Regular embeddings of $K_{n,n}$ where n is a power of 2. II: Non-metacyclic case

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Abstract

In this paper, a classification of the regular embeddings of $K_{n,n}$, where $n = 2^e$ is a power of two, is completed. The number of such regular maps is one or two for $e = 1, 2$, respectively. For $e \geq 3$ there are $2^{e-2} + 4$ regular embeddings of $K_{2^e,2^e}$. The method is based on classification of groups G which factorise as a product of two cyclic groups $G = XY$ of order n such that the cyclic factors are transposed by an involutory automorphism. The case of metacyclic G is analysed in the preceding paper. In the present paper we show that for each exponent $e \geq 3$ there are exactly three non-metacyclic groups $G = XY$ with $|X| = |Y| = 2^e$. Given $n = 2^e$ the three non-metacyclic groups G give rise to four regular regular embeddings of $K_{n,n}$.

1 Introduction

One of the central problems in topological graph theory is to classify all the regular embeddings in orientable surfaces of a given class of graphs. In this paper we exclusively consider orientably regular embeddings of graphs by which we mean 2-cell embeddings in oriented surfaces for which the orientation-preserving automorphism group of the resulting map acts transitively, or equivalently regularly, on the directed edges, called darts. In a general setting, the classification problem was treated in [7]. However, for particular classes of graphs, it has been solved only in a few cases. In particular, all regular embeddings of complete graphs K_n have been determined by James and Jones [12]. This classification has been extended to the 'cocktail party' graphs $K_n \otimes K_2$ in [19]. More recently, the classification has been achieved for n-dimensional cubes, n odd, in [5], for complete multipartite graphs $K_n[\bar{K}_p]$, p prime, in [4], for graphs with p and pq vertices, where p and q are not necessarily distinct primes [3], and for merged Johnson graphs in [14].

In this paper we consider the classification problem for the complete bipartite graph $K_{n,n}$. It is well known that for each integer n, the graph $K_{n,n}$ has at least one regular embedding in an orientable surface, namely the standard embedding, described by Biggs and White [1, §5.6.7] as a Cayley map for the group $\mathbb{Z}_{2n}$. A complete description of all regular embeddings of $K_{n,n}$ seems to be a hard problem. The classification of regular embeddings of $K_{n,n}$ was settled for some infinite sequences of n in [20, 15, 16]. In particular, there are exactly p^{e-1} regular embeddings of K_{p^e,p^e}, p an odd prime [16]. A more combinatorial approach has been taken by Kwak and Kwon in [17, 18], where they have determined all the reflexible regular embeddings of $K_{n,n}$.

Our method uses the fact, proved in [15], that if $\mathcal{M}$ is a regular orientable embedding of $K_{n,n}$ for any n, then the group $G = \mathrm{Aut}_0^+\mathcal{M}$ of automorphisms of $\mathcal{M}$, preserving orientation and vertex-colours, factorises as a product of two disjoint cyclic groups of order n. When n is an odd prime power, a result of Huppert [8] implies that such a group G must be metacyclic, and this fact was used in [16] to classify the possibilities for G, and hence for $\mathcal{M}$. When n is a power of 2, however, Huppert's result does not apply, and indeed for each $e \geq 2$ there are regular embeddings of $K_{n,n}$ ($n = 2^e$) for which G is not metacyclic, so these are not direct analogues of the maps arising when p is odd. Nevertheless, the techniques used in the odd case were applied for $n = 2^e$ in [6] where those embeddings for which G is metacyclic are classified. The purpose of the present paper is to give a classification of regular embeddings of $K_{n,n}$ for $n = 2^e$ in the non-metacyclic case.

Before we proceed further we recall some facts on regular maps. The automorphism group of a regular map $A = \mathrm{Aut}(\mathcal{M}) = \langle r, \ell \rangle$ is generated by a generator r of the stabiliser of a vertex v (which is necessarily cyclic) and by an involution ℓ inverting an edge incident to v, see [7]. Moreover, the embedding is determined by the group A and choice of the generators $\langle r, \ell \rangle$ [13, 7, 3]. A regular map given by $A = \langle r, \ell \rangle$, $\ell^2 = 1$, is called an *algebraic map* $\mathcal{M}(A; r, \ell)$. Two algebraic maps $\mathcal{M}(A; r, \ell)$, $\mathcal{M}(A; r', \ell')$ are isomorphic if and only if there is a group automorphism in $\mathrm{Aut}(A)$ taking $r \mapsto r'$ and $\ell \mapsto \ell'$. In the particular case, when the underlying graph of a regular map $\mathcal{M}(A; r, \ell)$ is $K_{n,n}$, the automorphism group $A = \mathrm{Aut}(\mathcal{M})$

contains an index two subgroup G which is known to be a product of two cyclic groups $G = \langle r\rangle\langle r^\ell\rangle$. Indeed, any $y \in \langle r\rangle \cap \langle r^\ell\rangle$ fixes two adjacent vertices. Since $K_{n,n}$ is a simple graph, by regularity of the action on darts, we derive $y = 1$. Hence $|G| \geq |r||r^\ell| = n^2$. By regularity of the action $|G| = n^2$, hence $G = \langle r\rangle\langle r^\ell\rangle$. Vice-versa, given product of cyclic groups $G = \langle a\rangle\langle b\rangle$ admitting $x \in \mathrm{Aut}(G)$, $x^2 = 1$, $a^x = b$, a regular embedding of $K_{n,n}$, $n = |a| = |b|$, can be defined via the algebraic map $(A; a, x)$, where A is the semidirect product $A = G : \langle x\rangle$.

Thus the problem to determine isomorphism classes of regular embeddings of complete bipartite graphs $K_{n,n}$ can be settled in two steps:

1. First we determine the products $G = \langle a\rangle\langle b\rangle$ of two cyclic groups of order n interchanged by an involutory automorphism taking $a \mapsto b$;

2. Given G of the above form we consider its 2-extension $A = \langle G, x\rangle$, where $b = a^x$, $x^2 = 1$. For each such A we determine all the generating pairs r, ℓ (up to the action of $\mathrm{Aut}(A)$) such that $G = \langle r\rangle\langle r^\ell\rangle$, $|r| = n$, $\ell^2 = 1$.

It follows that the problem to classify regular embeddings of $K_{n,n}$ is closely related to the problem of classification of products of cyclic groups. This problem has attracted group theorists some time ago, fundamental results were achieved by Wielandt in [21] (1951), by Huppert in [8] (1953), Itô in [10, 11] (1955, 1956). However, the classification was not completed till now. The main obstacle to describe the groups which are products of cyclic groups is a lack of understanding of the structure of the groups in the non-metacyclic case. The following two theorems present the main results of this and of the preceding paper [6].

Theorem 1.1 *Suppose that $G = \langle a\rangle\langle b\rangle$, where $|a| = |b| = 2^e$, $\langle a\rangle \cap \langle b\rangle = 1$, and $a^\alpha = b$ for some involution α in $\mathrm{Aut}(G)$. Then one of the following cases hold:*

(1) *G is metacyclic and G has presentation*

$$G_1(e, f) = \langle h, g \mid h^{2^e} = g^{2^e} = 1, h^g = h^{1+2^f}\rangle$$

where $f = 2, \cdots, e$, and we may set $a = g^m$ and $b = g^m h$, m is odd, $1 \leq m \leq 2^{e-f}$;

(2) *G is not metacyclic, $G' = C_2$, and G has presentation*

$$G_2 = \langle a, b \mid a^4 = b^4 = 1, [b, a] = a^2b^2, [a^2, b] = [b^2, a] = 1\rangle;$$

(3) *G' is generated by two elements, and G has presentation*

$$\begin{aligned} G_3(e,k,l) &= \langle a, b \mid a^{2^e} = b^{2^e} = 1,\ c = [b,a] = a^{2+k2^{e-1}} b^{-2-k2^{e-1}}, \\ & c^a = c^{-1+l2^{e-2}} a^4,\ c^b = c^{-1-l2^{e-2}} b^{-4} \rangle, \end{aligned}$$

where $e \geq 3$, and $k, l \in \{0,1\}$. Moreover, $G_3(e,0,1) \cong G_3(e,1,1)$.

Part (1) of Theorem 1.1 is proved in [6].

Corresponding to the above groups $G_1(e,f)$, G_2 and $G_3(e,k,l)$ in Theorem 1.1 respectively, we use $A_1(e,f,m)$, A_2 and $A_3(e,k,l)$ to denote the extension of the group by x, where $x^2 = 1$ and $a^x = b$, noting that $A_3(e,0,1) \not\cong A_3(e,1,1)$ (see Lemma 4.6). Moreover, $A_1(e,f_1,m_1) \cong A_1(e,f_2,m_2)$ if and only if $(f_1,m_1) = (f_2,m_2)$, $2 \leq f_1, f_2 \leq e$ and $1 \leq m_1, m_2 \leq 2^{e-f}$, m_1, m_2 odd (see [6]). Now we are ready to state the second main theorem.

Theorem 1.2 *Let $\mathcal{M}$ be a regular map which underlying graph is the complete bipartite graph $K_{2^e,2^e}$, $e \geq 2$. Then $\mathcal{M}$ is isomorphic to one of the following maps:*

(1) $e = 2 : \mathcal{M}(A_1(2,0); a, x)$ or $\mathcal{M}(A_2, ; a, x)$;

(2) $e \geq 3 : \mathcal{M}(A_1(e,f,m); a, x)$, or $\mathcal{M}(A_3(e,k,l); a, x)$, where $k, l \in \{0,1\}$, $f = 2, \cdots, e$, m is odd, $1 \leq m \leq 2^{e-f}$.

The analysis in case when the map automorphism group is $A_1(e,f,m)$ is done in [6].

Corollary 1.3 *The number of regular embeddings of $K_{2^e,2^e}$ is $2^{e-2}+4$ for any $e \geq 3$.*

The paper is organized as follows. In Section 2 we show that a group G satisfying the assumptions of Theorem 1.1 satisfies one of the following conditions:

1. G is metacyclic

2. G is not metacyclic but its derived subgroup is cyclic

3. G is not metacyclic and its derived subgroup is generated by two elements.

As already mentioned the metacyclic case is handled in [6]. Cases 2 and 3 will be analysed separately in the following sections. Proofs of Theorems 1.1,1.2 are completed by a sequence of lemmas analysing the non-metacyclic case (see the summary in the end of the paper).

2 Structure of the derived group

For a p-group G, let $\Phi(G)$ be the Frattini subgroup of G, and let $K_i(G) = [G, G, \cdots, G]$ (i times), $\mho_i(G) = \langle g^{p^i} \mid g \in G \rangle$, $\Omega_i(G) = \langle g^{p^i} = 1 \mid g \in G \rangle$ and $Z(G)$ the central subgroup of G. Then all of them are characteristic subgroups of G. In particular $K_2(G) = G'$. First we quote some group theoretical results.

Proposition 2.1 [10] *Let $G = AB$, where A and B are abelian subgroups. Then G' is abelian.*

Proposition 2.2 [2, Corollary C] *Let $G = AB$, where A and B are abelian and B is finite. If A or B is cyclic, then $G'/(G' \cap A)$ is isomorphic to a subgroup of B.*

Proposition 2.3 [9, III. Hilfssatz 11.3] *Let G be a p-group. Then G is metacyclic if and only if $G/\Phi(G')K_3(G)$ is cyclic.*

In what follows by G we shall mean a group $G = AB$, where $A = \langle a \rangle \cong C_{2^e} \cong B = \langle b \rangle$ for $e \geq 2$, $A \cap B = 1$ and there exists an involution $x \in \mathrm{Aut}(G)$ exchanging a and b.

Lemma 2.4 [6, Lemma 3.1] *For each i, we have*

$$\mho_i(G) = \langle a^{2^i} \rangle \langle b^{2^i} \rangle, \qquad \mho_i(G)/\mho_{i+1}(G) \cong C_2 \times C_2.$$

Proof First take $i = 1$. Since $|\langle a^2 \rangle \langle b^2 \rangle| = |\langle a^2 \rangle||\langle b^2 \rangle| = 2^{2(e-1)}$ and $|G/\mho_1(G)| \geq |G/\Phi(G)| = 4$, it follows that $\mho_1(G) = \langle a^2 \rangle \langle b^2 \rangle$. For $i \geq 2$, a similar argument to prove the statement by induction on i applies. Therefore $\mho_i(G)/\mho_{i+1}(G) \cong C_2 \times C_2$. $\square$

Lemma 2.5 *Let G be the group as above. Then $G' = \langle c \rangle \times \langle a^{2^v} \rangle = \langle c \rangle \times \langle b^{2^v} \rangle$, where $c = a^{i2^u} b^{-i2^u}$, for some odd i and $u \leq v$. In particular, G' is cyclic when $v = e$.*

Proof Since $G = AB$, each element can be written by $a^i b^j$. Let $c = [b, a] = a^r b^s$. Since x interchanges a and b, we have

$$b^r a^s = [b^x, a^x] = [a, b] = [b, a]^{-1} = b^{-s} a^{-r}.$$

Therefore, we get $r \equiv -s (\mathrm{mod}\ 2^e)$ and so $c = a^r b^{-r}$. By the same argument we get that for any integer k there is h such that $c^k = a^h b^{-h}$. In particular, $\langle c \rangle \cap A = \langle c \rangle \cap B = 1$.

Suppose that $G' \cap A = \langle a^{2^v} \rangle$. Since G' is abelian (by Proposition 2.1) and $G' = \langle c^g \mid g \in G \rangle$, c is one of the elements in G' with maximal order. Since

$\langle c\rangle \cap A = \langle c\rangle \cap B = 1$, $\langle c\rangle \times \langle a^{2^v}\rangle \leq G'$. By Proposition 2.2, $G'/\langle a^{2^v}\rangle$ is cyclic. Therefore, $G'/\langle a^{2^v}\rangle = \langle c\langle a^{2^v}\rangle\rangle$, which forces that $G' = \langle c\rangle \times \langle a^{2^v}\rangle$. Set $c = a^r b^{-r}$, where $r = i2^u$ and i is odd. Since c is an element of maximal order in G', we get $u \leq v$. □

Lemma 2.5 allows us to split the analysis of G into three cases:

Case 1. G is metacyclic (then G' is cyclic),

Case 2. G is not metacyclic but G' is cyclic,

Case 3. G' is generated by two elements.

As already mentioned Case 1 is analysed in [16]. In the subsequent two sections Cases 2 and 3 will be examined.

3 G is not metacyclic but G' is cyclic

Lemma 3.1 *Let G be not metacyclic and G' be cyclic. Then G is isomorphic to the group G_2 in Theorem 1.1 (2).*

Proof By the assumptions and by Proposition 2.5 $G' = \langle c\rangle$, where $c = [b, a] = a^r b^{-r}$, for some $r = i2^u$, i odd. First observe that for any k, $\langle c^k\rangle \lhd G$. Assume that $c^a = c^t$, where t must be odd. Since the automorphism α interchanges a and b, we have $c^b = c^t$ as well. It follows that $c^{ab^{-1}} = c$. Moreover, $[c, a] = [c, b] = c^{t-1} \in \langle c^2\rangle$, that means $\overline{c} = c\langle c^2\rangle$ is a central involution in $G/\langle c^2\rangle$. Now we have

$$(ab^{-1})^2 = ab^{-1}ab^{-1} = a^2(a^{-1}ba)^{-1}b^{-1} = a^2(bc)^{-1}b^{-1} = a^2c^{-1}b^{-2} \equiv a^2b^{-2}c \,(\mathrm{mod}\ \langle c^2\rangle)$$

and

$$a^2b^2 = abac^{-1}b = bac^{-1}ac^{-1}b \equiv baab \equiv babac^{-1} \equiv b^2ac^{-1}ac^{-1} \equiv b^2a^2(\mathrm{mod}\ \langle c^2\rangle).$$

Assume $u \geq 2$. Then

$$(ab^{-1})^r = ((ab^{-1})^2)^{i2^{u-1}} \equiv ((a^2b^{-2}c)^2)^{i2^{u-2}} \equiv (a^4b^{-4})^{i2^{u-2}} \equiv a^rb^{-r} \equiv c(\mathrm{mod}\ \langle c^2\rangle).$$

Therefore, $|(ab^{-1})^r| = |c| = 2^{e-u}$ and so $|ab^{-1}| = 2^e$. Since $c \in \langle ab^{-1}\rangle$, $\langle ab^{-1}\rangle \lhd G$. Hence we have a normal cyclic subgroup $\langle ab^{-1}\rangle$ of G of order 2^e. It follows that $G = \langle a, b\rangle = \langle ab^{-1}, a\rangle = \langle ab^{-1}\rangle : \langle a\rangle$, and consequently, G is metacyclic, a contradiction.

Assume $u = 1$. Then $c = a^{2i}b^{-2i}$ for an odd i. Now we consider the normal abelian subgroup $L = \langle c, ab^{-1}\rangle$. Since $|c| = 2^{e-1}$, $|L| = 2^e$ and G is not metacyclic,

$L \cong C_{2^{e-1}} \times C_2$. Let d be an involution of L different from $c^{2^{e-2}}$. Since $\langle a \rangle \cap L = 1$ we have $G = L : \langle a \rangle$. Let $c^a = c^t$ and $d^a = c^{k2^{e-2}} d$, where $k = 0$ or 1. Then $G' = \langle [c,a]^g, [d,a]^g \mid g \in G \rangle \leq \langle c^2, c^{k2^{e-2}} \rangle$. Therefore $G' = \langle c \rangle$ if and only if $k = 1$ and $e = 2$. In this case, $|G|$=16 and G is isomorphic to G_2. □

Lemma 3.2 *Let $A_2 = G_2 : \langle x \rangle = \langle a, b \rangle : \langle x \rangle$ where x is the involutory automorphism of G_2 interchanging a and b. Then there is exactly one regular embedding of $K_{4,4}$ with the automorphism group A_2. The embedding is defined by the algebraic map $\mathcal{M}(A_2; a, x)$.*

Proof It has been shown in [17] that there are only two maps for $K_{4,4}$, one has the automorphism group isomorphic to $(C_4 \times C_4) : C_2$. Thus we have only one map with $\mathrm{Aut}(\mathcal{M}) \cong A_2 = G_2 : C_2$. In this case we may choose $r = a$ and $\ell = x$ and so the map is isomorphic to $\mathcal{M}(A_2; a, x)$. □

4 G' is a two generated group

In this section we assume that G' is not cyclic. By Lemma 2.5 $G = \langle c \rangle \times \langle a^{2^v} \rangle$, where $c = [b, a] = a^{i2^u} b^{-i2^u}$, for some odd integer i, $u \leq v$. The following lemma determines the structure of G.

Lemma 4.1 *If G' is not cyclic then $G \cong G_3(e, k, l)$ for some $k, l \in \{0, 1\}$.*

Proof Recall that $G = \langle a \rangle \langle b \rangle$ and $b = a^\alpha$ for an involutory group automorphism α. Let $H = \Omega_{e-v}(G')$. Then $H = \langle a^{i2^v} b^{-i2^v} \rangle \times \langle a^{2^v} \rangle = \langle a^{2^v} \rangle \times \langle b^{2^v} \rangle$. Set $L = \Phi(G')K_3(G')$. Then $(G/H)' = G'/H \cong C_{2^{v-u}}$, is cyclic. Since G is not metacyclic, it follows from Proposition 2.3 that G/L is not metacyclic. Since $G/L \cong (G/H)/(L/H)$, we get that G/H is not metacyclic. It follows that $G/H = \langle aH \rangle \langle bH \rangle$ and $(aH)^\alpha = bH^\alpha = bH$ and G/H is not metacyclic but with a cyclic derived subgroup. By Lemma 3.1 the factor G/H is isomorphic to the group G_2. In particular, $|G/H| = 16$ and $|(G/H)'| = 2$. Since $G/H = \langle aH, bH \rangle$ and $|aH| = 4$, it follows that $a^4 \in H = \langle a^{2^v}, b^{2^v} \rangle$ and so $v = 2$. Since $(G/H)' \cong C_{2^{v-u}}$ and $|(G/H)'| = 2$ we derive $v - u = 1$, and consequently, $u = 1$.

It follows that $c = a^{2i} b^{-2i}$ for some odd i and $G' = \langle c \rangle \times \langle a^4 \rangle$. Thus $c^a = c^s a^{4t}$ for some odd s and t. Taking into the account the action of α we get

$$c^a = c^s a^{4t}, \qquad c^b = c^s b^{-4t}, \tag{1}$$

where s and t are odd. In what follows we determine the integers i, s, t, up to group automorphisms.

Set $i=1+2k$. From $a^4c=ca^4$, we get that $[a^4,b^i]=1$. Hence $[a^4,\langle b^2\rangle]=1$, and in particular, $[a^4,b^2]=1$. Symmetrically, $[b^2,a^4]=1$. Moreover,

$$c^{a^2b^{-2}}=b^2a^{-2}a^{2+4k}b^{-2-4k}a^2b^{-2}=a^{2+4k}b^{-2-4k}=c.$$

Therefore, $c^{a^2}=c^{b^2}$. Since

$$c^{a^2}=(c^a)^a=(c^sa^{4t})^a=(c^a)^sa^{4t}=c^{s^2}a^{4t(s+1)}$$

and

$$c^{b^2}=(c^b)^b=(c^sb^{-4t})^b=(c^b)^sb^{-4t}=c^{s^2}b^{-4t(s+1)},$$

we get $s\equiv -1(2^{e-2})$. Since t is odd, we have $c^{s^2}=c$ and $a^{4t(1+s)}=1$. Therefore, we get $c^{a^2}=c$, which forces

$$[a^2,b^2]=1. \tag{2}$$

Using (1) we get

$$\begin{aligned} & b^{-1}a^{2j}b=((a^b)^2)^j=((ac^{-1})^2)^j=(ac^{-1}ac^{-1})^j=(a^2(c^a)^{-1}c^{-1})^j \\ = \; & (a^2a^{-4t}c^{-(s+1)})^j=a^{2(1-2t)j}c^{-j(s+1)}, \end{aligned} \tag{3}$$

for any positive integer j. In particular, for $j=2^{e-2}$ we get $[b,a^{2^{e-1}}]=1$, which implies $a^{2^{e-1}}\in Z(G)$ and for the symmetry reasons we have $b^{2^{e-1}}\in Z(G)$ as well. In what follows we set $z=a^{2^{e-1}}b^{2^{e-1}}=c^{2^{e-2}}$ and $s=-1+l2^{e-2}$, $l=0,1$. By (1) and (3) we have

$$c^a=c^{-1}a^{4t}z^l,\quad c^b=c^{-1}b^{-4t}z^l,\qquad b^{-1}a^{2j}b=a^{2(1-2t)j}z^{lj}. \tag{4}$$

By $c^b=(a^{2(1+2k)}b^{-2(1+2k)})^b$ and (4), we get

$$a^{-2(1+2k)}b^{2(1+2k)}b^{-4t}z^l=a^{2(1-2t)(1+2k)}z^{l(1+2k)}b^{-2(1+2k)},$$

that is

$$a^{4(1-t+2k-2kt)}=b^{4(1-t+2k)}.$$

Therefore we get

$$1-t+2k-2kt\equiv 1-t+2k\equiv 0(\text{mod } 2^{e-2}).$$

Solving this equation we get $2k\equiv 0(\text{mod } 2^{e-2})$ and $t\equiv 1(\text{mod } 2^{e-2})$. With these results in hand, we rewrite the relations (4) as follows:

$$\begin{aligned} & c=a^2b^{-2}z^k,\quad c^a=c^{-1}a^4z^l=a^2b^2z^l,\quad c^b=c^{-1}b^{-4}z^j=a^{-2}b^{-2}z^l, \\ & b^{-1}a^{2j}b=a^{-2j}z^{lj},\quad a^{-1}b^{2j}a=b^{-2j}z^{lj}, \end{aligned} \tag{5}$$

where $l, k = 0, 1$. This shows that G is isomorphic to one of the groups $G_3(e, k, l)$ in Theorem 1.1 (3). □

For any element $g \in G$ we use $Inn(g)$ to denote the inner automorphism induced by g.

Lemma 4.2 *The groups $G_3(e, l, k)$ are well defined and satisfy the assumptions of Theorem 1.1.*

Proof Let $K = \langle c\rangle \times \langle d\rangle \cong C_{2^{e-1}} \times C_{2^{e-2}}$, where $e \geq 3$. Set $z = c^{2^{e-2}}$. It is clear that for $l = 0$ or 1, the mapping $\tau : c^i d^j \to (c^{-1}dz^l)^i d^j$ on K defines an automorphism of K, for $i \in C_{2^{e-1}}$ and $j \in C_{2^{e-2}}$. Clearly, $\tau^4 = \mathrm{Inn}(d) = 1, \quad d^\tau = d$. By the group extension theory, we may define a cyclic extension N of K by $\langle a\rangle$ as follows: $a^4 = d$ and $g^a = g^\tau$, for $g \in K$. Moreover, one can check

$$N = K\langle a\rangle = \langle c, a\rangle, \quad N' = \langle c^{-2}dz^l\rangle, \quad [c, a^2] = 1.$$

For $k = 0$ or 1, define a mapping $\sigma : N \to N$ by $a^i c^j \to (ac^{-1})^i (ca^{-4}z^k)^j$. One can check $\sigma \in \mathrm{Aut}(N)$, $\sigma^2 = 1$, and

$$\sigma^2 = \mathrm{Inn}(a^2c^{-1}z^l), \quad (a^2c^{-1}z^l)^\sigma = a^2c^{-1}z^l.$$

Therefore, by the group extension theory, we may define a cyclic extension M of N by $\langle b\rangle$ as follows: $b^2 = a^2c^{-1}z^l$ and $g^b = g^\sigma$ for $g \in N$.

Now $M = \langle a, c, b\rangle = \langle a, b\rangle$ and $|M| = 2|N| = 8|K| = 2^{2e}$. Also, $|a| = 4|d| = 2^e$, $|b| = 2|a^2c^{-1}z^l| = 2^e$, $\langle a\rangle \cap \langle b\rangle = 1$ and the mapping α interchanging a and b extends to an automorphism of M. From the definition of the group M, one can see that $M = G_3(e, k, l)$. In other words, $G_3(e, k, l)$ is well defined. □

The following lemma gives some formulas for the group $G_3(e, k, l)$ for the later use.

Lemma 4.3 *For any $a^i, b^j \in G_3(e, k, l)$, we have*

1. $b^j a^i = a^i b^j$ *for* $2 \mid i, j$;
2. $b^j a^i = a^i b^{-j} z^{\frac{lj}{2}}$ *for* $2 \nmid i$ *and* $2 \mid j$;
3. $b^j a^i = a^{-i} b^j z^{\frac{li}{2}}$ *for* $2 \mid i$ *and* $2 \nmid j$;
4. $b^j a^i = a^{-i} b^{-j} z^{l(\frac{i+j}{2})+k}$ *for* $2 \nmid i$ *and* $2 \nmid j$;

where $z = a^{2^{e-1}} b^{2^{e-1}}$.

Proof We prove Lemma 4.3(4). The remaining statements can be proved by similar arguments. By using (5), we have

$$\begin{aligned} & b^j a^i = b^{j-1}abca^{i-1} = b^{j-1}ab(b^{-2}a^2z_k)a^{i-1} = a(b^{j-1})^a(a^{i+1})^b b^{-1}z^k \\ = \; & ab^{1-j}z^{\frac{l(j-1)}{2}}a^{-i-1}z^{\frac{l(i+1)}{2}}b^{-1}z^k = a^{-i}b^{-j}z^{l(\frac{i+j}{2})+k}. \end{aligned}$$ □

Lemma 4.4 *$G_3(e,1,1) \cong G_3(e,0,1)$; and the groups $G_3(e,0,0), G_3(e,1,0)$ and $G_3(e,1,0)$ are pairwise nonisomorphic.*

Proof Let

$$\begin{aligned} G_3(e,k,l) \; &= \langle a,b \mid a^{2^e} = b^{2^e} = 1,\, c = [b,a] = a^{2+k2^{e-1}}b^{-2-k2^{e-1}}, \\ & c^a = c^{-1+l2^{e-1}}a^4,\, c^b = c^{-1-l2^{e-1}}b^{-4}\rangle. \end{aligned}$$

and

$$\begin{aligned} G_3(e,k_1,l_1) \; &= \langle a_1,b_1 \mid a_1^{2^e} = b_1^{2^e} = 1, c_1 = [b_1,a_1] = a_1^{2+k_12^{e-1}},\, b_1^{-2-k_12^{e-1}}, \\ & c_1^{a_1} = c_1^{-1+l_12^{e-1}}a_1^4,\, c_1^{b_1} = c_1^{-1-l_12^{e-1}}b_1^{-4}\rangle. \end{aligned}$$

Let σ be a mapping from $G_3(e,k_1,l_1)$ to $G_3(e,k,l)$ defined by:

$$a_1^\sigma = a^ib^j, \qquad b_1^\sigma = a^ub^v,$$

for some integers i, j, u and v. Then σ is an isomorphism if and only if σ preserves the defining relations and maps the generating pair (a_1,b_1) to a generating pair of $G_3(e,k,l)$.

If i and j are odd, then by Lemma 4.3 $(a^ib^j)^2 = a^ib^ja^ib^j = z^{l\frac{i+j}{2}+k}$ and so $|a^ib^j| \le 4$. Therefore, either i and v are odd, and j and v are even; or j and v are odd and i and v are even.

As before, we set $z = a^{2^{e-1}}b^{2^{e-1}}$ and $z_1 = a_1^{2^{e-1}}b_1^{2^{e-1}}$. Clearly, $z_1^\sigma = z_1$. Now by Lemma 4.3, we get

$$c_1^\sigma = [b_1^\sigma, a_1^\sigma] = [a^ub^v, a^ib^j] = \begin{array}{l} a^{2i}b^{-2v}z^{\frac{l(v-i)}{2}+k}, \text{ for } 2\nmid i, 2\nmid v; \\ a^{-2u}b^{2j}z^{\frac{l(u-j)}{2}+k}, \text{ for } 2\nmid j, 2\nmid u, \end{array}$$

and

$$(a_1^\sigma)^2(b_1^\sigma)^{-2}(z_1^\sigma)^{k_1} = (a^ib^j)^2(a^jb^u)^{-2} = \begin{array}{l} a^{2i}b^{-2v}z^{\frac{l(j-u)}{2}+k_1}, \text{ for } 2\nmid i, 2\nmid v; \\ a^{-2u}b^{2j}z^{\frac{l(i-v)}{2}+k_1}, \text{ for } 2\nmid i, 2\nmid v. \end{array}$$

Therefore, $c_1^\sigma = (a_1^\sigma)^2(b_1^\sigma)^2(z_1^\sigma)^{k_1}$ if and only if

$$\frac{l((i+j)-(u+v))}{2} + k \equiv k_1 (\text{mod } 2). \tag{6}$$

Note that $c^a = c^{-1}a^4z^l$ is equivalent to $a^{-1}b^{-2}ab^{-2}z^l = 1$. From

$$(a_1^\sigma)^{-1}(b_1^\sigma)^{-2}a_1^\sigma(b_1^\sigma)^{-2}(z_1^\sigma)^{l_1} = 1,$$

we get

$$b^{-j}a^{-i}(a^ub^v)^{-2}a^ib^j(a^ub^v)^{-2}z^{l_1} = 1.$$

Applying Lemma 4.3 to the above equation we get $z^{l+l_1} = 1$, that is

$$l \equiv l_1 (\text{mod } 2). \tag{7}$$

Using similar arguments, starting from the relation $c^b = b^{-1}b^{-4}z^l$ we get (7).

Therefore σ is an isomorphism if and only if (6) and (7) hold, where either i, v are odd and j, u are even, or i, v are even and j, u are odd. By these conditions, one immediately gets that $G_3(e, k, 0) \neq G_3(e, k', 1)$ and $G_3(e, 0, 0) \neq G_3(e, 1, 0)$, and moreover, taking $i = 3$, $j = u = 0$ and $v = 1$ in the definition of σ we get an isomorphism from $G_3(e, 0, 1)$ to $G_3(e, 1, 1)$. □

The following lemma summarises information on automorphisms of groups $G_3(e, l, k)$ extracted from the above proof.

Lemma 4.5 *Each automorphism of $G_3(e, l, k)$ is given by $a^\sigma = a^ib^j, b^\sigma = a^ub^v$, where $4 \mid l((i+j) - (u+v))$ and either i and v are odd and j and u are even; or i and v are even and j and u are odd. Moreover, each choice of parameters i, j, u, v satisfying the above conditions determines an automorphism of $G_3(e, l, k)$.*

Recall that $A_3(e, k, l) = G_3(e, k, l) : \langle x \rangle$, where $x^2 = 1$ and $a^x = b$. Now we determine the orbits of $\mathrm{Aut}(A_3(e, k, l))$ acting on the generating pairs (r, ℓ) of $A_3(k, l)$ such that $|r| = n$ and $|\ell| = 2$, and $\langle r \rangle \cap \langle r^\ell \rangle = 1$.

Lemma 4.6 *With the above notation, $\mathrm{Aut}(A_3(e, k, l))$ is transitive on the admissible generating pairs (r, ℓ), with a representative (a, x). Moreover, given $e \geq 3$, the four groups $\mathrm{Aut}(A_3(e, k, l))$, $k, l \in \{0, 1\}$, are pairwise nonisomorphic. Correspondingly, for each such group we get exactly one regular embedding of $K_{2^e, 2^e}$.*

Proof First suppose that $r = a^ib^jx$ for some i, j. Then $r^2 = a^ib^{i+j}a^j$, implying $\langle r^{2^{e-1}} \rangle = \langle z \rangle$. Since $\langle z \rangle$ is characteristic in $A_3(e, k, l)$, we have $1 \neq \langle z \rangle \leq \langle r \rangle \cap \langle r \rangle^g$ for any $g \in A_3(e, k, l)$, a contradiction. Therefore, $r \in G_3(e, k, l)$. implying $r = a^ib^j$ for some i, j. Since r is an element of order $n = 2^e$, by Lemma 4.3 exactly one of i, j is odd.

Let $\ell = a^sb^tx$ for some s, t. From $\ell^2 = 1$ we get $s = -t$ and so $\ell = a^sb^{-s}x$.

Assume τ is an automorphism of $A_3(e,k,l) = \langle a, x\rangle$ taking $a^\tau = a^i b^j$ and $x^\tau = a^s b^{-s} x$, where exactly one of i, j is odd. Then by Lemma 4.3

$$b^\tau = (xax)^\tau = a^s b^{-s} x a^i b^j a^s b^{-s} x = a^s b^{-s} b^i a^j a b^s b a^{-s}$$
$$= \begin{cases} a^j b^{2s-i} z^{\frac{l(i-s)}{2}}, & \text{for } 2 \mid s, 2 \mid i, 2 \nmid j; \\ a^{2s-j} b^i z^{\frac{l(j-s)}{2}}, & \text{for } 2 \mid s, 2 \nmid i, 2 \mid j; \\ a^{-j} b^{i-2s} z^{\frac{l(s+j)}{2}+k}, & \text{for } 2 \nmid s, 2 \mid i, 2 \nmid j; \\ a^{2s+j} b^{-i} z^{\frac{l(i-s)}{2}+k}, & \text{for } 2 \nmid s, 2 \nmid i, 2 \mid j. \end{cases} \tag{8}$$

Since $a^\tau, b^\tau \in G_3(e,k,l)$, τ induces an automorphism of $G_3(e,k,l)$. It follows that there are u, v such that $b^\tau = a^u b^v$. Using equation (8) we can compute u and v in each of the four cases and check that $4 \mid l((i+j) - (u+v))$. By Lemma 4.5 such τ exists. That means that $\mathrm{Aut}(A_3(e,k,l))$ is transitive on the generating pairs with the representative (a, x).

It has been proved that $G_3(e,0,1) \cong G_3(e,1,1)$. Suppose σ is an isomorphism from $A_3(e,1,1) = \langle a_1, x_1\rangle$ to $A_3(e,0,1) = \langle a, x\rangle$. By the above arguments, we may assume that $a_1^\sigma = a^i b^j$ and $x_1^\sigma = a^s b^{-s} x$. Set $b_1^\sigma = a^u b^v$. By (8), we get that $4 \mid ((i+j) - (u+v))$. Since σ induces an isomorphism from $A_3(e,1,1)$ to $A_3(e,0,1)$, it follows from (6) that $1 = k_1 \equiv k = 0 \pmod 2$, a contradiction. Therefore, $A_3(e,1,1) \ncong A_3(e,0,1)$. □

Proof of Theorem 1.1. By Lemma 2.5 a group G satisfying the assumptions is either metacyclic, or non-metacyclic with cyclic G', or it is non-metacyclic with a 2-generator abelian derived group G'. If G is metacyclic then by [6] it has presentation as in Theorem 1.1 (1). If G is not metacyclic, but G' is, then the statement of Theorem 1.1 (2) follows from Lemma 3.1. Finally, the case (3) in Theorem 1.1 is covered by Lemma 4.1. □

Proof of Theorem 1.2. By Theorem 1.1 we deduce that the automorphism group A of a regular map with the underlying graph $K_{n,n}$, $n = 2^e$, is one of the required groups. For each of the groups, the generating pairs r, ℓ giving rise to non-isomorphic regular embeddings of $K_{n,n}$ are identified in [6] if $A = A_1(e,f,m)$; in Lemma 3.2 if $A = A_2$; and in Lemma 4.6 if $A = A_3(e,k,l)$. □

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