The Cyclic Antibandwidth Problem

(Extended Abstract)

Ondrej Sýkora
Department of Computer Science, Loughborough University
Loughborough, LE11 3TU, The United Kingdom

Lubomir Torok*
Institute of Mathematics and Computer Science
Severná 5, 974 01, Banská Bystrica, Slovak Republic

Imrich Vrťo
Institute of Mathematics, Slovak Academy of Sciences
Dúbravská 9, 841 04 Bratislava, Slovak Republic

Abstract

The cyclic antibandwidth problem is to embed an n-vertex graph into the cycle C_n, such that the minimum distance (measured in the cycle) of adjacent vertices is maximised. This is a variant of obnoxious facility location problems or a dual problem to the cyclic bandwidth problem. The problem is NP-hard. In the paper we start investigating this invariant for typical graphs. We prove basic facts and exact results for meshes, tori and asymptotics for hypercubes.

1 Introduction

For a nonempty graph $G = (V, E)$, let f be a one-to-one labelling $f : V \to \{1, 2, ...|V|\}$. Define the cyclic antibandwidth of G according to f as

$$\text{cab}(G, f) = \min_{uv \in E}\{|f(u) - f(v)|, |V| - |f(u) - f(v)|\}.$$

The antibandwidth of G is defined as

$$\text{cab}(G) = \max_{f} \text{cab}(G, f).$$

It is useful to imagine the cyclic antibandwidth problem as a cycle embedding problem. The vertices are mapped bijectively into $C_{|V|}$ such that the minimum distance, measured in the cycle, of adjacent vertices is maximised. The problem was first introduced by Leung

*This research was supported partially by the VEGA grants 2/2060/23, 1/3164/23 and the European Research Training Network COMBSTRU-HPRN-CT-2002-00278.

et al. [4], in connection with some scheduling problems. Since then the problem has remained almost unexplored. The cyclic antibandwidth problem is a variant of obnoxious facility location problems [1], a new graph labelling problem [2] or a dual problem to the cyclic bandwidth problem [3, 6], where the maximum distance of adjacent vertices is minimised. The decision problem, i.e., "Is cab(G) $\geq$ 2 ?", is simply equivalent to the existence of a Hamiltonian cycle in the complement of G. This belongs to standard textbooks on complexity and is well known as a King Arthur's round table problem: "Is it possible to place knights around a table such that no two enemies are neighbouring?"

The problem is closely related to the antibandwidth problem [4, 5, 7], (called also dual bandwidth or separation number) defined as follows

$$\mathrm{ab}(G) = \max_f \min_{uv \in E} \{|f(u) - f(v)|\}.$$

In [8] we showed exact results and tight estimations for the antibandwidth parameter for meshes, tori and hypercubes. In this paper we start investigating the cyclic antibandwidth parameter. We show basic facts about the problem and prove exact results for meshes, tori and asymptotics for hypercubes. It turns out that for the above classes of graphs both parameters are almost the same, but there are graphs for which they differ essentially.

2 Basic Observations

Lemma 2.1. *Let $G = (V, E), |V| = n$, Then*

$$\mathrm{cab}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Equality holds, e.g., for G consisting of a matching.

Lemma 2.2. *For n-vertex paths and cycles,*

$$\mathrm{cab}(P_n) = \mathrm{cab}(C_n) = \left\lceil \frac{n}{2} \right\rceil - 1.$$

Let $\overline{G}$ denote the complement of G. Let G^i, for $i \geq 1$, denote the graph obtained from G by joining all vertices in a distance of at most i.

Lemma 2.3. *For n-vertex graph G and $k \geq 2$*

$$\mathrm{cab}(G) \geq k \quad \text{iff } C_n^{k-1} \subseteq \overline{G},$$

$$\mathrm{cab}(G) = 1 \quad \text{iff } \overline{G} \text{ does not contain a Hamiltonian cycle},$$

Lemma 2.4.

$$\frac{1}{2}\mathrm{ab}(G) \leq \mathrm{cab}(G) \leq \mathrm{ab}(G),$$

and both bounds are attainable.

The next Lemma is obvious but usefull. All results for complete binary trees, meshes, tori and hypercubes are obtained by the Lemma.

Lemma 2.5. *Let $G = (E, V), |V| = n$, Assume there exists an optimal labelling of G with respect to* ab(G) *in which the maximum distance of adjacent vertices is at most $n - \mathrm{ab}(G)$. Then $\mathrm{ab}(G) = \mathrm{cab}(G)$.*

3 Meshes

The 2-dimensional mesh is defined by means of Cartesian product of two paths as $P_m \times P_n$.

Theorem 3.1. *Let $m \geq n$. If m is even and n is odd, then*

$$\left\lfloor \frac{n(m-1)}{2} \right\rfloor \leq \mathrm{cab}(P_m \times P_n) \leq \mathrm{ab}(P_m \times P_n) = \left\lceil \frac{n(m-1)}{2} \right\rceil,$$

otherwise

$$\mathrm{cab}(P_m \times P_n) = \frac{n(m-1)}{2}.$$

We conjecture that for m even and n odd $\mathrm{cab}(P_m \times P_n) = \lfloor n(m-1)/2 \rfloor$. An example of a type of numbering attaining all lower bounds in Theorem 3.1, for $P_8 \times P_7$, is in the next table.

38	13	45	20	51	25	55	28
7	39	14	46	21	52	26	56
33	8	40	15	47	22	53	27
3	34	9	41	16	48	23	54
30	4	35	10	42	17	49	24
1	31	5	36	11	43	18	50
29	2	32	6	37	12	44	19

4 Tori

The 2-dimensional toroidal mesh is defined as $C_m \times C_n$.

Theorem 4.1. *For even n*

$$cab(C_n \times C_n) = ab(C_n \times C_n) = \frac{n(n-2)}{2}.$$

Theorem 4.2. *For odd n*

$$cab(C_n \times C_n) = ab(C_n \times C_n) = \frac{(n-2)(n+1)}{2}.$$

5 Asymptotics for Hypercubes

The vertices of the hypercube are represented by binary numbers of length n and two vertices are adjacent iff the strings differ in one position.

Theorem 5.1. *For the n-dimensional hypercube Q_n*

$$\mathrm{cab}(Q_n) = 2^{n-1} - \frac{2^n}{\sqrt{2\pi n}} (1 + o(1)).$$

The same result holds for $\mathrm{ab}(Q_n)$, with possibly different third order term [8].

References

[1] Cappanera, P., A survey on obnoxious facility location problems, *Technical Report TR-99-11*, Dipartimento di Informatica, Uni. di Pisa, 1999.

[2] Gallian, J.A., Graph labeling, *Electronic Journal on Combinatorics*, (2005), DS6.

[3] Hromkovič, J., Müller, V., Sýkora, O., Vrťo, I., On embeddings in cycles, *Information and Computation* **118** (1995), 302-305.

[4] Leung, J.Y.-T., Vornberger, O., Witthoff, J.D. On some variants of the bandwidth minimization problem, *SIAM J. Computing* **13** (1984), 650-667.

[5] Lin Yixun, Yuan JinJiang, The dual bandwidth problem for graphs, *J. Zhengzhou Univ. Nat. Sci. Ed.* **35** (2003), 1-5.

[6] Lin Yixun, Minimum bandwidth problem for embedding graphs in cycles, *NETWORKS* **29** (1997), 135-140.

[7] Miller, Z., Pritikin, D., On the separation number of a graph, *NETWORKS* **19** (1989), 651-666.

[8] Raspaud, A., Schröder, H., Sýkora, O., Torok, L., Vrt'o, I., Antibandwidth of meshes and hypercubes, submitted to WG2005.