

# Antibandwidth and Cyclic Antibandwidth of Meshes and Hypercubes

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## Abstract

The antibandwidth problem consists of placing the vertices of a graph on a line in consecutive integer points in such a way that the minimum difference of adjacent vertices is maximized. The problem was originally introduced in [15] in connection with multiprocessor scheduling problems and can be also understood as a dual problem to the well known bandwidth problem, as a special radiocolouring problem or as a variant of obnoxious facility location problems. The antibandwidth problem is NP-hard, there are a few classes of graphs with polynomial time complexities. Exact results for nontrivial graphs are very rare. Miller and Pritikin [18] showed tight bounds for 2-dimensional meshes and hypercubes. We solve the antibandwidth problem precisely for two dimensional meshes, tori and estimate the antibandwidth value for hypercubes up to the third order term. The cyclic antibandwidth problem is to embed an  $n$ -vertex graph into the cycle  $C_n$ , such that the minimum distance (measured in the cycle) of adjacent vertices is maximised. This is a natural extension

of the antibandwidth problem or a dual problem to the cyclic bandwidth problem. We start investigating this invariant for typical graphs and prove basic facts and exact results for the same product graphs as for the antibandwidth.

*Key words:* antibandwidth, cyclic antibandwidth, mesh, toroidal mesh, hypercube

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## 1 Introduction

The antibandwidth problem consists of placing the vertices of a graph on a line in integer points in such a way that the minimum difference of adjacent vertices is maximised. The problem was originally introduced in [15] in connection with multiprocessor scheduling problems. It can be understood as a dual problem to the well known bandwidth problem [5] in which the maximum distance of adjacent vertices in the linear layout is minimised. Another motivation comes from the area of radio frequencies assignment problem [10]. Transmitters are assigned  $n$  different frequencies such that the physically neighbouring transmitters have as different frequencies as possible. The transmitters and their neighbourhood are given by an  $n$ -vertex graph. The problem also belongs to the family of obnoxious facility location problems: The "enemy graph" is represented by  $n$  persons. Two of them are joined by an edge iff they are enemies. The problem is to build a house for every person along a road such that the minimal distance between enemies is maximised [3]. From the graph theory point of view our problem is just a new labelling problem, see a survey [7]. The problem was originally studied under the term *separation number* [15]. However, in the meantime, this name was also used for another linear layout problem [5]. Lin and Yuan called it *dual bandwidth*. In this paper we propose and use a new term for the problem, (the most appropriate according to our opinion) *antibandwidth*.

The antibandwidth problem is NP-complete [15]. So far it is polynomially solvable for 3 classes of graphs: the complements of interval, arborescent comparability and threshold graphs [4,12]. Known results include simple relations of the antibandwidth invariant to the minimum, maximum degree, chromatic index and powers of hamiltonian paths in the complement graph [15,17,18]. Exact results are known for paths, cycles, special trees, complete and complete

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bipartite graphs [17,18,21]. The class of  $n$ -vertex forests with  $\text{ab}(F) = \lfloor n/2 \rfloor$  is characterized in [18], which covers, e.g., complete binary trees. The problem is also interesting for disconnected graphs. Exact values were proved for graphs consisting of copies of simple graphs [8,22].

Miller and Pritikin [18] proved the following tight bounds for  $m \times n$  mesh graphs,  $m \geq n$ , and  $n$ -dimensional hypercubes:

$$\left\lceil \frac{n(m-1)}{2} \right\rceil \leq \text{ab}(P_m \times P_n) \leq \left\lfloor \frac{mn}{2} \right\rfloor,$$

$$2^{n-1} - \frac{2^n}{\sqrt{2\pi n}} (1 + o(1)) \leq \text{ab}(Q_n) \leq 2^{n-1}.$$

The lower bounds are obtained by their general labelling scheme applicable to bipartite graphs. The upper bounds come from a simple observation that for  $n$ -vertex connected graphs  $\text{ab}(G) \leq n/2$ .

In our contribution we develop an upper bound method suitable for bipartite graphs and improve the above bounds:

$$\begin{aligned} \text{ab}(P_m \times P_n) &= \left\lceil \frac{n(m-1)}{2} \right\rceil, \\ \text{ab}(Q_n) &= 2^{n-1} - \frac{2^n}{\sqrt{2\pi n}} (1 + o(1)). \end{aligned}$$

In addition, for toroidal meshes  $C_n \times C_n$ , we show

$$\text{ad}(C_n \times C_n) = \begin{cases} \frac{(n-2)n}{2}, & \text{if } n \text{ is even,} \\ \frac{(n-2)(n+1)}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

The cyclic antibandwidth problem is to embed an  $n$ -vertex graph into the cycle  $C_n$ , such that the minimum distance (measured in the cycle) of adjacent vertices is maximised. This is a natural extension of the antibandwidth problem or a dual problem to the cyclic bandwidth problem. The problem was first introduced by Leung et al. [15], in connection with some scheduling problems. Since then the problem has remained almost unexplored. The cyclic antibandwidth problem is a new graph labelling problem [7] or a dual problem to the cyclic bandwidth problem [11,16], where the maximum distance of adjacent vertices is minimised. The decision problem, "Is  $\text{cab}(G) \geq 2$ ?", is simply equivalent to the existence of a Hamiltonian cycle in the complement of  $G$ . This implies the NP-completeness of the problem and belongs to textbooks on complexity as a King Arthur's round table problem: "Is it possible to place knights around a table such that no two enemies are neighbouring?" We show basic facts about the problem and prove exact results for meshes,

tori and asymptotics for hypercubes. It turns out that for the above classes of graphs, both parameters are almost the same, but there are graphs for which they differ essentially.

## 2 Preliminaries

For a nonempty graph  $G = (V, E)$ , let  $f$  be a one-to-one labelling

$$f : V \rightarrow \{0, 1, 2, 3, \dots, |V| - 1\}$$

Define the antibandwidth of  $G$  according to  $f$  as

$$\text{ab}(G, f) = \min_{uv \in E} |f(u) - f(v)|.$$

The antibandwidth of  $G$  is defined as

$$\text{ab}(G) = \max_f \text{ab}(G, f).$$

It is useful to imagine the antibandwidth problem as a linear layout problem. The vertices are mapped into integer points  $\{0, 1, \dots, |V| - 1\}$  on a line such that the minimal distance of adjacent vertices (in the graph) is maximised.

Define the cyclic antibandwidth of a connected graph  $G$  according to  $f$  as

$$\text{cab}(G) = \min_{uv \in E} \{|f(u) - f(v)|, |V| - |f(u) - f(v)|\}.$$

The antibandwidth of  $G$  is defined as

$$\text{cab}(G) = \max_f \text{cab}(G, f).$$

It is useful to imagine the cyclic antibandwidth problem as a cycle embedding problem. The vertices are mapped bijectively into  $C_{|V|}$  such that the minimal distance, measured in the cycle, of adjacent vertices is maximised.

Let  $G$  be a graph. Let  $\partial(A)$  denote the vertex boundary of a set  $A \subseteq V$ , i.e., the set of all vertices from  $V - A$  having a neighbour in  $A$ .

Let  $G = (V_1, V_2, E)$  be a bipartite graph. Let  $\partial_b(A)$  denote the vertex boundary of a set  $A \subseteq V_1$ , i.e. the set of all vertices from  $V_2$  having a neighbour in  $A$ . We call it the bipartite vertex boundary.

Let  $P_n$  and  $C_n$  denote the  $n$ -vertex path and the  $n$ -vertex cycle. Let  $V(P_n) = \{0, 1, 2, \dots, n - 1\}$  and  $E(P_n) = \{(i, i + 1) : i = 0, 1, 2, \dots, n - 1\}$ . Similarly, let  $V(C_n) = \{0, 1, 2, \dots, n - 1\}$  and  $E(C_n) = \{(i, (i + 1) \bmod n) : i = 0, 1, 2, \dots, n - 1\}$ .

### 3 Antibandwidth

#### 3.1 Antibandwidth of Meshes

The 2-dimensional mesh is defined by means of Cartesian product of two paths as  $P_m \times P_n$ . The mesh is a bipartite graph with  $|V_1| = \lceil mn/2 \rceil$ ,  $|V_2| = \lfloor mn/2 \rfloor$ .

**Theorem 3.1** For any  $m \geq n \geq 2$

$$\text{ab}(P_m \times P_n) = \left\lceil \frac{(m-1)n}{2} \right\rceil.$$

**Proof.** The lower bound is showed in [18]. An example of the numbering of  $P_8 \times P_7$  is in Table 1.

38	13	45	20	51	25	55	28
7	39	14	46	21	52	26	56
33	8	40	15	47	22	53	27
3	34	9	41	16	48	23	54
30	4	35	10	42	17	49	24
1	31	5	36	11	43	18	50
29	2	32	6	37	12	44	19

Table 1  
Optimal numbering of the vertices of  $P_8 \times P_7$  mesh.

Upper Bound. Define an interval  $I = [L, R]$ , where

$$L = \left( \left\lfloor \frac{n}{2} \right\rfloor - 1 \right)^2 + 1 \text{ and } R = \left\lceil \frac{mn}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor - 1.$$

A recent result of Bezrukov and Piotrowski [2] shows that  $p$ -vertex subsets of the mesh of minimal bipartite vertex boundary are obtained as the first  $p$  vertices in the order depicted in Table 2 for  $m = 11$  and  $n = 7$ .

This especially implies the following. Let  $A$  be a subset of one partition set of the mesh. If  $|A| \in I$  then

$$|\partial_b(A)| \geq |A| + \left\lfloor \frac{n}{2} \right\rfloor. \quad (1)$$

The interval  $I$  is shown in bold.

10		17		24		31		36		39
	11		18		25		32		37	
5		12		19		26		33		38
	6		13		20		27		34	
2		7		14		21		28		35
	3		8		15		22		29	
1		4		9		16		23		30

Table 2

Optimal numbering of the vertices of  $P_{11} \times P_7$  mesh with respect to the bipartite vertex boundary.

Denote  $k = ab(P_m \times P_n)$ . We know that  $\lceil (m-1)n/2 \rceil \leq k \leq mn/2$ . Consider a linear layout of the mesh. For a set  $S$  of consecutive  $k$  vertices of the layout denote  $A_1 = S \cap V_1$ , and  $A_2 = S \cap V_2$ .

- (1) Assume that there exist  $S$  such that the corresponding  $A_1$  satisfies  $|A_1| \in I$ . Observe that

$$|\partial_b(A_1)| + k - |A_1| \leq |V_2|.$$

Thus by (1)

$$k \leq \left\lfloor \frac{mn}{2} \right\rfloor - (|\partial_b(A_1)| - |A_1|) \leq \left\lfloor \frac{mn}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{(m-1)n}{2} \right\rfloor.$$

- (2) Assume that for all sets  $S$ ,  $|A_1| \notin I$ . We show that this leads to a contradiction. Distinguish 3 cases.
- (a) There exist  $S, S'$  such that  $|A_1| < L$  and  $|A'_1| > R$ . Then one can easily compute that  $|A_1| < |A_2|$  and  $|A'_1| > |A'_2|$ . This forces the existence of  $S''$  such that  $||A''_1| - |A''_2|| \leq 1$  which gives

$$\frac{k-1}{2} \leq |A''_1| \leq \frac{k+1}{2}.$$

Comparing this interval with  $I$  we conclude that  $|A''_1| \in I$ , a contradiction.

- (b) Assume that for all  $S$ ,  $|A_1| < L$ . Then  $|A_2| = k - |A_1| > k - L$ . Take two disjoint  $S$  and  $S'$  with  $|A_2| > k - L$  and  $|A'_2| > k - L$ . We get  $|A_2| + |A'_2| > 2k - 2L \geq mn/2$ , a contradiction.
- (c) Assume that for all  $S$ ,  $|A_1| > R$ . This case is symmetric to the previous case.

□

### 3.2 Antibandwidth of Tori

Consider a 2-dimensional toroidal mesh  $C_n \times C_n$ , defined by means of the Cartesian product of two cycles.

**Theorem 3.2** For even  $n$

$$\text{ab}(C_n \times C_n) = \frac{n(n-2)}{2}.$$

**Proof.**

35	11	51	27	63	23	47	7
3	43	19	59	31	55	15	39
34	10	50	26	62	22	46	6
2	42	18	58	30	54	14	38
33	9	49	25	61	21	45	5
1	41	17	57	29	53	13	37
32	8	48	24	60	20	44	4
0	40	16	56	28	52	12	36

Table 3  
Optimal numbering of the vertices of  $C_8 \times C_8$  torus.

Lower bound. The optimal numbering is depicted in Table 3, for  $n = 8$ . The numbering in the first row from bottom is generally described by a vector

$$R_1 = (0, n + \frac{n^2}{2}, 2n, 3n + \frac{n^2}{2}, 4(n-1), \dots, (\frac{n}{2} - 1)n + \frac{n^2}{2}, (n-1)\frac{n}{2}, \\ (n-3)\frac{n}{2} + \frac{n^2}{2}, (n-5)\frac{n}{2}, (n-7)\frac{n}{2} + \frac{n^2}{2}, \dots, 3\frac{n}{2}, \frac{n}{2} + \frac{n^2}{2}).$$

Define

$$A_1 = (\frac{n^2}{2}, -\frac{n^2}{2}, \frac{n^2}{2}, -\frac{n^2}{2}, \dots, \frac{n^2}{2}, -\frac{n^2}{2}), \\ A_2 = (-\frac{n^2}{2} + 1, \frac{n^2}{2} + 1, -\frac{n^2}{2} + 1, \frac{n^2}{2} + 1, \dots, -\frac{n^2}{2} + 1, -\frac{n^2}{2} + 1).$$

The numberings in rows  $R_i, i > 1$ , are described recursively by

$$R_2 = R_1 + A_1, R_3 = R_2 + A_2, R_4 = R_3 + A_1, \dots$$

The numbering is bijective. In fact, in all rows, the even (odd) labels form increasing arithmetic progressions of length  $n/2$  and distance 1. The beginning elements of all progression are multiples of  $n/2$ , so the progressions are distinct.

The distances of neighbours in the first row, including the wrap-around edge, are at least  $\frac{n(n-2)}{2}$ . By definition, the same distances are preserved in all rows. The distances in columns, except for the wrap-around edges, are at least  $\frac{n^2}{2} - 1$ . The row  $R_n$  satisfies

$$R_n = R_1 + A,$$

where

$$A = \left(-1 + \frac{n}{2} + \frac{n^2}{2}, -1 + \frac{n}{2} - \frac{n^2}{2}, \dots, -1 + \frac{n}{2} + \frac{n^2}{2}, -1 + \frac{n}{2} - \frac{n^2}{2}\right).$$

The components of  $A$  taken  $(\text{mod } n^2)$  are the distances of labels of rows  $R_1$  and  $R_n$ .

**Remark.** Another optimal numbering is obtained from the numbering of the even torus in [13,19]. See for illustration Table 4. First we number the vertices of the first partition of the torus by restricting the original order to the first partition using numbers  $1, 2, \dots, n^2/2$  and then repeating this process for the second partition. One can formally prove that the minimal difference is  $n^2/2 - n$ .

99	46	91	34	75	30	88	44	98	50
45	90	33	74	16	70	29	87	43	97
89	32	73	15	59	12	69	28	86	42
31	72	14	58	4	56	11	68	27	85
71	13	57	3	51	2	55	10	67	26
21	63	7	53	1	52	5	60	17	76
81	22	64	8	54	6	61	18	77	35
39	82	23	65	9	62	19	78	36	92
95	40	83	24	66	20	79	37	93	47
49	96	41	84	25	80	38	94	48	100

Table 4  
Optimal numbering of the vertices of  $C_{10} \times C_{10}$  torus.

Upper bound. The even torus is a bipartite graph with partite sets  $V_1$  and  $V_2$ ,  $|V_1| = |V_2| = n^2/2$ . We use the same ideas as for the mesh. Denote by  $I = [L, R]$ , an interval with  $L = n^2/4 - n + 2$  and  $R = n^2/4 - 1$ . It is known that an  $m$ -vertex set with the minimal vertex boundary in any even torus is given by the first  $m$  vertices in the numbering given in [13,19]. Bezrukov and Piotrowski [2] proved that an  $m$ -vertex set in any even torus with the minimal bipartite vertex boundary is given by taking the first  $m$  vertices in the above numbering restricted to one partite set only. An example of such numbering is shown in Table 4, for  $n = 10$ . A simple analysis shows that if  $A$

is a subgraph of one of the partite sets and  $n^2/4 - n + 2 \leq |A| \leq n^2/4 - 1$ , then  $|\partial_b(A)| - |A| \geq n$ .

Consider a linear layout. Let  $k = \text{ad}(C_n \times C_n)$ . We know that  $n^2/2 - n \leq k \leq n^2/2 - 2$ . Assume indirectly that  $k \geq n^2/2 - n + 1$ . For any set  $S$  of  $k$  consecutive vertices on the line let  $A_i = V_i \cap S$  for  $i = 1, 2$ . Let  $S$  be a set of consecutive  $k$  vertices on the line.

- (1) Assume that there exist  $S$  such that the corresponding  $A_1$  satisfies  $|A_1| \in I$ . Observe that

$$|\partial_b(A_1)| + k - |A_1| \leq |V_2|.$$

Hence similarly as for the mesh

$$k \leq \frac{n^2}{2} - (|\partial_b(A_1)| - |A_1|) \leq \frac{n^2}{2} - n,$$

a contradiction.

- (2) Assume that for all  $S$ , it holds  $|A_1| \notin I$ . Distinguish 3 cases.  
(a) There exist  $S, S'$  such that  $|A_1| < L$  and  $|A'_1| > R$ . Then one can easily compute that  $|A_1| \leq |A_2|$  and  $|A'_1| \geq |A'_2|$ . This forces the existence of  $S''$  such that  $||A''_1| - |A''_2|| \leq 1$  which gives

$$\frac{k-1}{2} \leq |A''_1| \leq \frac{k+1}{2}.$$

Comparing this interval with  $I$  we conclude that  $|A''_1| \in I$ , a contradiction.

- (b) Assume that for all  $S$ ,  $|A_1| < L$ . Then  $|A_2| = k - |A_1| \geq n^2/4$ . Take two disjoint  $S$  and  $S'$  with  $|A_2| \geq n^2/4$  and  $|A'_2| \geq n^2/4$ . As  $|V_2| = n^2/2$ , we have  $|A_2| = |A'_2| = n^2/4$  and consequently if  $v \notin S \cup S'$  then  $v \in V_1$ . Consider now 2 special choices of the sets  $S$  and  $S'$ .
- $S \cup S'$  is the set of the first (last)  $2k$  vertices on the line.
  - $S$  (resp.  $S'$ ) is the set of the first (last)  $k$  vertices on the line.

This implies the following structure of vertices on the line: The vertices are divided into consecutive sets  $Z_1, Z_2, Z_3, Z_4, Z_5$  from the left, where  $Z_1, Z_3, Z_5 \subseteq V_1$ ,  $|Z_1| = |Z_3| = |Z_5| = n^2 - 2k$ , and  $|Z_2| = |Z_4| = 3k - n^2 < n^2/2$ . Consider a Hamiltonian cycle in the torus. Let  $u, v \in V_2$  be two vertices in the distance 2 on the cycle. These two vertices have a common neighbour  $w \in V_1$ . If  $u, v$  are from different sets, say  $u \in Z_2, v \in Z_4$ . Then for any position of  $w$  we get a distance smaller than  $k$ , a contradiction. This forces that if say,  $u \in Z_2$  and  $u \in V_2$  then its neighbour on the cycle in the distance 2 must lie in  $Z_2$  too. Continuing this process we conclude that  $V_2 \subseteq Z_2$ , a contradiction again.

- (c) Assume that for all  $S$ ,  $|A_1| > R$ . This case is symmetric to the previous case.

□

**Theorem 3.3** For odd  $n$

$$\text{ab}(C_n \times C_n) = \frac{(n-2)(n+1)}{2}.$$

**Proof.** Lower bound. Consider a labelling  $f : V(C_n \times C_n) \rightarrow \{0, 1, 2, \dots, n^2-1\}$  defined by

$$f(i, j) = \frac{(n^2 + n + 2)i + (n^2 - n)j}{2} \bmod n^2.$$

First we show that  $f$  is a one-to-one mapping. Assume indirectly that for some  $(i, j) \neq (i', j')$ :  $f(i, j) - f(i', j') = 0$ . This implies

$$\frac{(n^2 + n + 2)(i - i') + (n^2 - n)(j - j')}{2} \equiv 0 \bmod n^2 \equiv 0 \bmod n. \quad (2)$$

As  $(n^2+n+2)/2 \equiv 1 \bmod n$  and  $(n^2-n)/2 \equiv 0 \bmod n$  we have  $i-i' \equiv 0 \bmod n$ , hence  $i = i'$ . Substituting this into (2) we get

$$\frac{n(n-1)}{2}(j - j') \equiv 0 \bmod n^2.$$

Then

$$\begin{aligned} \frac{n-1}{2}(j - j') &\equiv 0 \bmod n, \\ (n-1)(j - j') &\equiv 0 \bmod 2n \equiv 0 \bmod n, \\ j' - j &\equiv 0 \bmod n, \end{aligned}$$

which implies  $j' = j$ , a contradiction.

Now we compute the absolute difference of two adjacent labels in the toroidal mesh. By the definition of  $f$

$$\begin{aligned} |f((i+1) \bmod n, j) - f(i, j)| &= \\ &= \frac{n^2 + n + 2}{2}((i+1) \bmod n - i) \bmod n^2. \end{aligned} \quad (3)$$

If  $(i+1) \bmod n - i = 1$  then the right hand side of (3) equals  $(n^2 + n + 2)/2$ . If  $(i+1) \bmod n - i = -1$  then the right hand side of (3) equals  $n^2 - (n^2 + n + 2)/2 = (n-2)(n+1)/2$ .

The second case, i.e., the labels are column-adjacent, is similar. Table 5 shows the optimal numbering of torus the  $C_7 \times C_7$ .

Upper bound. Consider any labelling of the vertices of the toroidal mesh by  $\{0, 1, \dots, n^2-1\}$ . We say that a label is small if its value is at most  $(n^2-1)/2$ ,

27	48	20	41	13	34	6
47	19	40	12	33	5	26
18	39	11	32	4	25	46
38	10	31	3	24	45	17
9	30	2	23	44	16	37
29	1	22	43	15	36	8
0	21	42	14	35	7	28

Table 5

Optimal numbering of the vertices of  $C_7 \times C_7$  torus.

otherwise it is large. Consider  $n$  row cycles of the mesh. In any row cycle there is at least one pair  $xy$  of adjacent labels such that  $x > y$  and either  $y > (n^2 - 1)/2$  or  $x \leq (n^2 - 1)/2$ , i.e. any row contains a pair of adjacent labels which are both either small or large. Distinguish 2 cases.

Assume that there are at least  $(n + 1)/2$  pairs of large row adjacent labels. Among them find a pair  $x_0y_0$  such that  $y_0$  is the largest number. Then

$$y_0 \geq \frac{n^2 - 1}{2} + \frac{n + 1}{2} \text{ and } x_0 - y_0 \leq n^2 - 1 - \frac{n^2 + n}{2} = \frac{(n - 2)(n + 1)}{2}.$$

This gives

$$\min_{uv \in E} |f(u) - f(v)| \leq \frac{(n - 2)(n + 1)}{2},$$

for any  $f$ , which implies the result.

Assume that there are at least  $(n + 1)/2$  pairs of small row adjacent labels. Repeating the above argument we get a pair  $x_0y_0$  such that

$$x_0 - y_0 \leq \frac{n^2 - n}{2},$$

with equality for  $x_0 = (n^2 - 1)/2$  and  $y_0 = (n - 1)/2$ . If either  $x_0 \neq (n^2 - 1)/2$  or  $y_0 \neq (n - 1)/2$  then the claim follows. So assume that  $x_0 = (n^2 - 1)/2$  and  $y_0 = (n - 1)/2$ . Now we repeat the previous arguments for column cycles and in the worst case we find a pair  $x'_0y'_0$  with  $x'_0 = (n^2 - 1)/2$  and  $y'_0 = (n - 1)/2$ , a contradiction.  $\square$

### 3.3 Antibandwidth of Hypercubes

The vertices of the hypercube are represented by binary strings of length  $n$  and two vertices are adjacent iff the strings differ in one position. The vertices of  $Q_n$  can be naturally partitioned into sets  $X_i, i = 0, 1, 2, \dots, n$  according to

their distance from the vertex  $0\dots 00$ . Note that there are edges between  $X_i$  and  $X_{i+1}$ ,  $i = 0, 1, 2, \dots, n - 1$  only and  $|X_i| = \binom{n}{i}$ .

Before proving the upper bound we recall some knowledge about vertex isoperimetric properties of hypercubes. It is known [10] that the set of cardinality  $m$  of the minimal vertex boundary in the hypercube  $Q_n$  is obtained by taking the first  $m$  vertices in the following order:  $X_0, X_1, \dots, X_n$ , while the order in the set  $X_i$  is lexicographic. Moreover  $m$  can be uniquely expressed in the form

$$m = \binom{n}{n} + \binom{n}{n-1} + \dots + \binom{n}{j+1} + \binom{x}{j},$$

for some real  $x$ . Then for any set  $A$ ,  $|A| = m$ , following Frankl [6]

$$|\partial(A)| \geq \binom{n}{j} + \binom{x}{j-1}.$$

It is known [1,14,20] that the set of cardinality  $m$  of minimal vertex boundary of a set  $A \subseteq V_1$  in the hypercube  $Q_n$  is obtained by taking the first  $m$  vertices in the following order:  $X_0, X_2, X_4, \dots$ , while the order in the set  $X_{2i}$  is lexicographic. Moreover  $m$  can be uniquely expressed in the form

$$m = \binom{n}{n} + \binom{n}{n-2} + \dots + \binom{n}{j+2} + \binom{x}{j},$$

for some real  $x$  and  $j$  of the same parity as  $n$ . One can easily prove an analogue of the Frankl's result for the bipartite vertex boundaries in hypercubes.

**Lemma 3.1** *For any set  $A \subseteq V_1$  of the hypercube  $Q_n$ , with  $|A| = m$ ,*

$$|\partial_b(A)| - |A| \geq \binom{n-1}{j} + \binom{x}{j-1} - \binom{x}{j}. \quad (4)$$

Now we are prepared to prove the result.

**Theorem 3.4** *For the  $n$ -dimensional hypercube  $Q_n$*

$$\text{ab}(Q_n) = 2^{n-1} - \frac{2^n}{\sqrt{2\pi n}} (1 + o(1)).$$

**Proof.** The lower bound was constructed in [18]. For the upper bound we

prove that

$$\text{ab}(Q_n) \leq \begin{cases} 2^{n-1} - \binom{n-1}{\frac{n}{2}}, & \text{for } n \equiv 0 \pmod{4}, \\ 2^{n-1} - \frac{1}{2} \binom{n}{\frac{n-1}{2}}, & \text{for } n \equiv 1 \pmod{4}, \\ 2^{n-1} - \binom{n-1}{\frac{n}{2}-2}, & \text{for } n \equiv 2 \pmod{4}, \\ 2^{n-1} - \binom{n-1}{\frac{n-1}{2}} & \text{for } n \equiv 3 \pmod{4}. \end{cases}$$

Let  $n$  be divisible by 4. Other cases differ in details only. We use a similar method as for meshes and even tori. Denote  $k = \text{ad}(Q_n)$ . Let  $k \leq 2^{n-1} - \binom{n-1}{\frac{n}{2}}$ . Then we are done and note that

$$\binom{n-1}{\frac{n}{2}} = \frac{2^n}{\sqrt{2\pi n}}(1 + o(1)).$$

Assume indirectly that  $k \geq 2^{n-1} - \binom{n-1}{\frac{n}{2}} + 1$ . Note that  $k \leq 2^{n-1} - 1$ . Let  $I = [L, R]$ , where

$$L = 2^{n-2} - \binom{n-1}{\frac{n}{2}} + 1 \text{ and } R = 2^{n-2} - 1.$$

Consider a linear layout. For a set  $S$  of consecutive  $k$  vertices of the linear layout denote  $A_1 = S \cap V_1$ , and  $A_2 = S \cap V_2$ .

- (1) Assume that there exist  $S$  such that the corresponding  $A_1$  satisfies  $|A_1| \in I$ . Then

$$\begin{aligned} |A_1| &= 2^{n-2} - \binom{n-1}{\frac{n}{2}} + 1 \\ &= \binom{n}{n} + \binom{n}{n-2} + \dots + \binom{n}{\frac{n}{2}+2} + \frac{1}{2} \binom{n}{\frac{n}{2}} - \binom{n-1}{\frac{n}{2}} + 1 \\ &= \binom{n}{n} + \binom{n}{n-2} + \dots + \binom{n}{\frac{n}{2}+2} + 1, \end{aligned}$$

and consequently  $|A_1|$  can be expressed as

$$|A_1| = \binom{n}{n} + \binom{n}{n-2} + \dots + \binom{n}{\frac{n}{2}+2} + \binom{x}{\frac{n}{2}},$$

for some real  $n/2 \leq x \leq n-1$ . The last inequality comes from the fact that  $|A_1| \leq \frac{k}{2} < 2^{n-2}$ . From Lemma 3.1 we have

$$|\partial_b(A_1)| - |A_1| \geq \binom{n-1}{\frac{n}{2}} + \binom{x}{\frac{n}{2}-1} - \binom{x}{\frac{n}{2}} \geq \binom{n-1}{\frac{n}{2}}.$$

Observe that  $|\partial_b(A_1)| + k - |A_1| \leq |V_2|$ , which implies

$$k \leq 2^{n-1} - (|\partial_b(A_1)| - |A_1|) \leq 2^{n-1} - \binom{n-1}{\frac{n}{2}},$$

a contradiction.

- (2) Assume that for all  $S$ , it holds  $|A_1| \notin I$ . Distinguish 3 cases.
- (a) There exist  $S, S'$  such that  $|A_1| < L$  and  $|A'_1| > R$ . Then one can easily compute that  $|A_1| \leq |A_2|$  and  $|A'_1| \geq |A'_2|$ . This forces the existence of  $S''$  such that  $||A''_1| - |A''_2|| \leq 1$  which gives

$$\frac{k-1}{2} \leq |A''_1| \leq \frac{k+1}{2}.$$

Comparing this interval with  $I$  we conclude that  $|A''_1| \in I$ , a contradiction.

- (b) Assume that for all  $S$ ,  $|A_1| < L$ . Then  $|A_2| = k - |A_1| \geq 2^{n-2} + 1$ . Take two such disjoint  $S$  and  $S'$  we get  $|A_2| + |A'_2| > 2^{n-1}$ , a contradiction.
- (c) Assume that for all  $S$ ,  $|A_1| > R$ . This case is symmetric to the previous one.

□

## 4 Cyclic Antibandwidth

### 4.1 Basic Observations

**Lemma 4.1** *Let  $G = (V, E)$ ,  $|V| = n$ , Then*

$$\text{cab}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Equality holds, e.g., for  $G$  consisting of a matching.

**Lemma 4.2** *For  $n$ -vertex paths and cycles,*

$$\text{cab}(P_n) = \text{cab}(C_n) = \left\lceil \frac{n}{2} \right\rceil - 1.$$

Let  $\overline{G}$  denote the complement of  $G$ . Let  $G^i$ , for  $i \geq 1$ , denote the graph obtained from  $G$  by joining all vertices in a distance of at most  $i$ .

**Lemma 4.3** For  $n$ -vertex graph  $G$  and  $k \geq 2$

$$\text{ab}(G) \geq k \quad \text{iff } P_n^{k-1} \subseteq \overline{G},$$

$$\text{ab}(G) = 1 \quad \text{iff } \overline{G} \text{ does not contain a Hamiltonian path,}$$

$$\text{cab}(G) \geq k \quad \text{iff } C_n^{k-1} \subseteq \overline{G},$$

$$\text{cab}(G) = 1 \quad \text{iff } \overline{G} \text{ does not contain a Hamiltonian cycle,}$$

**Lemma 4.4**

$$\frac{1}{2}\text{ab}(G) \leq \text{cab}(G) \leq \text{ab}(G),$$

and both bounds are attainable.

**Proof.** The right inequality is evident and the equality can be obtained by a path. Let  $\text{ab}(G) = k \geq 2$ . This implies that  $\overline{G}$  contains  $P_n^{k-1}$  and  $P_n^{2\lfloor \frac{k-1}{2} \rfloor}$ . As  $C_n \subseteq P_n^2$ , we have  $C_n^{\lfloor \frac{k-1}{2} \rfloor} \subseteq P_n^{2\lfloor \frac{k-1}{2} \rfloor}$ . By Lemma 4.3 it holds  $C_n^{\lfloor \frac{k-1}{2} \rfloor} \subseteq \overline{G}$  and hence

$$\text{cab}(G) \geq \lfloor \frac{k-1}{2} \rfloor + 1 = \lfloor \frac{\text{ab}(G) + 1}{2} \rfloor.$$

The left inequality is sharp what can be proved in the following way. Let  $G$  be the complement of  $P_n^i$ , for  $i \geq 2$ . We claim that

$$\text{ab}(G) = i + 1 \quad \text{and} \quad \text{cab}(G) = \lfloor \frac{i}{2} \rfloor + 1.$$

The first equality is trivial. For the second one, clearly,  $P_n^i$  contains  $C_n^{\lfloor \frac{i}{2} \rfloor}$  which implies  $\text{cab}(G) \geq \lfloor \frac{i}{2} \rfloor + 1$ . The last inequality is equality from the following reason. The degree of the first vertex in  $P_n^i$  is  $2i$ . Hence the maximum  $C_n^j$  contained in  $P_n^i$  satisfies  $2j \leq i$ , i.e.,  $j = \lfloor \frac{i}{2} \rfloor$ .  $\square$

The next Lemma is obvious but usefull. All results for meshes, tori and hypercubes are obtained by the Lemma.

**Lemma 4.5** Let  $G = (E, V)$ ,  $|V| = n$ . Let  $f$  be a labeling of  $G$ . then

$$\text{cab}(G) \geq \min\{\text{ab}(G, f), n - \max_{uv \in E} |f(u) - f(v)|\}. \quad (5)$$

#### 4.2 Cyclic Antibandwidth of Meshes

**Theorem 4.1** Let  $m$  be even and  $n$  be odd and  $m \geq n$ . Then

$$\left\lfloor \frac{n(m-1)}{2} \right\rfloor \leq \text{cab}(P_m \times P_n) \leq \left\lceil \frac{n(m-1)}{2} \right\rceil$$

Otherwise

$$\text{cab}(P_m \times P_n) = \frac{n(m-1)}{2}$$

**Proof.** Upper bound follows from  $\text{cab}(P_m \times P_n) \leq \text{ab}(P_m \times P_n) = \lceil \frac{(m-1)n}{2} \rceil$ .

Lower bounds is based on Lemma 4.5. Distinguish two cases according to parity of  $n$ .

Len  $n$  be even. For this case we define a new optimal labeling  $f$  with respect to antibandwidth (see Table 6).

3	27	9	33	15	39	21
24	6	30	12	36	18	42
2	26	8	32	14	38	20
23	5	29	11	35	17	41
1	25	7	31	13	37	19
22	4	28	10	34	16	40

Table 6  
Optimal labeling of  $7 \times 6$  mesh

We describe this labeling formally and then we show that this labeling is optimal with respect to antibandwidth. The first column of the labeling in the Table 6 is described by the vector

$$K_1 = \left( \frac{n}{2}, \frac{mn}{2} + \frac{n}{2}, \frac{n}{2} - 1, \dots, \frac{mn}{2} + 2, 1, \frac{mn}{2} + 1 \right) \quad (6)$$

Define

$$A_1 = \left( \frac{mn}{2} + \frac{n}{2}, -\frac{mn}{2} + \frac{n}{2}, \frac{mn}{2} + \frac{n}{2}, -\frac{mn}{2} + \frac{n}{2}, \dots, -\frac{mn}{2} + \frac{n}{2} \right)$$

$$A_2 = (n, n, n, \dots, n)$$

The numbering in columns  $K_i, i > 1$  is defined as follows.

$$K_2 = K_1 + A_1, K_3 = K_1 + A_2, K_4 = K_2 + A_1,$$

$$K_5 = K_2 + A_2, \dots, K_n = K_{n-3} + A_{n-3}$$

The smallest difference of labels of adjacent vertices is given by the smallest absolute value in the vector  $A_1$  which is  $|\frac{mn}{2} + \frac{n}{2}| = \frac{n(m-1)}{2}$ . Hence the labeling is optimal with respect to antibandwidth. The maximum difference of labels of adjacent vertices is  $\lfloor mn/2 + n/2 \rfloor = \lfloor (m+1)n/2 \rfloor$ . Substituting this into (5) we get the lower bound.

Let  $n$  be odd.

We use the same way of labeling as in previous case (see Table 7).

20	5	25	10	30	15	35
2	22	7	27	12	32	17
19	4	24	9	29	14	34
1	21	6	26	11	31	16
18	3	23	8	28	13	33

Table 7

Optimal labeling of  $7 \times 5$  mesh

Formally the first column is described by the vector:

$$K_1 = \left( \left\lfloor \frac{mn}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{mn}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil - 1, \left\lfloor \frac{n}{2} \right\rfloor - 1, \dots, \left\lfloor \frac{mn}{2} \right\rfloor + 1, 1, \left\lfloor \frac{mn}{2} \right\rfloor \right)$$

Define

$$A_1 = \left( -\left\lfloor \frac{mn}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{mn}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil, -\left\lfloor \frac{mn}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor, \dots, -\left\lfloor \frac{mn}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \right)$$

$$A_2 = (n, n, n, \dots, n)$$

The following terms define the labeling for columns  $K_i, i > 1$ :

$$\begin{aligned} K_2 &= K_1 + A_1, K_3 = K_1 + A_2, K_4 = K_2 + A_1, \\ K_5 &= K_2 + A_2, \dots, K_n = K_{n-3} + A_{n-3} \end{aligned}$$

One can check that the smallest difference of adjacent labels is  $\lceil (m-1)n/2 \rceil$ , i.e. the labeling is antibandwidth optimal. The maximum difference of adjacent labels is  $\lceil (m+1)n/2 \rceil$ . Substituting this into (5) we get the result.  $\square$

### 4.3 Cyclic Antibandwidth of Tori

**Theorem 4.2** For even  $n$

$$\text{cab}(C_n \times C_n) = \text{ab}(C_n \times C_n) = \frac{n(n-2)}{2}$$

**Proof.** Upper bound follows from  $\text{cab}(C_n \times C_n) \leq \text{ab}(C_n \times C_n) = \frac{n(n-2)}{2}$ . We apply Lemma 4.5 again. The antibandwidth optimal labeling  $f$  is shown in

Theorem 3.2. The maximum difference between adjacent labels is  $n + \frac{n^2}{2}$ . After substitution of this value into (4.5) we get the claimed result.  $\square$

**Theorem 4.3** For odd  $n$

$$\text{cab}(C_n \times C_n) = \frac{(n-2)(n+1)}{2}$$

**Proof.** Analogically as in previous case we determine the maximum difference of adjacent labels in the antibandwidth optimal labeling described in Theorem 3.3. To see the differences between any two adjacent labels in the torus we examine the following four cases:

(1) Neighbours in a column.

$$\begin{aligned} |f(i+1, j) - f(i, j)| &= \frac{(n^2 + n + 2)(i+1) + (n^2 - n)j}{2} \\ &\quad - \frac{(n^2 + n + 2)i + (n^2 - n)j}{2} \pmod{n^2} \\ |f(i+1, j) - f(i, j)| &= \frac{n^2 + n + 2}{2} \end{aligned}$$

(2) Neighbours in a row.

$$|f(i, j+1) - f(i, j)| = \frac{n^2 - n}{2}$$

(3) First and last vertex in a row.

$$\begin{aligned} |f(i, (j+1) \bmod n) - f(i, j)| &= \tag{7} \\ &= \frac{(n^2 - n)((j+1) \bmod n - j)}{2} \pmod{n^2} \end{aligned}$$

If  $(j+1) \bmod n - j = 1$  then the right hand side of (7) equals  $(n^2 - n)/2$ .

If  $(j+1) \bmod n - j = -1$  then the right hand side of (7) equals  $(n^2 + n)/2$ .

(4) First and last vertex in a column.

$$\begin{aligned} |f((i+1) \bmod n, j) - f(i, j)| &= \tag{8} \\ &= \frac{n^2 + n + 2}{2}((i+1) \bmod n - i) \pmod{n^2} \end{aligned}$$

If  $(i+1) \bmod n - i = 1$  then the right hand side of (8) equals  $(n^2 + n + 2)/2$ .

If  $(i+1) \bmod n - i = -1$  then the right hand side of (8) equals  $n^2 - (n^2 + n + 2)/2 = (n-2)(n+1)/2$ .

Thus, the maximum difference of adjacent labels is  $(n^2 + n + 2)/2$ , which by Lemma 4.5 implies the claim.  $\square$

#### 4.4 Cyclic Antibandwidth of Hypercubes

**Theorem 4.4** For the  $n$ -dimensional hypercube  $Q_n$

$$\text{cab}(Q_n) = 2^{n-1} - \frac{2^{n-1}}{\sqrt{2\pi n}} (1 + o(1)).$$

**Proof.** The upper bound comes from Theorem 3.4. For the lower bound we apply Lemma 4.5. Miller and Pritikin [18] proposed the following labelling  $f$  of the hypercube. Place the sets  $X_i$  on the line in the order  $X_1, X_3, X_5, \dots, X_0, X_4, X_6, \dots$ , while the order of vertices in the sets is decreasing according to the corresponding binary numbers. They proved

$$\text{ab}(Q_n, f) = 2^{n-1} - \frac{2^{n-1}}{\sqrt{2\pi n}} (1 + o(1)).$$

Now we estimate the maximum distance of any two adjacent vertices under the labelling  $f$ . Realize that

$$\max_{uv \in E(Q_n)} |f(u) - f(v)| = \max\{\max_{k \geq 1} \{D_1(k) + D_2(k)\}, 2^{n-1}\}, \quad (9)$$

where

$$D_1(k) = |X_{2k+1}| + |X_{2k+3}| + \dots + |X_0| + |X_2| + \dots + |X_{2k-2}|.$$

and  $D_2(k)$  is the maximum distance between two adjacent vertices  $u$  and  $v$ , where  $u \in X_{2k}$  and  $v \in X_{2k-1}$ , in the situation where the set  $X_{2k}$  is placed immediately after  $X_{2k-1}$ . First we determine  $D_1(k)$ .

$$\begin{aligned} D_1(k) &= \binom{n}{2k+1} + \binom{n}{2k+3} + \dots + \binom{n}{0} + \binom{n}{2} + \dots + \binom{n}{2k-2} \quad (10) \\ &= \binom{n}{1} + \binom{n}{3} + \dots + \binom{n}{2k-1} + \binom{n}{2k+1} + \dots + \binom{n}{0} + \binom{n}{2} \\ &\quad + \dots + \binom{n}{2k-2} - \left( \binom{n}{1} + \binom{n}{3} + \dots + \binom{n}{2k-1} \right) \\ &= 2^{n-1} - \binom{n-1}{2k-1}. \end{aligned}$$

Now we determine  $D_2$ . Let the vertex  $u$  be represented by a number containing ones on positions  $p_{2k}, \dots, p_2, p_1$ . Note that  $i \leq p_i \leq n - 2k + i$ . The position  $p_1$  is the least significant. The rank of the vertex  $u$  in  $X_{2k}$  counted from the

rightmost vertex is

$$\binom{p_{2k}-1}{2k} + \binom{p_{2k-1}-1}{2k-1} + \dots + \binom{p_2-1}{2} + \binom{p_1-1}{1} + 1.$$

The furthest neighbour of  $u$  in  $X_{2k-1}$  is the vertex  $v$  which contains ones on positions

$p_{2k}, p_{2k-1}, \dots, p_2$ . Its rank in  $X_{2k-1}$  counted from the rightmost vertex is

$$\binom{p_{2k}-1}{2k-1} + \binom{p_{2k-1}-1}{2k-2} + \dots + \binom{p_2-1}{1} + 1.$$

Hence

$$\begin{aligned} D_2(k) &= \binom{n}{2k} - \sum_{i=1}^{2k} \binom{p_i-1}{i} + \sum_{i=2}^{2k} \binom{p_i-1}{i-1} \\ &= \binom{n}{2k} - p_1 + 1 + \sum_{i=2}^{2k} \binom{p_i-1}{i-1} - \binom{p_i-1}{i}. \end{aligned} \quad (11)$$

Adding (10) and (11) and setting  $p_1 = 1$

$$D_1(k) + D_2(k) = 2^{n-1} + \binom{n-1}{2k} + \sum_{i=2}^{2k} \binom{p_i-1}{i-1} - \binom{p_i-1}{i}.$$

Note that for  $j \geq 1$ , and integer  $x$ , the function

$$\binom{x}{j} - \binom{x}{j+1}$$

is nonnegative and nondecreasing in the interval  $[j, 2j]$  which can be easily checked by computing the ratio of two consecutive values for  $x$  and  $x-1$ .

Then

$$\max_{[j, 2j]} \left\{ \binom{x}{j} - \binom{x}{j+1} \right\} = \binom{2j}{j} - \binom{2j}{j+1} = \frac{1}{j+1} \binom{2j}{j}. \quad (12)$$

It follows that

$$\binom{p_i-1}{i-1} - \binom{p_i-1}{i}$$

is maximized when  $p_i - 1 = \min\{2i - 2, n - 2k - 1 + i\}$ . Distinguish 2 cases:

- Let  $4k \leq n + 1$ . Then  $2i - 2 \leq n - 2k - 1 + i$  and consequently

$$D_1(k) + D_2(k) = 2^{n-1} + \binom{n-1}{2k} + \sum_{i=1}^{2k-1} \frac{1}{i+1} \binom{2i}{i}.$$

The right hand side is maximised when  $k = \lfloor (n+1)/4 \rfloor$ . Hence

$$\max_{4k \leq n+1} \{D_1(k) + D_2(k)\} = 2^{n-1} + \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} + \sum_{i=1}^{2^{\lfloor \frac{n+1}{4} \rfloor - 1}} \frac{1}{i+1} \binom{2i}{i}. \quad (13)$$

- Let  $4k \geq n+2$ . If  $i \geq n-2k+2$  then  $p_i - 1 = n-2k-1+i$ , otherwise  $p_i = 2(i-1)$ . This gives

$$\begin{aligned} D_2(k) &= \binom{n}{2k} + \binom{n-1}{2k-1} - \binom{n-1}{2k} + \binom{n-2}{2k-2} - \binom{n-2}{2k-1} + \dots \\ &+ \binom{2n-4k+1}{n-2k+1} - \binom{2n-4k+1}{n-2k+2} + \sum_{i=1}^{n-2k} \frac{1}{i+1} \binom{2i}{i} \\ &= \binom{n}{2k-1} + \sum_{i=1}^{n-2k} \frac{1}{i+1} \binom{2i}{i}. \end{aligned}$$

Hence

$$D_1(k) + D_2(k) = 2^{n-1} + \binom{n-1}{2k} + \sum_{i=1}^{n-2k} \frac{1}{i+1} \binom{2i}{i}.$$

The right hand side is maximised when  $k \geq \lceil (n+2)/4 \rceil$ . Then

$$\max_{4k \geq n+2} \{D_1(k) + D_2(k)\} = 2^{n-1} + \binom{n-1}{2\lfloor \frac{n+2}{4} \rfloor} + \sum_{i=1}^{n-2\lfloor \frac{n+2}{4} \rfloor} \frac{1}{i+1} \binom{2i}{i}. \quad (14)$$

Now realize that the righthand side of (14) does not exceed the right hand side of (13). It is easy to see that the sum of Catalan numbers in (13) is of order  $\theta(2^n/n^{\frac{3}{2}})$  and by Stirling approximation:

$$\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} = \frac{2^{n-1}}{\sqrt{2\pi n}} (1 + o(1)). \quad (15)$$

Then

$$\max_k \{D_1(k) + D_2(k)\} \leq 2^{n-1} + \frac{2^n}{\sqrt{2\pi n}} (1 + o(1)).$$

Finally, substituting the last expression into (9) and (5) we get the result.  $\square$

## 5 Conclusions

We proved exact results for antibandwidths and cyclic antibandwidth of two dimensional meshes and tori and asymptotics for hypercube graphs. The general (cyclic) antibandwidth problem is still rather unexplored and several questions remain open:

- Find at least tight bounds for other standard graph. Non-bipartite graphs are of special interest.
- Find a new upper bound argument.
- Find a "natural" class of graphs with polynomial time complexity.
- Find approximation algorithms or at least good practical heuristics.
- The "average" version of the problem is the so called maximal linear arrangement problem where the sum of differences of adjacent vertices is maximised [9]. However, this problem is equivalent to the optimal linear arrangement problem, i.e., the sum of differences of adjacent vertices is minimised [5].

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