manifolds with respect to these operations. It happened that two operations turned to be useful. First is the connected sum, the other one is the so called Johanson-Jaco-Shalen decomposition [23].

Let us consider a prime 3-manifold $\mathcal{M}$ with respect to these two operations. Clearly $\mathcal{M} \approx \widetilde{\mathcal{M}}/\pi_1(\mathcal{M})$, where $\widetilde{\mathcal{M}}$ is the universal cover. A goal is that the action of the fundamental group on the universal cover have a geometric meaning. Thurston [23] discovered that there are *eight* types of geometries, including the well-known elliptic on S^3 and Euclidean on E^3, which can be attached with $\widetilde{\mathcal{M}}$. The action of the fundamental group $\pi(\mathcal{M})$ on $\widetilde{\mathcal{M}}$ is interpreted as the action of a group of "geometric transformations". In this case $\mathcal{M}$ is said to be *geometric*. A 3-manifold is said to have a *geometric decomposition* if it decomposes into prime 3-manifolds $\mathcal{M}_i$ such that each $\mathcal{M}_i$ is geometric.

A celebrated Thurston's conjecture reads as follows.

Conjecture 1.10 (Thurston's Geometrisation Conjecture) *Every compact connected orientable 3-manifold has a geometric decomposition.*

More information on the geometrisation can be found in a book of W.P. Thurston [23]. Geometrisation of genus two 3-manifolds is discussed in [5]. Using Thurston's Symmetry Theorem [23] the following two statements are proved.

Theorem 1.11 *[5] Every irreducible 3-manifold with Heegaard genus two is geometrisable.*

Theorem 1.12 *[5] Any reducible manifold $\mathcal{M}$ with Heegaard splitting of genus two is expressible as a connected sum of two 3-manifolds with Heegard splittings of genus one. These are lens spaces or $S^1 \times S^2$*

The above two theorems imply that a genus two 3-manifold has a geometric decomposition is sense of Thurston. Another consequence reads as follows.

Theorem 1.13 *Let $\mathcal{M}$ be a compact connected orientable 3-manifold of genus two. Let the fundamental group of $\mathcal{M}$ be finite. Then the universal cover of $\mathcal{M}$ is the Poincaré sphere S^3.*

Proof. Assume, on the contrary, that the universal cover $\widetilde{M}$ is not S^3. By its definition it is a counterexample to the Poincaré Conjecture. By Theorem 1.11 this is impossible. □

Corrolary 1.14 *Let $\mathcal{M}$ be a compact, connected, orientable 3-manifold of genus at most two. Let $\mathcal{M}$ has finite cyclic fundamental group of order p. Then $\mathcal{M}$ is homeomorphic to a lens space $\mathcal{L}(p, q)$, for some $q \in \mathbb{N}$.*

2 Graph encoded 3-manifolds

Every compact connected n-manifold, $n \leq 3$, can be expressed as a *simplicial complex* containing a finite set of simplices of dimension n. A general problem is to decide, whether two different simplicial complexes represent the same n-manifold. The structure of a 3-dimensional simplicial complex representing a 3-manifold without boundary can be described by means of the associated dual graph. To be more precise, Pezzana [20] proved that a closed connected 3-manifold can be represented by particular edge-coloured graph called crystallisation. The concept of crystallisations plays a crucial role in our next considerations.

Each orientable 3-manifold $\mathcal{M}$ can be represented by a bipartite 4-edge-coloured graph [1]. Let T be any simplicial triangulation of $\mathcal{M}$ and T' be its first barycentric subdivision. Each vertex $\hat{\omega}$, which is the barycenter of the simplex ω of T is labelled by the dimension of ω. Take the dual graph Γ of T' and if uv is an edge and $\{i, j, k\}$ are the colours of respective triangle in T use the colour complementary to $\{i, j, k\}$ to colour the edge uv. The labelling of vertices of T induces a decomposition of the tetrahedrons of T into two classes distinguished by orientation, where adjacent tetrahedrons belong to different classes. Thus Γ is bipartite. The dual graph Γ of T', together with the edge-colouring ν, is a 4-coloured graph, representing T. Conversely, given bipartite 4-edge-coloured graph one can construct an associated 3-dimensional complex T. However, in general, T need not to be homeomorphic to a 3-manifold.

4-edge-coloured graph. Let $\Gamma = \{V(\Gamma), E(\Gamma)\}$ be a bipartite graph and let there exist a mapping $\nu : E(\Gamma) \to \Delta_4 = \{1, 2, 3, 4\}$ such that for all pairs of incident edges $f, g \in E(\Gamma) : \nu(f) \neq \nu(g)$. This mapping is called a graph colouring and the graph Γ a 4-edge-coloured graph.

Dipole-move. Let Γ be a 4-coloured graph and let Θ is a subgraph of Γ containing vertices X, Y joined by h edges ($1 \leq h \leq 3$) coloured by colours $c_1, \ldots, c_h$. If X and Y are in two different components of graph $\Gamma_{\Delta_4 - \{c_1,\ldots,c_h\}}$ induced by the set of complementary colours $\Delta_4 - \{c_1, \ldots, c_h\}$ then the subgraph Θ will be called a *dipole of type h*.

There is a well defined operator over the set of 4-coloured graphs [7] called *dipole-move*. Note that a dipole-move can be defined generally for $(n + 1)$-coloured, connected and bipartite graphs coding n-manifolds.

Construction 2.1 (Elementary dipole-move) If Θ is a dipole of type h in Γ_{Δ_4} coloured by colours $\{c_1, \ldots, c_h\}$ we define a dipole-move as follows:

a) *Cutting of* Θ

- remove edges and vertices of Θ
- glue "hanging" edges of graph Γ_{Δ_4} of same colour

b) *Adding of* Θ as inverse to cutting

Main result of [7] states that graphs Γ and Γ' represent homeomorphic 3-dimensional spaces if and only if there is a finite sequence of dipole-moves transforming Γ to Γ'. Hence the "homeomorphism problem" reduces to the problem to decide whether two 4-coloured graphs are "dipole-move equivalent". Not all 4-coloured graphs represent (orientable) 3-manifolds. The characterisation of the subclass that represents 3-manifolds follows.

Residual graph. Take a proper subset of colours $\mathcal{B} \subset \Delta_4$. The *residual graph* $\Gamma_{\mathcal{B}}$ is obtained by deleting the edges coloured by colours from $\mathcal{B}$. We use the symbol $\Gamma_{\hat{c}}$ for residual graph created by deleting the edges coloured by the colour c.

The following characterisation theorem was proved by Pezzana.

Theorem 2.2 *[8] A* 4*-coloured graph* Γ *represents a compact, connected and orientable* 3*-manifold if and only if it is contracted and the residual* $\Gamma_{\hat{c}}$ *with the induced colouration admits a regular embedding into the* 2*-sphere* S^2 *for each* $c \in \Delta_4$*.*

Regular genus. Let Γ be a bipartite 4-edge-coloured graph. Since the colouring is regular a factor induced by two colours is a disjoint union of bicoloured cycles. Let $\mathcal{I}$ denotes the set of 2-cell embeddings of Γ_{Δ_4} into a closed orientable surface such that the local rotation of colours induced by the embedding in "black" vertices is the same, say ρ, while the local rotation of colours in "white" vertices is ρ^{-1}. Note that there are six possibilities for choosing ρ. It follows that faces of such embedding are bounded by bicoloured cycles. Out of the six possibilities for ρ we choose ρ such that the genus g of the underlying surface is minimal in $\mathcal{I}$. Let us call $g = g(\Gamma)$ the *regular genus of graph* Γ. The *Regular genus of a* 3*-manifold* $\mathcal{M}$ is the minimum of genera $g(\Gamma)$, where the minimum is taken through all bipartite 4-edge-coloured graphs Γ representing $\mathcal{M}$. Shortly we talk about the genus of $\mathcal{M}$. It is known that regular genus of $\mathcal{M}$ is equal to the Heegaard genus of $\mathcal{M}$ [1].

Crystallisations. As already mentioned, not all 4-coloured bipartite graphs represent compact, connected and orientable 3-manifolds. A 4-coloured graph Γ is said to be *contracted* if for each colour $c \in \Delta_4$ the residual graph $\Gamma_{\hat{c}}$ is connected. Contracted 4-coloured graphs are called *crystallisations*.

Crystallisations given by $\mathbf{2(n+1)}$-tuples. Let n be a non-negative integer. It follows from [3] that each (closed) orientable 3-manifold of genus n can be represented by a crystallisation Γ which structure can be coded by a $2(n+1)$-tuple of integers satisfying certain conditions. Let $\widetilde{\mathcal{F}}_n$ is set of $2(n+1)$-tuples:

$$f = (h_0, h_1 \dots h_n, q_0, q_1, \dots q_n),\ \ h_i, q_i \in \mathbb{N} \cup \{0\},$$

satisfying the following axioms:

(i) $\forall i \in \mathbb{Z}_{n+1} : h_i > 0$,

(ii) all h_i has the same parity,

(iii) $\forall i \in \mathbb{Z}_{n+1} : 0 \leq q_i < h_{i-1} + h_i = 2l_i$,

(iv) all q_i has the same parity.

Remark. From here all operations with numbers q_i will be considered modulo $2l_i$, and according to (iii), q_i will be always the least non negative integer of the class.
Now let us define the set $V(f)$ for a $2(n+1)$-tuple $f \in \widetilde{\mathcal{F}_n}$:

$$V(f) = \bigcup_{i \in \mathbb{Z}_{n+1}} \{i\} \times \mathbb{Z}_{2l_i}$$

and the following involutory permutations on $V(f)$:

$$\alpha_0(i,j) = (i, j + (-1)^j),$$

$$\alpha_1(i,j) = (i, j - (-1)^j),$$
$$\alpha_2(i,j) = \begin{cases} (i+1, 2l_{i+1} - j - 1); & 0 < j < h_i \\ (i-1, 2l_i \quad - j - 1); & h_i \le j < 2l_i \end{cases},$$
$$\alpha_3(i,j) = \rho \circ \alpha_2 \circ \rho^{-1},$$

where $\rho : V(f) \to V(f)$ is a bijection defined by the rule

$$\rho(i,j) = (i, j + q_i).$$

Now let $f \in \widetilde{\mathcal{F}_n}$ and satisfies the following conditions:

(v) $\forall i \in \mathbb{Z}_{n+1} : h_i + q_i$ is odd, h_i and q_i have different parity,

(vi) the group $\langle \alpha_2, \alpha_3 \rangle$ has exactly three orbits.

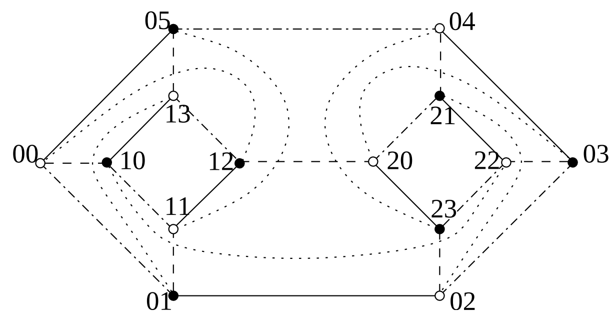

Figure 1: Crystallisation represented by `(3,1,3,2,2,0)`

Definition 5 (Admissible $2(n+1)$-tuple) *The elements of the set $\mathcal{F}_n \subset \widetilde{\mathcal{F}}_n$ satisfying conditions (i) – (vi) will be called admissible $2(n+1)$-tuples.*

An admissible $2(n+1)$-tuple f determines a 4-edge-coloured graph $\Gamma(f)$ with vertex set $V(f)$ and edge-set $E = E_0 \cup E_1 \cup E_2 \cup E_3$, where E_i is the set of orbits of the involution α_i, $i = 0, 1, 2, 3$. The sets E_i form colour classes on the set of edges. Moreover, the cyclic rotation of colours $(0, 2, 1, 3)$ determines an embedding of Γ into a surface of genus n (see the definition of regular genus of Γ). It follows that Γ is a crystallisation of a 3-manifold of genus at most n. Moreover, each 3-manifold of genus n can be coded by an admissible $2(n+1)$-tuple.

Complexity. Given admissible $2(n+1)$-tuple

$$f = (h_0, h_1, \ldots, h_n, q_0, q_1, \ldots, q_n)$$

the sum $c = h_0 + h_1 + \cdots + h_n$ is called complexity of f. In other words, complexity is half of the number of vertices of the crystallisation $\Gamma(f)$. Complexity of a 3-manifold $\mathcal{M}$ of genus n is the minimum of complexities of admissible $2(n+1)$-tuples representing $\mathcal{M}$.

Admissible $2(n+1)$-tuples for $n \le 1$ represent 3-manifolds of genus at most one which are well-known. In what follows we shall restrict ourselves onto the case $n = 2$ representing 3-manifolds of genus at most two. The number of admissible 6-tuples growths enormously with the complexity. Using some equivalences defined on the set of 3-manifolds (based on dipole-move equivalences) a reduced catalogue of 6-tuples containing representatives of all 3-manifolds of complexity at most 21 was obtained in [15]. The disadvantage of this catalogue consists in fact that it still contains different 6-tuples representing the same genus two 3-manifold. In what follows we solve this problem using an analysis of the associated fundamental groups.

3 Isomorphism classes of fundamental groups of 3-manifolds of genus two

Finite groups which appear in the following text are called `[n,k]` using the notation of GAP [10], meaning `SmallGroup(n,k)`. Here `n` means the order of a group and `k` means the position in the GAP library of small groups.

Presentation of the fundamental group. The following algorithm to derive the fundamental group of a 3-manifold represented by an admissible 6-tuple is based on [9]. Given admissible 6-tuple f we define a group $\pi_1(f)$ as follows. Let $\Gamma(f)$ be a crystallisation of 3-manifold given by a 6-tuple f. Set $G = \langle a, b, c \mid R_0, R_1, R_2 \rangle$ to be a 3-generated group, where a, b, c be the three cycles of the 2-factor in $\Gamma(f)$ induced by edges coloured by $\{0, 1\}$. Hence we have a mapping $\beta : V(\Gamma) \to \{a, b, c\}$. By definition, the $\{2, 3\}$-factor in $\Gamma(f)$ consists of 3-cycles C_0, C_1, C_2. The relator R_i is defined by $C_i = (v_{0,i}v_{1,i}v_{2,i} \dots v_{k_i,i})$, $i = 0, 1, 2$, where the edge $v_{0,i}v_{1,i}$ is coloured by the colour 2. The relator given by C_i is:

$$R_i = \prod_{j=0}^{(k_i-1)/2} \beta(v_{2j,i})(\beta(v_{2j+1,i}))^{-1} \ , \quad i \in \{0, 1, 2\}.$$

Add the relator $c = 1$ into the presentation of G. The group $\pi(f) = G/\langle\langle c \rangle\rangle$, where $\langle\langle c \rangle\rangle$ denotes the normal closure of $\langle c \rangle$, is a 2-generated abstract group. The presentation of $\pi(f)$ can be derived from the presentation of G by deleting each appearance of the generator c in the presentation of G.

Example 3.1 Let $f = (3, 3, 9, 2, 0, 4)$. Define the mapping $\beta : V(\Gamma) \to \{a, b, c\}$ as follows: $\beta(v) = a, b, c$ depending on whether v belongs to C_0, C_1 or C_2, respectively. The $\{2, 3\}$-factor is a union of the connectivity components:

```
C0:
   ([0, 0][1, 5][0, 10][2, 1][0, 4][2, 7][1, 0][2, 11]
    [0, 6][2, 5][1, 2][2, 9][0, 8][2, 3][0, 2][1, 3])
C1:
   ([0, 1][1, 4][0, 11][2, 0][0, 5][2, 6][1, 1][2, 10]
    [0, 7][2, 4])
C2:
   ([0, 3][2, 8][0, 9][2, 2])
```

A vertex in a connectivity component is coded by a pair of integers; the first one codes the cycle, which belongs to, the second one codes the order in the cycle and can be 0 for now. We apply β on the above cycles C_0, C_1, C_2 to find the respective relators. In this way we get the following the presentation of the group $\pi_1(f)$:

$$\pi_1(f) = \langle a, b, c \mid c = 1, ab^{-1}ac^{-1}ac^{-1}bc^{-1}ac^{-1}bc^{-1}ac^{-1}ab^{-1} = 1,$$
$$ab^{-1}ac^{-1}ac^{-1}bc^{-1}ac^{-1} = 1, ac^{-1}ac^{-1} = 1\rangle.$$

Cancelling c we get

$$\pi_1(f) = \langle a, b \mid ab^{-1}a^2baba^2b^{-1} = 1, ab^{-1}a^2ba = 1, a^2 = 1\rangle.$$

Using Tietze transformations we finally get the normal form of presentation

$$\pi_1(f) = \langle a, b \mid a^2 = 1\rangle.$$

Our aim is to determine isomorphism classes of fundamental groups $\pi(f)$ where f ranges through the catalogue in [15]. To distinguish the groups we consider some standard invariants which can be derived from the presentations.

Homology groups from 6-tuples. The first considered invariant is the homology group $H_1(f)$. We obtain the homology group from the fundamental group by factorising $\pi_1(f)$ by the derived subgroup. In our case, this can be done by adding the relator $[a, b]$ in the presentation of fundamental group.

Example 3.2 (Determining homology group H_1) Choose $f_1 = (7, 7, 7, 2, 2, 2)$. Using the above construction we derive the presentation of $\pi(f_1)$:

$$\pi_1(f_1) = \langle a, b \mid b^2a^2b^2 = ab^{-1}a, a^2b^2a^2 = ba^{-1}b\rangle.$$

Hence the abelianisation is

$$H_1(f_1) = \langle a, b \mid b^2a^2b^2 = ab^{-1}a, a^2b^2a^2 = ba^{-1}b, ab = ba\rangle,$$

written in additive form:

$$H_1(f_1) = \langle a, b \mid 5a + 5b = 0, 5a = 0\rangle.$$

One can easily determine the homology group from previous presentation as $H_1(f_1) = \mathbb{Z}_5 \times \mathbb{Z}_5$.

The 6-tuple $f_2 = (3, 3, 9, 2, 0, 4)$ from Example 3.1 gives the fundamental group $\pi_1(f_2) = \langle a, b \mid a^2 = 1\rangle$. The respective homology group is $H_1(f_2) = \mathbb{Z}_2 \times \mathbb{Z}$.

The structure of the homology group $H(f)$ in the finite case can be determined using the software GAP. Generally, an abelian group is a product of some infinite and finite cyclic groups. The number of infinite cyclic groups in the decomposition is an invariant called a rank of the group. In case of 2-generator groups the rank is bounded by 2. The following statement can be applied to determine the structure of 2-generator groups of rank one.

Lemma 3.3 Let H be an infinite Abelian group $H = \langle x, y \mid ax + by = 0, cx + dy = 0\rangle$. Let $\bar{a} = (a, c) = au + cv$ for some $u, v \in \mathbb{Z}$ and let $\bar{b} = bu + dv$.Then the group $H \cong \langle x, y \mid \bar{a}x + \bar{b}y = 0\rangle$ and H is one of the following groups

a) $\mathbb{Z} \times \mathbb{Z}$, if and only if $\bar{a} = 0$ and $\bar{b} = 0$

b) $\mathbb{Z} \times \mathbb{Z}_{|\bar{a}|}$, if and only if $\bar{b} = 0$

c) $\mathbb{Z} \times \mathbb{Z}_{|\bar{b}|}$, if and only if $\bar{a} = 0$

d) $\mathbb{Z} \times \mathbb{Z}_{\frac{|\bar{a}||\bar{b}|}{(\bar{a},\bar{b})}}$, if and only if $\bar{a}\bar{b} \neq 0$

Proof. Since H is infinite, the equations following from the presentation of the group H are linearly dependent. Note that division is not allowed over $\mathbb{Z}$. By the Euclidean algorithm there are $u, v \in \mathbb{Z}$ such that $(a, c) = q = au + cv$. Note that $c = pq, p \in \mathbb{Z}$. Rewrite the system of equations in the matrix form. Using Gauss elimination method we get:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} au & bu \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} q & bu+dv \\ c & d \end{pmatrix} \Rightarrow$$
$$\Rightarrow \begin{pmatrix} q & bu+dv \\ 0 & d-p(bu+dv) \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{d} \end{pmatrix}$$

Since the new equations are still linearly dependent, we know that $\bar{a}\bar{d} = 0$, and consequently $\bar{a} = q = (a, c) \neq 0 \Rightarrow \bar{d} = 0$. Thus two equations transform into the following one:

$$\bar{a}x + \bar{b}y = 0.$$

To prove the equivalence of the above equation and the system we have begun with, do the reverse transformation. This can be done as follows:

$$\begin{pmatrix} \bar{a} & \bar{b} \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} q & bu+dv \\ 0 & d-p(bu+dv) \end{pmatrix} \Rightarrow \begin{pmatrix} q & bu+dv \\ c & d \end{pmatrix}$$

The first equation one gets by multiplying the first line in matrix by integer $k = a/q$. Hence $\bar{a}x + \bar{b}y = 0$ is equivalent with the original system of equations over $\mathbb{Z}$.

In Case a) the relator is $0x + 0y = 0$, thus we have 2-generated Abelian free group isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

In Cases b) and c) we get the direct products of $\mathbb{Z} \times \mathbb{Z}_{\bar{a}}$, $\mathbb{Z} \times \mathbb{Z}_{\bar{b}}$, since the relators are $\bar{a}x = 0$, $\bar{b}y = 0$, respectively.

The equation $\bar{a}x = -\bar{b}y$, $\bar{a} \neq 0$, $\bar{b} \neq 0$, appears in Case d). The set of solutions of this equation over the field $\mathbb{Q}$ is $\{[q\bar{b}, -q\bar{a}],\ q \in \mathbb{Q}\}$. However, the set of admissible solutions must be a subset of $\mathbb{Z}\times\mathbb{Z}$. Choosing $q = 1/(-\bar{a}, \bar{b})$ we get the solution with the smallest positive value. Thus, all other solutions over $\mathbb{Z}$ are multiplicands of the chosen one. Set $m = |-\bar{a}|/(\bar{a}, \bar{b})$ and $n = |\bar{b}|/(\bar{a}, \bar{b})$. The group $A = \langle [m, n] \rangle$ is a normal subgroup of $\mathbb{Z} \times \mathbb{Z}$ and $H \cong \mathbb{Z} \times \mathbb{Z}/A$. Let $H \trianglerighteq K = \langle [0, m] \oplus H \rangle$. Every coset has the representative in the set $\{[x, y] \in \mathbb{Z}\times\mathbb{Z};\ 0 \leq x < n, 0 \leq y < n\}$, the order of $|H/K| = mn$. It follows that H/K is isomorphic to the torsion subgroup T of H and $H \cong K \times T \cong \mathbb{Z} \times (\mathbb{Z}_m \times \mathbb{Z}_n)$ [22, Th. 4.2.10]. Since m and n are relatively prime $H/K = \mathbb{Z} \times \mathbb{Z}_{mn}$. □

Low-index subgroups. To distinguish fundamental groups in the same homology class we have used their lists of subgroups of bounded index. This method is based on coset enumeration with an upper bound, GAP [10] can compute it in real time for some small index n. The input are two finitely presented groups, possibly non isomorphic. Using GAP command `LowIndexSubgroupsFpGroup(G,n)` (see [6]) we obtain the list of representatives of conjugacy classes of subgroups

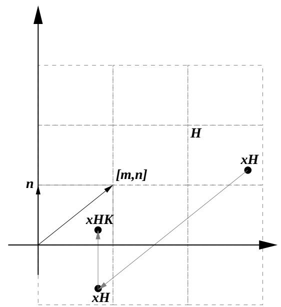

Figure 2: Illustration of cosets $\mathbb{Z}_{mn} \times \mathbb{Z}$

of index lower or equal to `n` for each of the considered groups. Obviously, two groups with different lists of subgroups of index $\leq n$ are not isomorphic. In many considered cases the comparison of lengths of the lists turned to be sufficient to distinguish non-isomorphic groups.

In some instances a particular improvement of the above method selecting only normal subgroups from the list of low index subgroups is useful. Clearly, we can factorise the group by each member of this restricted set to get all factors of bounded order. Since the considered normal subgroups have small indexes (usually up to 10), it is easy to compare the lists of the factor groups and test isomorphisms of the respective factors.

The small program in GAP script and one example follow.

```
LINormSubPrintGens := function (g,n)
% takes two args: the group and the maximal
% index of the subgroup
        local l,d;
        l:=LowIndexSubgroupsFpGroup(g,n);
        % create list of subgroup up to index n
        for d in l do
        % for all subgroups in the list
                if IsNormal(g,d)=true then
                % if the subgroup is normal
                        Print(IdSmallGroup(g/d));
                        % print identification of factor
                        Print(" ");
                        Print(GeneratorsOfGroup(d));
                        % print the subgroup generators
                        Print("\n");
                fi;
        od;
end;;
```

Example 3.4 (Show non-isomorphism of two groups) Let G_1, G_2 be two finitely presented groups

$$\begin{aligned} G_1 &= \langle a, b \mid a = b^2a^2b^2, b = a^3ba^3 \rangle, \\ G_2 &= \langle a, b \mid a^{-3} = ba^3b^{-1}, a^4 = b^2a^{-1}b^2 \rangle. \end{aligned}$$

We suppose that these groups are not isomorphic and we check it by using the function `LINormSubPrint`, which is a modification of above mentioned procedure.

```
gap> g:=F/ [a*b^-2*a^-2*b^-2,a^3*b*a^3*b^-1];
<fp group on the generators [ a, b ]>
gap> h:=F/[a*b^-1*a^3*b*a^2,a^4*b^-2*a*b^-2];
<fp group on the generators [ a, b ]>
gap> LINormSubPrint(g,12);
[ 1, 1 ] [ 2, 1 ] [ 4, 1 ] [ 8, 1 ] [ 3, 1 ] [ 6, 2 ] [ 12, 2 ]
[ 12, 3 ]
gap> LINormSubPrint(h,12);
[ 1, 1 ] [ 2, 1 ] [ 4, 1 ] [ 8, 1 ] [ 3, 1 ] [ 6, 2 ] [ 12, 2 ]
[ 10, 1 ] [ 12, 3 ]
```

From the lengths of the lists we deduce that the group are not isomorphic. Comparing the lists we see that the group G_2 contains the normal subgroup N such that $G/N \cong D_{10}$ which corresponds to the vector `[10,1]`. However, the group G_1 does not contain such a normal subgroup.

Some families of groups have appeared frequently as a result of the analysis. Some simple tricks described in the following lemmas turned to be useful.

Oriented triangle groups. The oriented triangle group $\Delta^+(k,m,n)$ is the group with the presentation $\Delta^+(k,m,n) = \langle a, b \mid a^k = b^m = (ab)^n = 1\rangle$; $k \geq m \geq n$. The groups $\Delta^+(k,2,2)$ are the dihedral groups D_{2k} of order $2k$.

Oriented triangle groups can be regarded as the groups grooving up from regular tessellations of simply connected surfaces. The generators of these groups can be associated with some rotations around a vertex named O (a) and around the barycenter of face incident with O (b) (see Fig. 3). The form of the group is closely related to the geometry of the tessellated surface induced by the action of $\Delta^+(k,m,n)$. The geometry of the surface is:

- elliptic, iff $1/k + 1/m + 1/n > 1$
- Euclidean, iff $1/k + 1/m + 1/n = 1$
- hyperbolic, iff $1/k + 1/m + 1/n < 1$

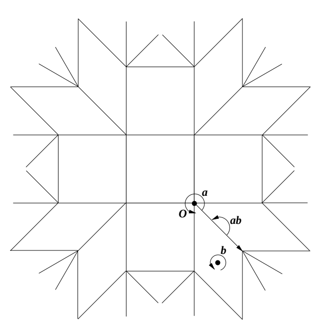

Figure 3: Action of the group $\Delta^+(5,4,2)$ on tessellation of type $\{5,4\}$.

Lemma 3.5 Oriented triangle groups $\Delta^+(k,m,n)$ have trivial centers, except $\Delta^+(k,2,2)$, where k is even.

Proof. [†] Any central element $z \in \zeta(\Delta^+(k,m,n))$ must commute with x and y, hence fixes the unique fixed points of x and y in the surface onto which the group acts on; since z preserves orientation and fixes two points, $z = 1$. This works for hyperbolic and Euclidean triangle groups, and with a little extra effort (since rotations of the sphere have 2 fixed points, permuted by z), it also tells that the centre of a spherical triangle group is trivial except for a dihedral group $D_{4n}, n \in \mathbb{N}$. □

Lemma 3.6 Let Z' be a central subgroup $Z' \trianglelefteq G$. Let $G/Z' \cong \Delta^+(k,m,n)$, $k \geq m \geq n$, $m > 2$ if k even. Then the center $\zeta(G) = Z'$.

Proof. Clearly, $\zeta(G)/Z'$ is a central subgroup of G/Z'. By Lemma 3.5 $\zeta(G)/Z' = 1$. It follows $\zeta(G) = Z'$. □

Extended triangle groups. The groups $\Delta(k,m,n) = \langle a,b \mid a^k = b^m = (ab)^n\rangle$ $k \geq m \geq n$ will be called *extended triangle groups*. Note that these groups has the center $\zeta\Delta(k,m,n) \cong \mathbb{Z}$, generated by the element a^k and the factor by the center is the oriented triangle group $\Delta^+(k,m,n)$.

Generalised quaternion groups (Dicyclic groups). The generalised quaternion group Q_{4n}, $n \geq 2$, is the group with the presentation $Q_{4k} = \langle a,b \mid a^k = b^2 = (ab)^2\rangle$[‡].

Lemma 3.7 Let $G = \langle a,b \mid a^k = b^l = (a^r b^s)^m = 1\rangle$ be a group and $(k,r) = 1$ and $(l,s) = 1$. Then $G \cong H = \langle x,y \mid x^k = y^l = (xy)^m = 1\rangle$.

Proof. Set the map $\phi : a^r \mapsto x, b^s \mapsto y$. Since $(k,r) = 1$, $x \in \langle a\rangle$ and $|x| = |a|$. The situation is analogous for y. Thus $G \to H$ is a group isomorphism. □

Lemma 3.8 Let $G = \langle a,b \mid a^k = (a^l b^m)^n, \ldots\rangle$ be a group. Then $G = \langle a,b \mid a^k = (b^m a^l)^n, \ldots\rangle$.

Proof. Immediate, since conjugation by an element a^l of the group is an inner automorphism of the group taking $a^k \mapsto a^k$, $a^l b^m \mapsto b^m a^l$. □

4 A sample: homology classes 1, $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z} \times \mathbb{Z}_4$.

In this section we show how the analysis goes in the three homology classes in detail. A complete analysis can be found in [14]. In each homology class H_1 the input set of 6-tuples was obtained by a computer assisted search of the catalogue from [15]. The relators were obtained using the algorithm described above.

[†] thanks to Gareth Jones

[‡] these groups are noted as $\langle 2,2,k\rangle$ in [4]

Homology class $H_1 = 1$:

Table 1: Relators of $\pi(f)$ obtained from 6-tuples

No.	f	G
1	`( 5, 5, 5, 4, 4, 4)`	$\langle a, b \mid ab^{-1}a^{-1}b^{-1}ab, b^{-1}a^{-1}b^3a^{-1}b^{-1}\rangle$
2	`( 4, 4, 8, 3, 3, 7)`	$\langle a, b \mid ab^{-1}a^{-1}ba^{-1}b^{-1}, b^3ab^{-2}a\rangle$
3	`( 3, 5, 9, 4, 4,12)`	$\langle a, b \mid ab^{-2}ab, b^{-1}a^4b^{-1}a^{-1}\rangle$
4	`( 5, 5, 9, 4, 4, 8)`	$\langle a, b \mid ab^{-1}a^{-1}ba^{-1}b^{-1}a, a^2ba^{-1}b^{-1}a^{-1}ba\rangle$
5	`( 5, 7, 7, 6, 6, 4)`	$\langle a, b \mid a^{-1}b^{-1}abab^{-1}, b^{-1}a^{-1}b^4a^{-1}b^{-1}a\rangle$
6	`( 4, 4,12, 3, 3,11)`	$\langle a, b \mid b^{-1}aba^{-1}ba, bab^{-1}a^{-4}b^{-1}a\rangle$
7	`( 4, 6,10, 3, 5,15)`	$\langle a, b \mid ab^{-2}ab, b^{-1}a^4b^{-1}a^{-1}\rangle$
8	`( 3, 5,13, 4, 4,16)`	$\langle a, b \mid ab^{-2}ab, b^{-1}a^6b^{-1}a^{-1}\rangle$
9	`( 3, 7,11, 4, 6, 0)`	$\langle a, b \mid ab^{-3}a^3, ab^{-5}a^6, ab^{-2}a^2\rangle$
10	`( 7, 7, 7, 6, 6, 6)`	$\langle a, b \mid ab^{-1}a^{-1}baba^{-1}b^{-1}a, ab^{-1}a^{-1}b^2a^{-1}b^{-1}ab\rangle$

The group G_9 is the only trivial group in the list.

The orders of fundamental groups G_1, G_2, G_3 and G_7 are all 120. The mutual isomorphism of these groups can be checked by using GAP [10]. This group was recognised as `[120,5]`. The structure of this group can be described as follows. The center of this group is isomorphic to $\mathbb{Z}_2$ and factor $G/\zeta(G)$ is isomorphic to the alternating group on five elements. Hence, the group is a central extension of $\mathbb{Z}_2$ by A_5. The canonical presentation of the group can be obtained i.e. from the presentation of G_3. Rewrite relators into the form $b^2 = aba$ and $a^4 = bab$. Then right multiply the first relator by b and left multiply the second relator by a. Thus the presentation of the group is $\langle a, b \mid a^5 = b^3 = (ab)^2\rangle$.

Let us examine the presentation of the group G_4. At first rewrite the presentation in generators $c = a$ and $d = a^2b^{-1}$. Note that c and d generates the group G_4. The first relator turns into $d^2c^{-1}d^{-1}c^{-1} = 1$ and further simplifies into $d^3 = (cd)^2$. Second relator changes into the form $c^2d^{-1}c^{-1}dc^{-1}d^{-1} = 1$. Using the first relator we derive $d^{-1}c^{-1} = cd^{-2}$. Substituting it into the second relator we get $c^6d^{-1}c^{-1}d^{-1} = 1$. Hence $c^7 = (cd)^2$. Finally, we rewrite everything in the letters a and b to get the presentation $G_4 = \langle a, b \mid a^7 = b^3 = (ab)^2\rangle$.
$\mathbf{G_5 \cong G_4}$. To recognise the type of G_5 we rewrite the presentation in the generators b and $c = ab$. From the first relation we get $c^{-1}b^{-1}c^2b = 1$, the second one transforms into the form $c^{-1}b^5c^{-1}b^{-1}cb^{-1} = 1$. Rewrite the first relator to the form $c^{-1}b^{-1} = bc^{-2}$ and substitute it twice into the second relator to get $c^{-1}b^7c^{-2} = 1$. Rewrite the fist relator to the form $c^2 = bcb$ and by right multiplication by c we obtain $c^3 = (bc)^2$. The assignment $\phi : b \mapsto a,\ c \mapsto b$ extends to an isomorphism $G_5 \cong G_4$, since the generators are mapped onto the generators and the relations are preserved.
$\mathbf{G_6 \cong G_4}$. To examine G_6 we first rewrite the relations in the generators a and $c = b^{-1}a$. The relations transform into $ac^{-2}ac = 1$ and $cac^{-1}aca^{-5} = 1$. Rewriting the first relation into the form $cac^{-1} = a^{-1}c$ and substituting it into the second one we get $caca^{-6} = 1$. It follows that $c^2 = aca$ and $a^6 = cac$. The assignment $\phi : a \mapsto a, c \mapsto b$ extends to a group isomorphism $G_6 \cong G_4$, since it maps the generators onto the generators and the relations are preserved.
$\mathbf{G_8 \cong G_4}$. An isomorphism of these groups can be easily checked by rewriting the relators of G_8 into the form $b^2 = aba$ for the first one, and $a^6 = bab$ for the second one. The isomorphism $G_8 \cong G_4$ can be easily set by taking

$\phi : a \mapsto a, b \mapsto b$.
$\mathbf{G_{10}} \cong \mathbf{G_4}$. The situation is more complicated in the case of the 6-tuple $(7,7,7,6,6,6)$. It is not easy to define a group isomorphism proving $G_{10} \cong G_4$. However, it is noted in [2] that the 3-manifold represented by this 6-tuple is homeomorphic to the manifold represented by the 6-tuple $(4,4,12,3,3,11)$: "The $\Gamma(7,7,7;6,6,6)$ represents the 2-fold covering space S^3 branched over the torus link $\{3,7\}$. The same 3-manifold is also represented by $\Gamma(4,4,12;5,3,9)$, as the 2-fold covering space of S^3 branched over the knot K_1'" [2]. Since $(4,4,12,5,3,9)$ is equivalent to $(4,4,12,3,3,11)$ (one can see it applying the σ-operator defined in [11]), it follows that the fundamental groups $G_{10} \cong G_6 \cong G_4$.

It transpires that the homology class $H_1 = 1$ consists of the following three isomorphism classes of fundamental groups:

Table 2: Isomorphism classes in $H_1 = 1$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(3, 7,11, 6, 0,12)	1	1	1	1
(5, 5, 5, 4, 4, 4)	$\langle a,b \mid a^5 = b^3 = (ab)^2 \rangle$	$\mathbb{Z}_2$	A_5	4
(5, 5, 9, 4, 4, 8)	$\langle a,b \mid a^7 = b^3 = (ab)^2 \rangle$	$\mathbb{Z}$	$\Delta^+(7,3,2)$	5

Homology class $H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$: Homology class consists of following manifolds represented by 6-tuples.

Table 3: Input 6-tuples and relators for $H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$

No.	f	G
1	(5, 5, 5, 2, 2, 2)	$\langle a,b \mid ab^{-2}a^{-1}b^{-2}, ab^{-1}a^2ba \rangle$
2	(5, 5, 7, 2, 4, 2)	$\langle a,b \mid b^{-1}a^2ba^2, a^3b^{-2}ab^{-2} \rangle$
3	(5, 5, 9, 4, 8, 4)	$\langle a,b \mid ab^{-1}a^2ba, b^{-1}a^{-3}b^{-2}a^{-1}b^{-1} \rangle$

Rewrite the relators of G_1 in the following way. The first relator rewrite to $a = b^2ab^2$ and left multiply it to get $a^2 = (ab^2)^2$. The second relator rewrite to $b = a^2ba^2$ and right multiply it to get $b^2 = (a^2b)^2$. Hence the presentation of the group G_1 is $\langle a,b \mid a^2 = (ab^2)^2, b^2 = (a^2b)^2 \rangle$.
$\mathbf{G_3} \cong \mathbf{G_2}$. Define an isomorphism $G_2 \cong G_3$ by setting $\phi : a \mapsto a, b \mapsto b^{-1}$. Since the relations are mapped onto the relations, the above assignment extends to an isomorphism.

Now we simplify the presentation of G_2 as follows. Rewrite the first relator into the form $b = a^2ba^2$ and by right multiplication by b get $b^2 = (a^2b)^2$. The second relator rewrite into the form $a^3 = b^2a^{-1}b^2$ and by left multiplication by a^{-1} get $a^2 = (a^{-1}b^2)^2$. We get $G_2 = \langle a,b \mid a^2 = (a^{-1}b^2)^2, b^2 = (a^2b)^2 \rangle$

The groups G_1 and G_2 are not isomorphic. To prove it, let us browse the normal subgroups of low index of the respective groups. The group G_1 contains three normal subgroups of index 10, while the group G_2 has only two such subgroups.

Remark.. Note that the group G_1 is isomorphic to a subgroup of the group of isometries of E_3. Set $\alpha : [x,y,z] \mapsto [x+1,-y,-z]$ and $\beta : [x,y,z] \mapsto [-x,y+1,1-z]$. The isomorphism can be set by mapping $\psi : a \mapsto \alpha$, $b \mapsto \beta$. Using this identification one can prove that the respective 3-manifold is Euclidean. Similarly, the group $G_3 \cong G_2$ is isomorphic to a subgroup of the group of affine transformations of E_3. Let $\alpha : [x, y, z] \mapsto [x-2y+1, -y, -z]$ and $\beta : [x, y, z] \mapsto [-x, y+1, 1-z]$ be transformations of the Euclidean space E_3. The embedding of G_3 into the group of transformations can be set by mapping $\phi : a \mapsto \alpha,\ b \mapsto \beta$. However, one can prove that the 3-manifold represented by $(5, 5, 7, 2, 4, 2)$ is not Euclidean.

Table 4: Isomorphism classes in $H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(5, 5, 5, 2, 2, 2)	$\langle a, b \mid a^2 = (ab^2)^2, b^2 = (a^2b)^2\rangle$	1	G	1
(5, 5, 7, 2, 4, 2)	$\langle a, b \mid a^2 = (a^{-1}b^2)^2, b^2 = (a^2b)^2\rangle$	1	G	2

Homology class $H_1 = \mathbb{Z}_4 \times \mathbb{Z}$**:**

Table 5: Input 6-tuples and relators for homology class $H_1 = \mathbb{Z}_4 \times \mathbb{Z}$

No.	f	G
1	(1, 1,15, 2, 0, 2)	$\langle a, b \mid a^4b^{-1}a^4b, a^4\rangle$
2	(6, 6, 6, 1, 7, 7)	$\langle a, b \mid ab^{-1}ab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}, ab^{-2}ab^{-1}a^{-2}b^{-1},$ $ab^{-1}a^{-1}baba^{-1}b^{-1}\rangle$

Obviously $G_1 = \mathbb{Z}_4 * \mathbb{Z}$.

Rewrite the presentation of the group G_2 in the generators a and $c = ab^{-1}$. The presentation in these generators is $G_2 = \langle a, c \mid c^3 = a^2c^{-1}a, a^3 = c^2a^{-1}c^2\rangle$. Third relator can be derived from the previous two.

The non-isomorphism of the groups G_2 and G_1 can be proved by browsing the list of subgroups of index three[§]. The group G_1 contains six conjugacy classes of subgroups of index three, while G_2 has seven conjugacy classes of subgroups of index three. Thus $G_2 \not\cong G_1$.

Table 6: Isomorphism classes in $H_1 = \mathbb{Z} \times \mathbb{Z}_4$

f	G	$\zeta(G)$	$G/\zeta(G)$	#
(6, 6, 6, 1, 7, 7)	$\langle a, b \mid a^3 = b^2a^{-1}b^2, b^3 = a^2b^{-1}a^2\rangle$	?	?	1
(1, 1,15, 2, 0, 2)	$\mathbb{Z}_4 * \mathbb{Z}$	1	$\mathbb{Z}_4 * \mathbb{Z}$	1

5 3-manifolds of genus two with complexity at most 21

To interpret the obtained results we first formulate some consequences of general results mentioned in the introductory sections.

Theorem 5.1 *Let f be an admissible 6-tuple. If $\pi(f) = 1$ then $\mathcal{M}(f)$ is the 3-sphere. If $\pi_1(f)$ is cyclic, then $\mathcal{M}(f)$ is either a lens space or $S^1 \times S^2$.*

The following statement gives an algebraic criterion to recognise admissible 6-tuples representing decomposable 3-manifolds of genus two.

[§]these subgroups are the representatives of conjugacy classes

Theorem 5.2 *Let f be an admissible 6-tuple. Let $\pi_1(f) = C_1 * C_2$ be a free product of cyclic groups. Then $\mathcal{M} = \mathcal{M}_1 \# \mathcal{M}_2$ with $\pi_1(\mathcal{M}_1) = C_1$ and $\pi_1(\mathcal{M}_2) = C_2$. Moreover, either $C_i = \mathbb{Z}_p$ and $\mathcal{M}_i \cong \mathcal{L}(p,q)$ for some q coprime to p, or $C_i = \mathbb{Z}$ and $\mathcal{M}_i \cong S_1 \times S_2$, for $i = 1, 2$.*

Theorem 5.3 *Let f be an admissible 6-tuple. Then $\mathcal{M}(f)$ is elliptic of genus two if and only if $\pi_1(f)$ is a finite non-abelian group. The 3-manifolds are determined by the groups uniquely.*

Let us now summarize our analysis of fundamental groups of 3-manifolds given by admissible 6-tuples.

We have found **78** isomorphism classes of fundamental groups of prime 3-manifolds of genus two coming from our catalogue [15]. The remaining six-tuples in our catalogue give rise either to cyclic groups or to free products of cyclic groups. These six-tuples code either 3-manifolds of genus at most one or decomposable 3-manifolds of genus two. More detailed description of fundamental groups follows.

Cyclic groups – 3-manifolds of genus 1. There are **1019** admissible 6-tuples up to complexity 21 giving rise to cyclic groups $\mathbb{Z}_n, 2 \leq n \leq 29$, and to the infinite cyclic group.

Free products of cyclic groups – decomposable 3-manifolds. There are **137** non-prime 3-manifolds of genus 2 coded by admissible 6-tuples up to complexity 21. They are the connected sums of lens spaces and $S^1 \times S^2$ respectively. The respective free products are of the form $\mathbb{Z}_m * \mathbb{Z}_n$ and $\mathbb{Z} * \mathbb{Z}_n$ for some parameters m, n.

Acyclic groups – Prime 3-manifolds of genus two There are **78** isomorphism classes of acyclic fundamental groups represented by admissible 6-tuples up to complexity 21. These 6-tuples represent prime 3-manifolds of genus 2. Among them, there are **71** 6-tuples admitting finite homology groups and **7** ones admitting infinite homology groups. Further, we have recognised **39** classes with a finite fundamental group and **39** classes with an infinite fundamental group. Finite fundamental groups, we found, are all mentioned by Milnor [18, p. 405]. Moreover, we have found also **4** 6-tuples representing compact, connected Euclidean 3-manifolds. See Appendix for details.

The main result of our analysis follows.

Theorem 5.4 *Prime 3-manifolds of genus at most two, represented by admissible 6-tuples of complexity ≤ 21 have fundamental groups of one of the following types:*

trivial group,

cyclic groups $\mathbb{Z}_n, 2 \leq n \leq 29$,

infinite cyclic group $\mathbb{Z}$,

acyclic groups (see Appendix for the list).

*The list includes **78** isomorphism classes of acyclic fundamental groups of prime 3-manifolds of genus two. Among them, there are **39** elliptic manifolds with finite groups, **4** Euclidean manifolds and **35** other 3-manifolds with infinite fundamental groups. Moreover, the 3-manifolds are determined by their fundamental groups.*

It happened that a non-abelian, infinite fundamental group came from different 6-tuples, say f and g, from the list. So far, the 3-manifolds in the list were determined up to fundamental groups. Although it seems that 3-manifolds of genus two are (in general) determined by their fundamental groups (see the discussion in next section), to complete the proof we need an argument showing that the structure of a 3-manifold in our list is fully determined by its fundamental group. We have solved this problem by checking that the respective 4-coloured graphs $\Gamma(f)$, $\Gamma(g)$ are equivalent up to colour-preserving graph isomorphisms and dipole-move equivalence. The result was achieved by direct computations using the software DUKE developed at University of Modena under the auspices of C. Gagliardi and P. Bandieri. Hence the 3-manifolds represented by 6-tuples up to complexity 21 are determined by their fundamental groups.

As concerns, Euclidean 3-manifolds of genus two it is well-known that they are exactly four and they can be distinguished by their homology groups. They can be represented by 6-tuples up to complexity 17 as follows.

Theorem 5.5 *There are 4 isomorphism classes of fundamental groups of orientable Eucleidean 3-manifolds of genus 2 [16]. The list of representing 6-tuples with minimum complexity follows:*

(5,5,5,2,2,2) with homology group $H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$,

(5,5,7,4,4,6) with homology group $H_1 = \mathbb{Z}$,

(4,6,6,5,5,3) with homology group $H_1 = \mathbb{Z} \times \mathbb{Z}_2$,

(5,5,7,2,6,6) with homology group $H_1 = \mathbb{Z} \times \mathbb{Z}_3$.

6 Concluding remarks

It turned out that the prime 3-manifolds of genus two up to complexity 21 are determined by their fundamental groups. Let us discuss this phenomenon from a general point of view. Till now no counterexamples are known at all. The following statements have arisen as a result of a discussion of this problem with S. Matveev.

1. If $\mathcal{M}$ is Haken, then it is determined by the fundamental group [18].
2. If $\mathcal{M}$ is hyperbolic, then the same is true.
3. If $\mathcal{M}$ is not Haken and not hyperbolic then by Theorem 1.11 it is Seifert over the sphere with three exceptional fibers.

In the latter case we have two possibilities:

a.) $\mathcal{M}$ has a finite fundamental group, hence it is elliptic, and consequently, it is determined by the fundamental group;

b.) $\mathcal{M}$ has an infinite fundamental group. Then its homotopy type is determined by the fundamental group. It is not clear how to convert this homotopy statement into a topological one.

The proof of Theorem 5.4 was completed by using the dipole-move equivalence, the following question can be viewed as a combinatorial reformulation of the homeomorphism problem.

Problem 1: Is there an algorithm to decide whether two 4-edge-coloured graphs are dipole-move equivalent?

Four Euclidean 3-manifolds of genus two are represented by 6-tuples of complexity at most 17. Among the prime 3-manifolds of genus 2 of complexity at most 21 there are 39 elliptic 3-manifolds. They are all quotients of S^3 by their fundamental groups acting freely as groups of symmetries on S^3. The classification of finite non-abelian fundamental groups freely acting on S^3 is due to Milnor [18, 19]. In fact, it is known that an elliptic 3-manifold is of genus at most two [18]. Since any 3-manifold of genus at most two can be represented by a 6-tuple [3], the following problem could be of interest.

Problem 2: For each elliptic 3-manifold $\mathcal{M}$ with given Milnor's group find an admissible 6-tuple (preferably with minimum complexity) representing $\mathcal{M}$.

Some particular results in this direction were established yet. We have found "canonical" representatives for elliptic 3-manifolds given by generalised quaternion groups.

Finally, let us remark that our results were independently confirmed by S. Matveev and his collaborators using a software developed on ideas from [18].

Acknowledgements

We want to thank Carlo Gagliardi and Paola Bandieri (Univ. Modena) for useful discussions on the topic and for a possibility to use the software DUKE. As well we want to thank to George Havas who helped us to find an isomorphism between two presentations of fundamental groups in the homology class $\mathbb{Z}_5 \times \mathbb{Z}_5$. Finally, we want to thank to Sergei Matveev and his collaborators for the output data produced by their (independently made) software.

The research of the first and third author is partially supported by the Ministry of Education of the Slovak Republic (APVT-51-012502, VEGA).

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Appendix: Fundamental groups of prime 3-manifolds with genus two

The Appendix contains a list of 6-tuples representing prime 3-manifolds of genus 2. Each record consists of 6-tuple, respective homology group H_1 and respective fundamental group π_1. If π_1 is finite (and contaned in the GAP library), the isomorphism with a small group from GAP [10] is listed.

P.1	`( 5, 5, 5, 4, 4, 4)`	$H_1(f) = 1$ $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab)^2 \rangle \cong [120, 5]$
P.2	`( 5, 5, 9, 4, 4, 8)`	$H_1(f) = 1$ $\pi_1(f) = \langle a, b \mid a^7 = b^3 = (ab)^2 \rangle \cong [\infty, ?]$
P.3	`( 4, 4, 6, 3, 3, 5)`	$H_1(f) = \mathbb{Z}_2$ $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab)^2 \rangle \cong [48, 28]$
P.4	`( 4, 6, 8, 5, 5,11)`	$H_1(f) = \mathbb{Z}_2$ $\pi_1(f) = \langle a, b \mid a^5 = b^4 = (ab)^2 \rangle \cong [\infty, ?]$
P.5	`( 5, 5,11, 4, 4,10)`	$H_1(f) = \mathbb{Z}_2$ $\pi_1(f) = \langle a, b \mid a^8 = b^3 = (ab)^2 \rangle \cong [\infty, ?]$
P.6	`( 3, 3, 3, 2, 2, 2)`	$H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\pi_1(f) = \langle a, b \mid a^2 = b^2 = (ab)^2 \rangle \cong [8, 4]$
P.7	`( 3, 3, 7, 2, 2, 6)`	$H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\pi_1(f) = \langle a, b \mid a^4 = b^2 = (ab)^2 \rangle \cong [16, 9]$
P.8	`( 3, 3,11, 2, 2,10)`	$H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\pi_1(f) = \langle a, b \mid a^6 = b^2 = (ab)^2 \rangle \cong [24, 4]$
P.9	`( 3, 3,15, 2, 2,14)`	$H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\pi_1(f) = \langle a, b \mid a^8 = b^2 = (ab)^2 \rangle \cong [32, 20]$
P.10	`( 4, 6,10, 5, 5,13)`	$H_1(f) = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\pi_1(f) = \langle a, b \mid a^6 = b^4 = (ab)^2 \rangle \cong [\infty, ?]$
P.11	`( 4, 4, 4, 3, 3, 3)`	$H_1(f) = \mathbb{Z}_3$ $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^2 \rangle \cong [24, 3]$
P.12	`( 5, 7, 7, 4, 6,12)`	$H_1(f) = \mathbb{Z}_3$ $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab^{-2})^{-3} \rangle \cong [\infty, ?]$
P.13	`( 4, 6, 6, 1, 1, 9)`	$H_1(f) = \mathbb{Z}_3 \times \mathbb{Z}_3$ $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^3 \rangle \cong [\infty, ?]$
P.14	`( 3, 3, 5, 2, 2, 4)`	$H_1(f) = \mathbb{Z}_4$ $\pi_1(f) = \langle a, b \mid a^3 = b^2 = (ab)^2 \rangle \cong [12, 1]$
P.15	`( 3, 3, 9, 2, 2, 8)`	$H_1(f) = \mathbb{Z}_4$ $\pi_1(f) = \langle a, b \mid a^5 = b^2 = (ab)^2 \rangle \cong [20, 1]$
P.16	`( 3, 3,13, 2, 2,12)`	$H_1(f) = \mathbb{Z}_4$ $\pi_1(f) = \langle a, b \mid a^7 = b^2 = (ab)^2 \rangle \cong [28, 1]$
P.17	`( 7, 7, 7, 2, 6,10)`	$H_1(f) = \mathbb{Z}_4$ $\pi_1(f) = \langle a, b \mid a^4 = b^2 a^{-1} b^2, b^3 = a^3 b^{-1} a^3 \rangle \cong [\infty, ?]$
P.18	`( 5, 5, 5, 2, 2, 2)`	$H_1(f) = \mathbb{Z}_4 \times \mathbb{Z}_4$ $\pi_1(f) = \langle a, b \mid a^2 = (ab^2)^2, b^2 = (a^2 b)^2 \rangle \cong [\infty, ?]$
P.19	`( 5, 5, 7, 2, 4, 2)`	$H_1(f) = \mathbb{Z}_4 \times \mathbb{Z}_4$ $\pi_1(f) = \langle a, b \mid a^2 = (a^{-1} b^2)^2, b^2 = (a^2 b)^2 \rangle \cong [\infty, ?]$
P.20	`( 4, 8, 8, 5, 5,13)`	$H_1(f) = \mathbb{Z}_5$ $\pi_1(f) = \langle a, b \mid a^5 = b^5 = (ab)^2 \rangle \cong [\infty, ?]$
P.21	`( 5, 5,11, 4, 8, 4)`	$H_1(f) = \mathbb{Z}_5$ $\pi_1(f) = \langle a, b \mid a^7 = b^2 = (a^{-2} b)^3 \rangle \cong [\infty, ?]$
P.22	`( 7, 7, 7, 2, 2, 2)`	$H_1(f) = \mathbb{Z}_5 \times \mathbb{Z}_5$ $\pi_1(f) = \langle a, b \mid a^3 b^{-1} a = b a^{-1} b^3 = (a^2 b^2)^2 \rangle \cong [\infty, ?]$
P.23	`( 4, 6,10, 3, 5, 3)`	$H_1(f) = \mathbb{Z}_6$ $\pi_1(f) = \langle a, b \mid a^5 = b^4 = (a^2 b)^2 \rangle \cong [\infty, ?]$

P.24	`( 5, 7, 9, 4, 6,14)`	$H_1(f) = \mathbb{Z}_6$ $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab^{-2})^{-3} \rangle \cong [\infty, ?]$
P.25	`( 4, 4, 4, 1, 1, 1)`	$H_1(f) = \mathbb{Z}_6 \times \mathbb{Z}_2$ $\pi_1(f) = \langle a, b \mid a^2 = b^2 = (ab)^{-2} \rangle \cong [24, 11]$
P.26	`( 4, 4, 6, 1, 5, 1)`	$H_1(f) = \mathbb{Z}_6 \times \mathbb{Z}_2$ $\pi_1(f) = \langle a, b \mid a^4 = b^2 = (ab^{-1})^2 \rangle \cong [48, 27]$
P.27	`( 4, 6,10, 5, 9, 3)`	$H_1(f) = \mathbb{Z}_6 \times \mathbb{Z}_3$ $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (a^2b)^3 \rangle \cong [\infty, ?]$
P.28	`( 4, 4,10, 3, 3, 3)`	$H_1(f) = \mathbb{Z}_7$ $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (a^2b)^2 \rangle \cong [840, 13]$
P.29	`( 4, 4, 4, 1, 1, 5)`	$H_1(f) = \mathbb{Z}_8$ $\pi_1(f) = \langle a, b \mid a^3 = b^2 = (a^2b)^2 \rangle \cong [24, 1]$
P.30	`( 3, 3,11, 2, 2, 4)`	$H_1(f) = \mathbb{Z}_8$ $\pi_1(f) = \langle a, b \mid a^5 = b^2 = (a^2b)^2 \rangle \cong [40, 1]$
P.31	`( 3, 3,15, 2, 2, 6)`	$H_1(f) = \mathbb{Z}_8$ $\pi_1(f) = \langle a, b \mid a^7 = (a^4b)^2, b^2 = (a^2b)^2 \rangle \cong [56, 1]$
P.32	`( 4, 8, 8, 1, 1,13)`	$H_1(f) = \mathbb{Z}_8$ $\pi_1(f) = \langle a, b \mid a^4 = b^4 = (ab)^3 \rangle \cong [\infty, ?]$
P.33	`( 4, 6, 8, 3, 9,13)`	$H_1(f) = \mathbb{Z}_8 \times \mathbb{Z}_2$ $\pi_1(f) = \langle a, b \mid a^4 = b^4 = (a^{-1}b)^2 \rangle \cong [\infty, ?]$
P.34	`( 4, 4, 6, 1, 1, 7)`	$H_1(f) = \mathbb{Z}_9$ $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (a^{-1}b)^2 \rangle \cong [72, 3]$
P.35	`( 4, 6, 8, 1, 1,11)`	$H_1(f) = \mathbb{Z}_9$ $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab)^3 \rangle \cong [\infty, ?]$
P.36	`( 4, 6,10, 1, 1,13)`	$H_1(f) = \mathbb{Z}_9$ $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab)^3 \rangle \cong [\infty, ?]$
P.37	`( 6, 6, 6, 1, 1, 1)`	$H_1(f) = \mathbb{Z}_9 \times \mathbb{Z}_3$ $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^{-3} \rangle \cong [\infty, ?]$
P.38	`( 4, 4, 8, 1, 1, 9)`	$H_1(f) = \mathbb{Z}_{10}$ $\pi_1(f) = \langle a, b \mid a^4 = b^2 = (ab)^3 \rangle \cong [240, 102]$
P.39	`( 4, 4,12, 1, 1, 5)`	$H_1(f) = \mathbb{Z}_{10} \times \mathbb{Z}_2$ $\pi_1(f) = \langle a, b \mid a^2 = b^2 = (a^3b^3)^2 \rangle \cong [40, 11]$
P.40	`( 4, 4, 8, 1, 1, 1)`	$H_1(f) = \mathbb{Z}_{10} \times \mathbb{Z}_2$ $\pi_1(f) = \langle a, b \mid a^4 = b^2 = (ab)^{-2} \rangle \cong [80, 27]$
P.41	`( 4, 4,10, 1, 5, 1)`	$H_1(f) = \mathbb{Z}_{10} \times \mathbb{Z}_2$ $\pi_1(f) = \langle a, b \mid {}^6 = b^2 = (ab^{-1})^2 \rangle \cong [120, 21]$
P.42	`( 4, 4,10, 1, 1,11)`	$H_1(f) = \mathbb{Z}_{11}$ $\pi_1(f) = \langle a, b \mid a^5 = b^{-2} = (ab^{-1})^3 \rangle \cong [1320, 14]$
P.43	`( 4, 4,10, 1, 5, 7)`	$H_1(f) = \mathbb{Z}_{12}$ $\pi_1(f) = \langle a, b \mid a^5 = b^2 = (a^2b^{-1})^2 \rangle \cong [60, 2]$
P.44	`( 4, 4,12, 1, 1,13)`	$H_1(f) = \mathbb{Z}_{12}$ $\pi_1(f) = \langle a, b \mid a^6 = b^3 = (a^{-1}b)^2 \rangle \cong [\infty, ?]$
P.45	`( 4, 6,10, 7, 3,15)`	$H_1(f) = \mathbb{Z}_{13}$ $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (a^2b^{-1})^2 \rangle \cong [1560, 13]$
P.46	`( 4, 6, 6, 1, 7, 1)`	$H_1(f) = \mathbb{Z}_{14}$ $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab^{-1})^2 \rangle \cong [336, 115]$
P.47	`( 4, 4,12, 1, 1, 1)`	$H_1(f) = \mathbb{Z}_{14} \times \mathbb{Z}_2$ $\pi_1(f) = \langle a, b \mid a^6 = b^2 = (ab)^{-2} \rangle \cong [168, 29]$
P.48	`( 3, 7, 7, 2, 2, 2)`	$H_1(f) = \mathbb{Z}_{15}$ $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (a^{-2}b)^2 \rangle \cong [120, 15]$
P.49	`( 6, 6, 6, 1, 1, 9)`	$H_1(f) = \mathbb{Z}_{15}$ $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab^{-1})^3 \rangle \cong [\infty, ?]$
P.50	`( 4, 4, 6, 1, 1, 1)`	$H_1(f) = \mathbb{Z}_{16}$ $\pi_1(f) = \langle a, b \mid a^3 = b^2 = (ab)^{-2} \rangle \cong [48, 1]$

P.51	(4, 4, 8, 1, 5, 1)	$H_1(f) = \mathbb{Z}_{16}$ $\pi_1(f) = \langle a, b \mid a^5 = b^2 = (ab^{-1})^2 \rangle \cong [80, 1]$
P.52	(6, 6, 8, 1, 1,11)	$H_1(f) = \mathbb{Z}_{16}$ $\pi_1(f) = \langle a, b \mid a^4 = b^4 = (a^{-1}b)^3 \rangle \cong [\infty, ?]$
P.53	(4, 8, 8, 1, 1, 1)	$H_1(f) = \mathbb{Z}_{16} \times \mathbb{Z}_2$ $\pi_1(f) = \langle a, b \mid a^4 = b^4 = (ab)^{-2} \rangle \cong [\infty, ?]$
P.54	(3, 7,11, 4, 2, 2)	$H_1(f) = \mathbb{Z}_{17}$ $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (a^2b^2)^2 \rangle \cong [2040, ?]$
P.55	(4, 6,10, 3, 9,15)	$H_1(f) = \mathbb{Z}_{18}$ $\pi_1(f) = \langle a, b \mid a^5 = b^4 = (a^{-1}b)^2 \rangle \cong [\infty, ?]$
P.56	(4, 6, 8, 1, 7, 1)	$H_1(f) = \mathbb{Z}_{19}$ $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab^{-1})^2 \rangle \cong [2280, ?]$
P.57	(4, 4,10, 1, 1, 7)	$H_1(f) = \mathbb{Z}_{20}$ $\pi_1(f) = \langle a, b \mid a^3 = b^2 = (a^2b)^{-2} \rangle \cong [60, 1]$
P.58	(5, 7, 9, 2, 4, 4)	$H_1(f) = \mathbb{Z}_{20}$ $\pi_1(f) = \langle a, b \mid a^2 = b^2ab^2, a^3 = ba^{-3}b^3 \rangle \cong [\infty, ?]$
P.59	(4, 6, 6, 1, 1, 1)	$H_1(f) = \mathbb{Z}_{21}$ $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^{-2} \rangle \cong [168, 22]$
P.60	(6, 6, 8, 1, 9, 1)	$H_1(f) = \mathbb{Z}_{21}$ $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab^{-1})^3 \rangle \cong [\infty, ?]$
P.61	(4, 6,10, 5, 1, 1)	$H_1(f) = \mathbb{Z}_{22}$ $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab^{-2})^2 \rangle \cong [528, 87]$
P.62	(4, 8, 8, 1, 9, 1)	$H_1(f) = \mathbb{Z}_{22}$ $\pi_1(f) = \langle a, b \mid a^5 = b^4 = (ab^{-1})^2 \rangle \cong [\infty, ?]$
P.63	(4, 4,10, 1, 1, 1)	$H_1(f) = \mathbb{Z}_{24}$ $\pi_1(f) = \langle a, b \mid a^5 = b^2 = (ab)^{-2} \rangle \cong [120, 2]$
P.64	(4, 4,12, 1, 5, 1)	$H_1(f) = \mathbb{Z}_{24}$ $\pi_1(f) = \langle a, b \mid a^7 = b^2 = (ab^{-1})^2 \rangle \cong [168, 4]$
P.65	(4, 6,10, 1, 7, 1)	$H_1(f) = \mathbb{Z}_{24}$ $\pi_1(f) = \langle a, b \mid a^6 = b^3 = (ab^{-1})^2 \rangle \cong [\infty, ?]$
P.66	(5, 5, 9, 2, 2, 2)	$H_1(f) = \mathbb{Z}_{24}$ $\pi_1(f) = \langle a, b \mid a = b^2a^2b^2, b = a^3ba^3 \rangle \cong [\infty, ?]$
P.67	(5, 5,11, 2, 4, 2)	$H_1(f) = \mathbb{Z}_{24}$ $\pi_1(f) = \langle a, b \mid b = a^3ba^3, a^4 = b^2a^{-1}b^2 \rangle \cong [\infty, ?]$
P.68	(4, 6, 8, 1, 1, 1)	$H_1(f) = \mathbb{Z}_{26}$ $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab)^{-2} \rangle \cong [624, 131]$
P.69	(4, 6,10, 7, 1, 1)	$H_1(f) = \mathbb{Z}_{27}$ $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (a^2b)^{-2} \rangle \cong [216, 3]$
P.70	(4, 6,10, 1, 1, 1)	$H_1(f) = \mathbb{Z}_{31}$ $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab)^{-2} \rangle \cong [3720, ?]$
P.71	(6, 6, 8, 1, 1, 1)	$H_1(f) = \mathbb{Z}_{33}$ $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab)^{-3} \rangle \cong [\infty, ?]$
P.72	(5, 5, 7, 4, 4, 6)	$H_1(f) = \mathbb{Z}$ $\pi_1(f) = \langle a, b \mid a^6 = b^3 = (ab)^2 \rangle \cong [\infty, ?]$
P.73	(6, 6, 6, 5, 5, 5)	$H_1(f) = \mathbb{Z}$ $\pi_1(f) = \langle a, b \mid a^2ba^{-1} = b^{-1}ab^2, b^{-1}a^2b^{-1} = ab^{-2}a \rangle \cong [\infty, ?]$
P.74	(6, 6, 8, 5, 5, 7)	$H_1(f) = \mathbb{Z}$ $\pi_1(f) = \langle a, b \mid a^3 = b^{-1}ab^2ab^{-1}, ab^{-1}ab = b^{-1}a^{-1}ba^{-1} \rangle \cong [\infty, ?]$
P.75	(4, 6, 6, 5, 5, 3)	$H_1(f) = \mathbb{Z} \times \mathbb{Z}_2$ $\pi_1(f) = \langle a, b \mid a^4 = b^4 = (ab)^2 \rangle \cong [\infty, ?]$
P.76	(5, 5, 7, 2, 6, 6)	$H_1(f) = \mathbb{Z} \times \mathbb{Z}_3$ $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^3 \rangle \cong [\infty, ?]$
P.77	(6, 6, 6, 1, 7, 7)	$H_1(f) = \mathbb{Z} \times \mathbb{Z}_4$ $\pi_1(f) = \langle a, b \mid a^3 = b^2a^{-1}b^2, b^3 = a^2b^{-1}a^2 \rangle \cong [\infty, ?]$

P.78 `( 6, 6, 8, 5,11, 7)` $H_1(f) = \mathbb{Z} \times \mathbb{Z}_5$

$\pi_1(f) = \langle a, b \mid ab^{-2}a^2 = b^2a^{-2}b, ba^{-2}b = a^{-1}b^2a^{-1} \rangle \cong [\infty, ?]$

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