Enumeration of unrooted hypermaps of a given genus

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Abstract

In this paper we derive an enumeration formula for the number of hypermaps of a given genus g and given number of darts n in terms of the numbers of rooted hypermaps of genus $\gamma \leq g$ with m darts, where $m|n$. Explicit expressions for the number of rooted hypermaps of genus g with n darts were derived by Walsh (1975) for $g = 0$, and by Arques (1987) for $g = 1$. We apply our general counting formula to derive explicit expressions for the number of unrooted spherical hypermaps and for the number of unrooted toroidal hypermaps with given number of darts. The enumeration results can be expressed in terms of Fuchsian groups.

Key Words: Enumeration, Map, Surface, Orbifold, Rooted hypermap, Unrooted hypermap, Fuchsian group
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1 Introduction

An *oriented map* is a 2-cell decomposition of a closed orientable surface with a fixed global orientation. Generally, maps can be described combinatorially via graph embeddings. Oriented hypermaps are generalisations of oriented maps. While maps are 2-cell embeddings of graphs, hypermaps can be viewed as embeddings of hypermaps in closed orientable surfaces. Such a model was investigated by Walsh in [32], where the underlying hypergraph is described via the corresponding 2-coloured bipartite graph B, and the hypermap itself is determined by a 2-cell embedding $B \to S$.

By a *map* we mean a 2-cell decomposition of a compact connected surface. Enumeration of maps on surfaces has attracted a lot of attention during last decades [23]. Generally, problems of the following sort are considered:

Problem 1: How many isomorphism classes of maps with a given property $\mathcal{P}$ and a given number of edges (vertices, faces) there are?

The beginnings of the enumerative theory of maps are closely related to the enumeration of plane trees considered in 1960s by Tutte [28] and Harary, Prins and Tutte [6] (see [7, 22] as well). Later many other distinguished classes of maps including triangulations, outerplanar, cubic, Eulerian, nonseparable, simple, loopless, two-face maps etc. were considered. Although there are more than 100 published papers on map enumeration most of them deal with the enumeration of rooted maps with a given property. In particular, there is a lack of results on enumeration of unrooted maps of genus ≥ 1. Most of the results on map enumeration in the unrooted case are restricted to planar maps [17, 18, 33, 34, 20]. A recent paper [25] presents a breakthrough in the enumeration problem for unrooted maps of genus ≥ 1. In the present paper we apply the methods employed in [25] to solve an analogous problem for hypermaps.

The problem considered in this paper reads as follows.

Problem 2. What is the number $H_g(n)$ of isomorphism classes of oriented unrooted hypermaps of given genus g and given number of darts n?

An oriented map is called *rooted* if one of the darts (arcs) is distinguished as a root. By a *dart* of a map we mean an edge endowed with one of the two possible orientations. Isomorphisms between oriented rooted maps take root onto root. A rooted variant of Problem 2 follows.

Problem 3. What is the number $h_g(n)$ of isomorphism classes of oriented rooted hypermaps of given genus g and given number of edges e?

Problem 3 was solved by Walsh in 1975 [32] for $g = 0$, i.e. for the spherical case. A corresponding case of Problem 2 for $g = 0$, was settled by Bosquet-Melon and Schaeffer in terms of planar 2-constellations in 2000 [3]. As concerns genera $g \geq 1$, the solution of Problem 3 was obtained by Arqués for $g = 1$ [2], the other instances of Problem 3 remain unsettled.

The aim of the present paper is to show that Problem 2 can be reduced to Problem 3. More precisely, we prove that the numbers of unrooted oriented hypermaps with n darts and given genus g can be determined explicitly whenever the the numbers $h_\gamma(m)$ are known for each $m|n$ and $\gamma \leq g$ (see Theorem 3.5 for details). Since $h_\gamma(m)$ are known for $\gamma = 0, 1$ we are able to determine the numbers $H_0(n)$ and $H_1(n)$ in terms of arithmetic functions depending on n. All the derived results can be expressed in group theoretical language. Namely, the numbers $h_g(n)$ determine the numbers of subgroups of index n and genus g of a free group of rank two, seen as the universal triangle group $\Delta^+(\infty, \infty, \infty) = \langle x, y, z | xyz = 1\rangle$; while $H_g(n)$ give the numbers of conjugacy classes of such subgroups. Note that conjugacy classes of subgroups of free groups of given index were enumerated by Liskovets [16] (see [15, 27] as well).

2 Hypermaps on orbifolds

Hypermaps on surfaces. An *oriented combinatorial hypermap* is a triple $\mathcal{H} = (D; R, L)$, where D is a finite set of darts (called brins, blades, bits as well) and R, L are permutations of D such that $\langle R, L\rangle$ is transitive on D. The orbits of R are called *hypervertices*, the orbits of L are called *hyperedges* and the orbits of RL are called *hyperfaces*. The degree of a hypervertex (hyperedge, hyperface) is the size of the respective orbit.

Let $|D| = n$. Denote by v, e and f the numbers of hypervertices, hyperedges and hyperfaces. Then the genus g of $\mathcal{H}$ is given by Euler-Poincare formula as follows

$$v + e + f - n = 2 - 2g.$$

Given hypermaps $\mathcal{H}_i = (D_i; R_i, L_i)$, $i = 1, 2$ a mapping $\psi : D_1 \to D_2$ such that $R_2\psi = \psi R_1$ and $L_2\psi = \psi L_1$ is called a *morphism* (or a *covering*) $\mathcal{H}_1 \to \mathcal{H}_2$. Note that each morphism between hypermaps is by definition an epimorphism. If $\psi : \mathcal{H}_1 \to \mathcal{H}_2$ is a bijection, ψ is an *isomorphism.* The isomorphisms $\mathcal{H} \to \mathcal{H}$ form a group $\mathrm{Aut}(\mathcal{H})$ of *automorphisms* of $\mathcal{H}$. It is easily seen that $\mathrm{Aut}(\mathcal{H})$ acts semiregularly on D; equivalently, the stabiliser of a dart is trivial. A hypermap $\mathcal{H}$ is called *rooted* if one element x of D is chosen to be a root. Morphisms between rooted hypermaps take roots onto roots. It follows that a rooted hypermap admits no non-trivial automorphisms.

By *a surface* we mean a connected, orientable surface without boundary. A *topological map* is a 2-cell decomposition of a surface. Usually, maps on surfaces are described as 2-cell embeddings of graphs. Oriented combinatorial maps are hypermaps $(D; R, L)$ such that L is a fixed-point-free involution. Walsh observed that oriented hypermaps can be viewed as particular maps. Namely, he proved a one-to-one correspondence [32, Lemma 1] between hypermaps and the set of (oriented) 2-coloured bipartite maps. That means that one of the two global orientations of the underlying surface is fixed, and moreover, we assume that a colouring of vertices, say by black and white colours, is preserved by morphisms between maps. The correspondence is given as follows. Let $\mathcal{M}$ be 2-coloured bipartite map on an orientable surface S with a fixed global orientation. We set D to be the set of edges of $\mathcal{M}$. The orientation of S induces at each black vertex v of $\mathcal{M}$ a cyclic permutation R_v of edges incident with v. This way a permutation $R = \prod R_v$ of D is defined. Similarly, the orientation of S determines at each white vertex u a cyclic permutation L_u. Set $L = \prod L_u$. Hence we have a unique hypermap $(D; R, L)$ corresponding to $\mathcal{M}$. Conversely, given hypermap $(D; R, L)$ we first define a bipartite 2-colored graph X whose edges are elements of D, black vertices are orbits of R and white vertices are orbits of L. An edge $x \in D$ is incident to a (black or white) vertex u if $x \in u$. The permutation R and L induce local rotations of arcs outgoing from black and white vertices, respectively. It is well known (see Gross and Tucker [5, Section 3.2]) that the system of rotations determines a 2-cell embedding of X into an orientable surface.

Similarly as above, an oriented 2-coloured bipartite map is called *rooted* if one of the edges is selected to be a root. Morphisms between rooted 2-coloured bipartite maps take a root onto a root.

There is yet another way to describe hypermaps. Let $\mathcal{H} = (D; R, L)$ be a hypermap. Clearly, the permutation group $\langle R, L\rangle$ is an epimorphic image of the free product $\Delta^+ = C * C \cong \langle \rho\rangle * \langle \lambda\rangle$ of two infinite cyclic groups. The group Δ^+ acts on D via the epimorphism taking $\rho \mapsto R$ and $\lambda \mapsto L$. Thus by using some standard results in permutation group theory each hypermap can be described by a subgroup $F \leq \Delta^+$ [13, 30, 31, 9]. The subgroup F, called a *hypermap subgroup*, can be identified with a stabiliser of a dart in the action of Δ^+ on D. Since the action of Δ^+ on D is transitive, the number of darts $|D| = n$ coincides with index $[\Delta^+ : F]$ of F in Δ^+. Given $F \leq \Delta^+$ the corresponding hypermap can be constructed as an *algebraic hypermap* $\mathcal{H}(\Delta^+/F) = (D; R, L)$, where $D = \{xF | x \in \Delta^+\}$ is the set of left cosets, and the action of R, L on D is defined by $R(xF) = (\rho x)F$, $L(xF) = (\lambda x)F$. Note that the group Δ^+ is sometimes called a universal oriented triangle group. More precisely, Δ^+ is identified with the triangle group $T(\infty, \infty, \infty) =< x, y, z : x\,y\,z = 1 >$ acting on the hyperbolic plane $\mathbf{H}^2$ by orientation-preserving isometries (see G.Jones, D.Singerman [13]). In this case $\mathbf{H}^2/\Delta^+$ is a thrice punctured sphere and $\mathbf{H}^2/F$ is a punctured orientable surface whose genus g coincides with the genus of the corresponding hypermap.

We summarise the above discussion in the following propositions.

Proposition 2.1 *The following objects are in one-to-one correspondence:*

(1) rooted 2-coloured bipartite maps of genus g with n edges,

(2) rooted hypermaps $(D; R, L)$ of genus g with $|D| = n$,

(3) subgroups of the group $\Delta^+ = T(\infty, \infty, \infty)$ of index n and genus g .

Part (1) $\Leftrightarrow$ (2) follows from Walsh [32]. Part (2) $\Leftrightarrow$ (3) is in ([13, 4]).

By definition isomorphic hypermaps have conjugate hypermap subgroups. Hence isomorphism classes of hypermaps correspond to conjugacy classes of subgroups.

Proposition 2.2 *The following objects are in one-to-one correspondence:*

(1) isomorphism classes of 2-coloured bipartite maps of genus g with n edges,

(2) isomorphism classes of hypermaps $(D; R, L)$ of genus g with $|D| = n$,

(3) conjugacy classes subgroups of index n and genus g of the group $\Delta^+ = T(\infty, \infty, \infty)$.

Regular coverings. Let $\psi : \mathcal{H}_1 \to \mathcal{H}_2$ be a covering of hypermaps. The covering transformation group consists of automorphisms α of $\mathcal{H}_1$ satisfying the condition $\psi = \psi \circ \alpha$. A covering $\psi : \mathcal{H}_1 \to \mathcal{H}_2$ will be called *regular* if the covering transformation group acts transitively on a fibre $\psi^{-1}(x)$ over a dart x of $\mathcal{H}_2$. Regular coverings can be constructed by taking a subgroup $G \leq Aut(\mathcal{H}_1)$, $\mathcal{H}_1 = (D; R, L)$, and setting $\bar{D}$ to be the set of orbits of G, $\bar{R}[x] = [Rx]$, $\bar{L}[x] = [Lx]$. Then the natural projection $x \mapsto [x]$ defines a regular covering $M \to N$, where $\mathcal{H}_2 = (\bar{D}, \bar{R}, \bar{L})$.

Maps and hypermaps on orbifolds. Given regular covering $\psi : \mathcal{H} \to \mathcal{K}$, let be x be a hypervertex, hyperface or a hyperedge of $\mathcal{K}$. Let $\mathcal{H}$ be of genus g, $\mathcal{K}$ be of genus γ and let $G \leq \mathrm{Aut}(\mathcal{H})$ be a covering transformation group. The ratio of degrees $b(x) = deg(\tilde{x})/deg(x)$, where $\tilde{x} \in \psi^{-1}(x)$ is a lifting of x along ψ, will be called a *branch index* of x. By transitivity of the action of the group of covering transformations, a branch index is a well-defined positive integer not depending on the choice of the lift $\tilde{x}$. Hence b is a well defined integer function defined on the union $V(\mathcal{K}) \cup \mathcal{E}(\mathcal{K}) \cup \mathcal{F}(\mathcal{K})$. Writing all the values $b(x)$, $b(x) \geq 2$, in non-decreasing order we get an integer sequence $m_1, m_2, \ldots, m_r$. In this way an orbifold S_g/G with signature $[\gamma; m_1, m_2, \ldots, m_r]$ is defined.

For our purposes we define a topological 2-dimensional orbifold $O = O[\gamma; m_1, \ldots, m_r]$ to be a closed orientable surface of genus γ with a distinguished set of points $\mathcal{B}$, called branch points, and an integer function assigning to each $x \in \mathcal{B}$ an integer $b(x) \geq 2$. A 2-coloured bipartite map of genus γ is a map on O provided the following two conditions are satisfied:

(1) no branch point $x \in \mathcal{B}$ lies on an edge,

(2) each face contains at most one branch point $x \in \mathcal{B}$.

The operation associating a 2-coloured bipartite map to a hypermap is functorial. In particular the signature of an orbifold associated with a regular covering of hypermaps coincides with the signature of an orbifold determined by the corresponding regular covering of Walsh's 2-coloured bipartite maps. Note also that a regular covering $\psi : \mathcal{H} \to \mathcal{K}$, extends (uniquely) to a regular covering $S_g \to S_g/G$, where g is the genus of $\mathcal{H}$ and G is the group of covering transformations.

Let O be an orbifold with signature $[\gamma; m_1, m_2, \ldots, m_r]$. The *orbifold fundamental group* $\pi_1(O)$ is an F-group

$$\pi_1(M, \sigma) = F[\gamma; m_1, m_2, \ldots, m_r] =$$

$$\langle a_1, b_1, a_2, b_2, \ldots, a_\gamma, b_\gamma, e_1, \ldots, e_r | \prod_{i=1}^{\gamma}[a_i, b_i] \prod_{j=1}^{r} e_j = 1, e_1^{m_1} = \ldots e_r^{m_r} = 1\rangle. \quad (2.1)$$

Let $\mathcal{H} \to \mathcal{H}/G = \mathcal{K}$ be a regular covering between hypermaps with a covering transformation group G and suppose $\mathcal{H}$ is finite. Let the signature of the respective orbifold of $\mathcal{K} = \mathcal{H}/G$ be $[\gamma; m_1, m_2, \dots, m_r]$. Then the Euler characteristic of the underlying surface of $\mathcal{H}$ is given by the Riemann-Hurwitz equation:

$$\chi = |G|\big(2 - 2\gamma - \sum_{i=1}^{r}(1 - \frac{1}{m_i})\big). \tag{2.2}$$

3 The general counting formula.

The following theorem is the main result of [24].

Theorem 3.1 *Let Γ be a finitely generated group. Then the number of conjugacy classes of subgroups of index n in the group Γ is given by the formula*

$$N_\Gamma(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{\substack{K < \Gamma \\ [\Gamma : K] = m}} Epi(K, Z_\ell).$$

In fact, a slight modification of the proof allows us to generalise the above statement to subsets of subgroups of given index closed under conjugacy. Let $\mathcal{P}$ be a set of subgroups of a finitely generated group Γ closed under conjugation. Denote by $Epi_{\mathcal{P}}(K, Z_\ell)$ the number of epimorphisms $K \to Z_\ell$ with the kernel in $\mathcal{P}$.

Hence we have the following

Theorem 3.2 *Let Γ be a finitely generated group. Let $\mathcal{P}$ be a set of subgroups of Γ closed under conjugation. Then the number of conjugacy classes of subgroups of index n in $\mathcal{P}$ is given by the formula*

$$N_\Gamma^{\mathcal{P}}(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{\substack{K < \Gamma \\ [\Gamma : K] = m}} Epi_{\mathcal{P}}(K, Z_\ell).$$

A group epimorphism is called *order-preserving* if it preserves the orders of elements of finite order. Given a closed orientable surface S_g of genus g and a cyclic orbifold $O = S_g/Z_\ell$ we denote by $Epi_0(\pi_1(O), Z_\ell)$ the number of order-preserving epimorphisms $\pi_1(O) \to Z_\ell$. The following theorem gives a general counting formula for the numbers of unrooted hypermaps of given genus.

Theorem 3.3 *Let S_g be a closed orientable surface of genus g. Let $h_O(m)$ be the number of rooted hypermaps with m darts on a cyclic orbifold $O = S_g/Z_\ell$.*

Then the number of unrooted hypermaps of genus g having n darts is

$$H_g(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{O \in Orb(S/Z_\ell)} h_O(m) Epi_0(\pi_1(O), Z_\ell),$$

where the second sum runs through all admissible cyclic orbifolds S_g/Z_ℓ.

Proof. Given $S = S_g$, let $\mathcal{P} = \mathcal{P}_g$ be the set subgroups of genus g of $\Delta^+ = T(\infty,\infty,\infty)$. By Propositions 2.1 and 2.2 rooted hypermaps on S correspond subgroups in $\mathcal{P}$, and isomorphism classes of unrooted hypermaps on S correspond to conjugacy classes of subgroups in $\mathcal{P}$. Setting $\Gamma = \Delta^+$ in Theorem 3.2 we get

$$H_g(n) = N^{\mathcal{P}}_{\Delta^+}(n) = \frac{1}{n}\sum_{\substack{\ell|n\\ \ell m=n}}\ \sum_{\substack{K<\Delta^+\\ [\Delta^+:K]=m}} Epi_{\mathcal{P}}(K, Z_\ell).$$

A given epimorphism $\psi : K \to Z_\ell$ with kernel $H \in \mathcal{P}$ determines a regular covering of algebraic hypermaps $\psi^* : \mathcal{H}(\Delta^+/H) \to \mathcal{H}(\Delta^+/K)$ induced by $H \trianglelefteq K$ with the group of covering transformations isomorphic to Z_ℓ. Let σ be the signature of the orbifold $O = O(\sigma) = S_g/Z_\ell$ determined by the covering of hypermaps. Then the set of epimorphisms $\psi : K \to Z_\ell$ with $Ker(\psi) = H \in \mathcal{P}$ splits into classes characterised by the signatures of the cyclic orbifolds $O = S/Z_\ell$. Denote by $Epi_\sigma(K, Z_\ell)$ the number of epimorphisms $K \to Z_\ell$ with kernel $H \in \mathcal{P}$ and quotient orbifold $O = S/Z_\ell$ with signature σ. We set $\mathcal{P}_\sigma = \{K|K < \Delta^+, Epi_\sigma(K, Z_\ell) \neq 0\}$.

It is well known that the group Δ^+ acts on the universal covering surface $\mathcal{H}^2$ as a discontinuous group of conformal automorphisms. This enables us to introduce the structure of Riemann surface (as well as the orbifold structure) on the hypermaps $\mathcal{H}(\Delta^+/H)$, $\mathcal{H}(\Delta^+/K)$, respectively. A regular covering of hypermaps $\psi : \mathcal{H}(\Delta^+/H) \to \mathcal{H}(\Delta^+/K)$ extends to a branched regular covering $S \to O$ of the orbifold $O = O(\sigma)$ by the closed surface S. By the Riemann Extension Theorem there is a one-to-one correspondence between the coverings $\mathcal{H}^2/H \to \mathcal{H}^2/K$ and the coverings of the compactified quotient spaces $S = \overline{\mathcal{H}^2/H} \to O = \overline{\mathcal{H}^2/K}$ (see [12] for a more detailed explanation). We will show that $Epi_\sigma(K, Z_\ell) = Epi_0(\Gamma(\sigma), Z_\ell)$. Given $K \in \mathcal{P}_\sigma$ we calculate the number of regular Z_ℓ-coverings $\mathcal{H}^2/H \to \mathcal{H}^2/K$ with $H \trianglelefteq K$ and $H \in \mathcal{P}$. By G. Jones [11] there are $Epi_\sigma(K, Z_\ell)/\varphi(\ell)$ such coverings. On the other hand, we have $Epi_0(\Gamma(\sigma), Z_\ell)/\varphi(\ell)$ regular Z_ℓ-coverings $S = \overline{\mathcal{H}^2/H} \to O = \overline{\mathcal{H}^2/K}$ over the orbifold $O = O(\sigma)$ with the signature σ [11]. By virtue of the one-to-one correspondence these numbers coincide. Hence, we have $Epi_\sigma(K, Z_\ell) = Epi_0(\Gamma(\sigma), Z_\ell)$ as required. Given m, ℓ and σ denote by $\nu_\sigma(m)$ the number of subgroups $K < \Delta^+$ in $\mathcal{P}(\sigma)$ and by $Sign(S_g/Z_\ell)$ the set of signatures of cyclic g-admissible orbifolds. We have

$$H_g(n) = \frac{1}{n}\sum_{\substack{\ell|n\\ \ell m=n}}\ \sum_{\substack{K<\Delta^+\\ [\Delta^+:K]=m}} Epi_{\mathcal{P}}(K, Z_\ell) = \frac{1}{n}\sum_{\substack{\ell|n\\ \ell m=n}}\ \sum_{\sigma\in Sign(S_g/Z_\ell)} \nu_\sigma(m) Epi_\sigma(K, Z_\ell) =$$

$$\frac{1}{n}\sum_{\substack{\ell|n\\ \ell m=n}}\ \sum_{\sigma\in Sign(S_g/Z_\ell)} \nu_\sigma(m) Epi_0(\Gamma(\sigma), Z_\ell).$$

Taking into the account the correspondence between groups in $\mathcal{P}_\sigma$ and rooted hypermaps on the orbifold $O = O(\sigma)$ we get $\nu_\sigma(m) = h_O(m)$ and the proof is complete. □

In what follows we derive a formula enumerating numbers of rooted hypermaps on orbifolds in terms of numbers of rooted hypermaps on surfaces. Let $\mathcal{H}$ be a rooted hypermap on an orbifold O such that $\mathcal{H} = \tilde{\mathcal{H}}/Z_\ell = (D; R, L)$ is a quotient of an ordinary finite map $\tilde{\mathcal{H}}$ on a surface S_g. Thus $O = S_g/G$, where $G \cong Z_\ell$ is a discrete cyclic group of orientation-preserving symmetries of S_g of order ℓ. It follows that each branch index of the branched covering $S_g \to O$ is a divisor of ℓ. We can write $O = O[\gamma; 2^{q_2}, \ldots, \ell^{q_\ell}]$, where $q_i \geq 0$ denotes the number of branch points of index i for $i = 2, \ldots, \ell$. The genera γ and g are related by the Riemann-Hurwitz equation

$2-2g=\ell(2-2\gamma-\sum_{j=2}^{\ell}q_j(1-1/j))$. We use the convention $h_\gamma(m)=\nu_{[\gamma;\emptyset]}(m)$ denoting the number of rooted hypermaps with m darts on a surface of genus g. Clearly, the exponential notation $O=O[\gamma;2^{q_2},\dots,\ell^{q_\ell}]$ can be used for any oriented orbifold (not necessarily cyclic) provided the indexes of branch points are bounded by ℓ.

Given integers $x_1,x_2,\dots,x_q$ and $y\geq x_1+x_2+\dots+x_q$ we denote by

$$\binom{y}{x_1,x_2,\dots,x_q}=\frac{y!}{x_1!x_2!\dots x_q!(y-\sum_{j=1}^{q}x_j)!},$$

the multinomial coefficient.

Proposition 3.4 *The number of rooted hypermaps on an orbifold $O=O[\gamma;2^{q_2},\dots,\ell^{q_\ell}]$ with m darts is*

$$h_O(m)=\binom{m+2-2\gamma}{q_2,q_3,\dots,q_\ell}h_\gamma(m). \tag{3.1}$$

Proof. Let $\mathcal{H}$ be a rooted hypermap on S_γ with v hypervetices, e hyperedges and f hyperfaces. Then $\mathcal{H}$ gives rise to as many rooted hypermaps as is the number of partitions of the set $V(\mathcal{H})\cup E(\mathcal{H})\cup F(\mathcal{H})$ of cardinality $v+e+f=m+2-2\gamma$ into disjoint subsets of cardinalities $q_1,q_2,\dots,q_\ell$. This is exactly the number

$$\binom{m+2-2\gamma}{q_2,q_3,\dots,q_\ell}.$$

□

Combining Proposition 3.4 and Theorem 3.3 we get our main theorem.

Theorem 3.5 *The number of unrooted hypermaps on a closed surface S_g of genus g with n darts is given by*

$$H_g(n)=\frac{1}{n}\sum_{\substack{\ell|n\\ \ell m=n}}\sum_{\substack{O\in Orb(S/Z_\ell)\\ O=O[\gamma;2^{q_2},3^{q_3},\dots,\ell^{q_\ell}]}}Epi_0(\pi_1(O),Z_\ell)\binom{m+2-2\gamma}{q_2,q_3,\dots,q_\ell}h_\gamma(m),$$

where the second sum runs through all cyclic orbifolds S_g/Z_ℓ.

Note that the numbers $Epi_0(\pi_1(O),Z_\ell)$ were computed by the authors in [25] in terms of some standard arithmetical functions. The following section surveys results on $Epi_0(\pi_1(O),Z_\ell)$.

4 Number of epimorphisms from an F-group onto a cyclic group

As one can see from Theorems 3.3 and 3.5, to derive an explicit formula for the number of unrooted hypermaps with given genus and given number of darts one needs to deal with the numbers $Epi_0(\pi_1(O),\mathsf{Z}_\ell)$ of order-preserving epimorphisms from an F-group Γ onto a cyclic group Z_ℓ. These numbers are counted using some number-theoretical machinery in [25]. In what follows we recall some relevant results used in later computations.

Denote by $\mu(n)$, $\phi(n)$ and $\Phi(x,n)$ the Möbius, Euler and von Sterneck functions, respectively. The relationship between them is given by the formula

$$\Phi(x,n)=\frac{\phi(n)}{\phi(\frac{n}{(x,n)})}\mu\left(\frac{n}{(x,n)}\right),$$

where (x, n) is the greatest common divisor of x and n. It was shown by O. Hölder that $\Phi(x, n)$ coincides with the Ramanujan sum $\sum_{\substack{1 \le k \le n \\ (k,n)=1}} \exp(\frac{2\,ikx}{n})$. For the proof, see Apolstol [1, p.164] and [26]. An arithmetic function, called by Liskovets the *orbicyclic arithmetic function* [21], is a multivariate integer function defined in [25] by

$$E(m_1, m_2, \ldots, m_r) = \frac{1}{m} \sum_{k=1}^{m} \Phi(k,\, m_1) \cdot \Phi(k,\, m_2) \ldots \Phi(k,\, m_r).$$

Recall that the Jordan multiplicative function $\phi_k(n)$ of order k can be defined as follows:

$$\phi_k(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d^k.$$

The following proposition is proved in [25].

Proposition 4.1 *Let $\Gamma = F[g; m_1, \ldots, m_r]$ be an $F-$group of signature $[g; m_1, \ldots, m_r]$. Denote by $m = lcm\,(m_1, \ldots, m_r)$ the least common multiple of $m_1, \ldots, m_r$ and let $m|\ell$. Then the number of order-preserving epimorphisms of the group Γ onto a cyclic group Z_ℓ is given by the formula*

$$Epi_0(\Gamma, Z_\ell) = m^{2g} \phi_{2g}(\ell/m) E(m_1, m_2, \ldots, m_r).$$

In particular, if $\Gamma = F[g; \emptyset] = F[g; 1]$ is a surface group of genus g we have

$$Epi_0(\Gamma, Z_\ell) = \phi_{2g}(\ell).$$

We note that the condition $m|\ell$ in the above proposition gives no essential restriction, since $Epi_0(\Gamma, \mathsf{Z}_\ell) = 0$ by the definition, provided that m does not divide ℓ. An orbifold $O = O[g; m_1, \ldots, m_r]$ will be called γ-*admissible* if it can be represented in the form $O = S_\gamma/\mathsf{Z}_\ell$, where S_γ is an orientable surface of genus γ surface and Z_ℓ is a cyclic group of automorphisms of S_γ. There is an orbifold $O = S_\gamma/\mathsf{Z}_\ell$ with signature $[g; m_1, m_2, \ldots, m_r]$ if and only if there exists ℓ such that the number $Epi_0(\pi_1(O), Z_\ell) \neq 0$ and the numbers γ, g, $m_1, \ldots, m_r$ and ℓ are related by the Riemann-Hurwitz equation $2 - 2\gamma = \ell(2 - 2g - \sum_{i=1}^{r}(1 - 1/m_i))$. Although the condition $Epi_0(\pi_1(O), Z_\ell) \neq 0$ can be checked using Proposition 4.1, for practical use it is more convenient to employ the following result by Harvey [8]. The Wiman theorem ensures us that $1 \le \ell \le 4\gamma + 2$ for $\gamma > 1$.

Theorem 4.2 [8] *Let $O = O[g; m_1, \ldots, m_r]$ be an orbifold. Then O is γ-admissible if and only if there exists an integer ℓ such that following conditions are satisfied:*

(1) $m = lcm(m_1, m_2, \ldots, m_r)$ *divides* ℓ *and* $m = \ell$ *if* $g = 0$,

(2) $2 - 2\gamma = \ell(2 - 2g - \sum_{i=1}^{r}(1 - 1/m_i))$ *(Riemann-Hurwitz equation),*

(3) $lcm(m_1, \ldots, m_{i-1}, m_i, m_{i+1}, \ldots, m_r) = lcm(m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_r)$ *for each* $i = 1, 2, \ldots, r$,

(4) *if* $m = lcm(m_1, m_2, \ldots, m_r)$ *is even, then the number of* m_j *divisible by the maximal power of* 2 *dividing* m *is even,*

(5) *if* $\gamma \ge 2$, *then* $r \neq 1$ *and* $r \ge 3$ *for* $g = 0$; *if* $\gamma = 1$, *then* $r \in \{0, 3, 4\}$; *if* $\gamma = 0$, *then* $r = 2$ *or* $r = 0$.

If $\gamma > 1$, then the integer ℓ is bounded by $1 \leq \ell \leq 4\gamma + 2$.

Using Theorem 4.2 we derive the following lists of γ-admissible orbifolds, for $\gamma = 0, 1$. Employing Proposition 4.1 the numbers $Epi_O(\pi_1(O), Z_\ell)$ are calculated for each orbifold in the list below.

Corollary 4.3 *0-admissible orbifolds are $O = O[0; \ell^2]$, with $Epi_0(\pi_1(O), Z_\ell) = \phi(\ell)$ for any positive integer ℓ.*

Corollary 4.4 *Let $O = O[g; m_1, m_2, \ldots, m_r] = S_1/Z_\ell$ be a 1-admissible orbifold. Then one of the following cases happens:*

$O = O[1; \emptyset]$, with $Epi_0(\pi_1(O), Z_\ell) = \sum_{k|\ell} \mu(\ell/k)k^2 = \phi_2(\ell)$ for any ℓ,

$\ell = 2$ and $O = O[0; 2^4]$, with $Epi_0(\pi_1(O), Z_\ell) = 1$,

$\ell = 3$ and $O = O[0; 3^3]$, with $Epi_0(\pi_1(O), Z_\ell) = 2$,

$\ell = 4$ and $O = O[0; 4^2, 2]$, with $Epi_0(\pi_1(O), Z_\ell) = 2$,

$\ell = 6$ and $O = O[0; 6, 3, 2]$, with $Epi_0(\pi_1(O), Z_\ell) = 2$.

Explicit lists of cyclic orbifolds on surfaces of genus 2 and 3 can be found in [25].

5 Counting unrooted spherical hypermaps

In this section we apply the above results to calculate the number of unrooted hypermaps with given number of darts on the sphere.

Theorem 5.1 *The number of spherical unrooted hypermaps with n darts is given by the formula*

$$H_0(n) = \tfrac{1}{n}\Big(\tfrac{3 \cdot 2^{n-1}}{(n+1)(n+2)}\tbinom{2n}{n} + \sum_{\substack{\ell|n,\, \ell>1 \\ \ell m=n}} 3 \cdot 2^{m-2}\tbinom{2m}{m}\phi(\ell)\Big).$$

Proof. For $\ell > 1$ there is only one possible action of cyclic group Z_ℓ on the sphere S. The corresponding orbifold O has signature $[0; \ell, \ell]$ and by Corollary 4.3 we have $Epi_0(\pi_1(O), Z_\ell) = \phi(\ell)$. By Theorem 3.5 we obtain

$$H_0(n) = \tfrac{1}{n}\Big(h_0(n) + \sum_{\substack{\ell|n,\, \ell>1 \\ \ell m=n}} \phi(\ell)\tbinom{m+2}{2}h_0(m)\Big). \tag{5.1}$$

By T. Walsh [32] we have

$$h_0(m) = \frac{3 \cdot 2^{m-1}}{(m+1)(m+2)}\binom{2m}{m}. \tag{5.2}$$

Inserting (5.2) into (5.1) we get the statement. □

The numbers of rooted and unrooted spherical hypermaps with up to 30 darts is given in Table 1.

Table 1. Numbers of rooted and unrooted hypermaps on the sphere with at most 30 darts

No. of darts:	No. rooted hypermaps:	No. unrooted hypermaps:
01	1	1
02	3	3
03	12	6
04	56	20
05	288	60
06	1584	291
07	9152	1310
08	54912	6975
09	339456	37746
10	2149888	215602
11	13891584	1262874
12	91287552	7611156
13	608583680	46814132
14	4107939840	293447817
15	28030648320	1868710728
16	193100021760	12068905911
17	1341536993280	78913940784
18	9390758952960	521709872895
19	66182491668480	3483289035186
20	469294031831040	23464708686960
21	3346270487838720	159346213738020
22	23981605162844160	1090073011199451
23	172667557172477952	7507285094455566
24	1248519259554840576	52021636161126702
25	9063324995286990848	362532999811480604
26	66032796394233790464	2539722940697502966
27	482722511571640123392	17878611539691757938
28	3539965084858694238208	126427324476844560112
29	26035872237025235042304	897788697828456380772
30	192014557748061108436992	6400485258395785352796

We note that the numbers $H_0(n)$ were determined in terms of unrooted planar 2-constellations formed by n polygons by M. Bosquet-Melon and G. Schaeffer [3].

Enumeration of self-dual spherical hypermaps. It is well-known that a spherical map is Eulerian if and only if its dual map is bipartite. It follows that there is a 1-1 correspondence between isomorphism classes of oriented bipartite spherical maps and oriented Eulerian spherical maps. Following [20] we denote by $E_O^+(n)$ the number of isomorphism classes of oriented Eulerian spherical maps with n edges. These numbers were determined in [19]. A formula for the difference $2E_0^+(n) - C_2^+(n)$, where C_2^+ is the number of 2-constellations (see [20, page 7] for the definition), was investigated in [20]. Observing that $C_2^+(n) = H_0(n)$, the above formula can be interpreted as number of vertex-edge self-dual oriented spherical hypermaps. Note that the self-duality mappings extend by definition to orientation-preserving self-homeomorphisms of the underlying surface. By [20, page 7] the numbers of self-dual spherical hypermaps are 1, 1, 2, 4, 8, 17, 40, 93, 224, 538, 1344, 3352, 8448, 21573 for $n = 1, 2, \ldots, 14$.

6 Counting unrooted toroidal hypermaps

In this section we derive an explicit formula for counting unrooted maps on torus. The list of 1-admissible orbifolds and the respective numbers $Epi_o(\pi_1(O), C)$ were derived in Corollary 4.4. Rooted toroidal maps were enumerated by D. Arquès in [2]. He proved that

$$h_1(n) = \frac{1}{3}\sum_{k=0}^{n-3} 2^k(4^{n-2-k} - 1)\binom{n+k}{k}. \tag{6.1}$$

The main result of this section follows.

Theorem 6.1 *The number $H_1(n)$ of oriented unrooted toroidal hypermaps with n darts is*

$$\frac{1}{n}\left(\binom{\frac{n}{2}+2}{4}h_0\left(\frac{n}{2}\right) + 2\binom{\frac{n}{3}+2}{3}h_0\left(\frac{n}{3}\right) + 6\binom{\frac{n}{4}+2}{3}h_0\left(\frac{n}{4}\right) + 12\binom{\frac{n}{6}+2}{3}h_0\left(\frac{n}{6}\right) + \sum_{\substack{\ell|n,\\ \ell m=n}} \phi_2(\ell)h_1(m)\right),$$

where ϕ_2 is Jordan multiplicative function of second order and h_0, h_1 are determined by (5.2), (6.1), *respectively.*

Proof. Following Theorem 3.5 and Corollary 4.4 we have

$$H_1(n) = \frac{1}{n}\big(h_{[0;2^4]}(n/2) + 2h_{[0;3^3]}(n/3) + 2h_{[0;2,4^2]}(n/4) + 2h_{[0;2,3,6]}(n/6) + \sum_{\substack{\ell|n\\ n=\ell m}}\sum_{k|\ell}\mu(\ell/k)k^2h_1(n/\ell)\big). \tag{6.2}$$

It remains to calculate the numbers of rooted hypermaps on orbifolds $O[0;2^4]$, $O[0;3^3]$, $0[2;4^2]$ and $O[0;2,3,6]$.

By Proposition 3.4 we have

$$h_{[0;2^4]}(m) = \tbinom{m}{4}h_0(m),\ \ h_{[0;3^3]}(m) = \tbinom{m+2}{3}h_0(m),$$
$$h_{[0;2,3,6]}(m) = \tbinom{m+2}{1,1,1}h_0(m) = 6\tbinom{m+2}{3}h_0(m),$$
$$h_{[0;2,4^2]}(m) = \tbinom{m+2}{1,2}h_0(m) = 3\tbinom{m+2}{3}h_0(m).$$

Inserting the above numbers into (6.2) we get the statement.

The following list containing the numbers of rooted and oriented unrooted maps of genus 1 up to 30 edges follows.

Table 2. Numbers of rooted and unrooted hypermaps on the torus with at most 30 darts

No. of darts:	No. rooted hypermaps:	No. unrooted hypermaps:
03	1	1
04	15	6
05	165	33
06	1611	285
07	14805	2115
08	131307	16533
09	1138261	126501
10	9713835	972441
11	81968469	7451679
12	685888171	57167260
13	5702382933	438644841
14	47168678571	3369276867
15	388580070741	25905339483
16	3190523226795	199408447446
17	26124382262613	1536728368389
18	213415462218411	11856420991413
19	1740019150443861	91579955286519
20	14162920013474475	708146055343668
21	115112250539595093	5481535740059577
22	934419385591442091	42473608898628639
23	7576722323539318101	329422709719100787
24	61375749135369153195	2557322884534185500
25	496747833856061953365	19869913354242478293
26	4017349254284543961771	154513432889706455145
27	32467023775647069984085	1202482362061007078175
28	262225359776626483309227	9365191420865873023026
29	2116714406654571321840981	72990151953605907649689
30	17077642118698511054318251	569254737292213025378571

The above tables were computed using MATHEMATICA, Ver. 5. The input numbers of rooted maps come from [2].

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