

Antibandwidth of Three-Dimensional Meshes

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Abstract

The antibandwidth problem is to label vertices of a graph $G = (V, E)$ bijectively by $1, 2, 3, \dots, |V|$ such that the minimal difference of labels of adjacent vertices is maximised. In this paper we discuss the antibandwidth of three-dimensional meshes. Provided results are extensions of the two-dimensional case and an analog of the result for the bandwidth of three-dimensional meshes obtained by FitzGerald.

Keywords: antibandwidth, mesh, three-dimensional

1 Introduction

The antibandwidth problem consists of placing the vertices of a graph on a line in an integer points in such a way that the minimum difference of adjacent vertices is maximised. The problem was originally introduced in [13] in a connection with multiprocessor scheduling problems. It can be understood as a dual problem to the well known bandwidth problem [7] in which the maximum distance of adjacent vertices in the linear layout is minimised. Another motivation comes from the area of radio frequencies assignment problem [11]. Transmitters are assigned n different frequencies such that the physically neighbouring transmitters have as different frequencies as possible. The transmitters and their neighbourhood are given by an n -vertex graph. The problem also belongs to the family of obnoxious facility location problems: The "enemy

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graph” is represented by n persons. Two of them are joined by an edge iff they are enemies. The problem is to build a house for every person along a road such that the minimal distance between enemies is maximised [4]. From the graph theory point of view our problem is just a new labelling problem, see a survey [9]. The problem was originally studied under the term *separation number* [13]. However, in the meantime, this name was also used for another linear layout problem [7]. Lin and Yuan called it *dual bandwidth*. In our older paper [16] we proposed a new term for the problem, (the most appropriate according to our opinion) *antibandwidth*.

The antibandwidth problem is NP-complete [13]. So far it is polynomially solvable for 3 classes of graphs: the complements of interval, arborescent comparability and treshold graphs [6,12]. Known results include simple relations of the antibandwidth invariant to the minimum, maximum degree, chromatic index and powers of hamiltonian paths in the complement graph [13,14,15]. Exact results are known for paths, cycles, special trees, complete and complete bipartite graphs [14,15,17]. The class of n -vertex forests with $\text{ab}(F) = \lfloor n/2 \rfloor$ is characterized in [15], which covers, e.g., complete binary trees. The invariant for complete k -ary trees is discussed in [5].

The problem is also interesting for disconnected graphs. Exact values were proved for graphs consisting of copies of simple graphs [10,18]. The problem is worth studying also in the case of general host graph. For example, the case where the host graph is cycle is analogical to cyclic bandwidth problem, see [16]. Another motivation for this comes from coding theory. In this case the host graph is hypercube Q_n and guest graph is a complete graph K_p . The antibandwidth of K_p embedded in Q_n is then equal to minimal Hamming distance of a code with p words which is interesting parameter connected with code reliability, especially error-correcting property.

This paper extends our previous results for 2D meshes [16] where we proved that for the two-dimensional mesh M_2 of type $m \times n$

$$\text{ab}(M_2) = \left\lfloor \frac{(m-1)n}{2} \right\rfloor.$$

We show that for the three dimensional mesh M_3 of type $n \times n \times n$

$$\text{ab}(M_3) = \frac{4n^3 - 3n^2}{8} + O(n).$$

Moreover, this result is an analogue of result by FitzGerald determining the bandwidth of 3D meshes [8].

2 Preliminaries

Let G be a graph. Let $\partial(A)$ denote the vertex boundary of a set $A \subseteq V$, i.e., the set of all vertices from $V - A$ having a neighbour in A .

Let $G = (V_1, V_2, E)$ be a bipartite graph. Let $\partial_b(A)$ denote the vertex boundary of a set $A \subseteq V_1$, i.e. the set of all vertices from V_2 having a neighbour in A . We call it the bipartite vertex boundary.

Denote $M_3 = P_n \times P_n \times P_n$, for $n \geq 3$. The vertices of P_n are $\{0, 1, 2, \dots, n-1\}$ and edges $\{(i, i+1) | i = 0, 1, 2, \dots, n-2\}$. Let $M_3 = (V_1, V_2, E)$, where V_1 and V_2 are the partition sets and $(0, 0, 0) \in V_1$.

Let us define the simplicial order in M_3 as follows [2]. Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Then $x < y$ if either $\sum x_i < \sum y_i$, or $\sum x_i = \sum y_i$, and for some j we have $x_j > y_j$ and $x_i = y_i$ for all $i < j$.

The diameter of M_3 is $3(n-1)$. For $r = 0, 1, 2, \dots, 3(n-1)$, let $B(r)$ denote the set of vertices

of M_3 in the distance r from $(0, 0, 0)$. It is easy to see that for $r = 0, 1, 2, \dots, 3(n-1)$

$$(1) \quad |B(r)| = |B(3(n-1) - r)|,$$

and

$$(2) \quad |B(r)| = \binom{r+2}{2} - 3 \binom{r-n+2}{2},$$

for $r \leq 2(n-1)$. Further it holds

$$(3) \quad \sum_{r=0}^{3(n-1)} (-1)^r |B(r)| = n \pmod{2}.$$

3 Upper bound

In the following paragraphs we prove the upper bound for antibandwidth of three-dimensional mesh. The proof is simplified to one case while the proof of other cases runs similarly.

Theorem 3.1 For $n \geq 3$

$$\text{ab}(M_3) \leq \frac{n^3}{2} - \frac{3n^2}{8} + O(1).$$

Proof. Assume $n \equiv 1 \pmod{4}$. Other cases are similar. Denote $t = 3(n-1)/4$. In this case (1) and (3) imply

$$(4) \quad |B(0)| + |B(2)| + |B(4)| + \dots + |B(2t-2)| + \frac{1}{2}|B(2t)| = \frac{n^3}{4}.$$

Further, (2) gives

$$|B(2t)| = |B(\frac{3(n-1)}{2})| = \frac{3n^2 + 1}{4}.$$

Consider an optimal linear layout of M_3 . Denote $k = \text{ab}(M_3)$. If

$$k \leq \frac{n^3 - |B(2t)|}{2} + 2,$$

then we are done. Suppose indirectly that

$$k \geq \frac{n^3 - |B(2t)|}{2} + 3,$$

We know that $k < n^3/2$. Let S be a set of consecutive k vertices on the line. Denote $A_i = V_i \cap S$, for $i = 1, 2$. Denote by $J = [L, R]$, an interval with

$$L = \frac{n^3}{4} - \frac{1}{2}|B(2t)| + 1, \quad R = \frac{n^3}{4}.$$

Distinguish two cases.

Case 1. Assume that there exist S such that the corresponding A_1 satisfies $|A_1| \in J$. Observe that

$$|\partial_b(A_1)| + k - |A_1| \leq |V_2|.$$

Hence

$$k \leq \frac{n^3 - 1}{2} - (|\partial_b(A_1)| - |A_1|).$$

In what follows we will show that $|\partial_b(A_1)| - |A_1| \geq |B(2t)|/2 - 2$, which immediately gives a contradiction. The equation (4) implies that

$$|A_1| = |B(0)| + |B(2)| + |B(4)| + \dots + |B(2t-2)| + \alpha|B(2t)|,$$

for some constant $0 < \alpha \leq 1/2$. Let I_1 be the set of the first $|A_1|$ vertices from V_1 in the simplicial order. Bezrukov and Piotrowski [1] proved that

$$(5) \quad |\partial_b(A_1)| \geq |\partial_b(I_1)|.$$

Let $A \in V_1 \cup V_2$ be a set of cardinality

$$|A| = |B(0)| + |B(1)| + |B(2)| + \dots + |B(2t-1)| + \alpha|B(2t)|.$$

Let I be the set of the first $|A|$ vertices from $V_1 \cup V_2$ in the simplicial order. Bollobás and Leader [2] proved

$$(6) \quad |\partial_b(A)| \geq |\partial_b(I)|.$$

From the definitions of I_1 and I we have

$$(7) \quad |\partial_b(I_1)| = |B(1)| + |B(3)| + |B(5)| + \dots + |B(2t-1)| + |\partial(I)| - (1-\alpha)|B(2t)|$$

Moreover Bollobás and Leader [2,3] showed that

$$(8) \quad |\partial(I)| \geq (1-\alpha)|B(2t)| + \alpha|B(2t+1)|.$$

Combining (5),(6),(7) and (8) we obtain

$$\begin{aligned} |\partial_b(A_1)| - |A_1| &\geq |\partial_b(I_1)| - (|B(0)| + |B(2)| + \dots + |B(2t-2)| + \alpha|B(2t)|) \\ &\geq \sum_{i=0}^{2t-1} (-1)^{i+1} |B(i)| - \alpha(|B(2t)| - |B(2t+1)|) \\ &= -\frac{1}{2} + \frac{1}{2}|B(2t)| - \alpha(|B(2t)| - |B(2t+1)|) \\ &\geq \frac{1}{2}|B(2t)| - \frac{3}{2}. \end{aligned}$$

Case 2. This case will be discussed in detail in the full version of this paper. □

4 Lower bounds

In this section we describe the optimal (up to the third order term) labeling of three-dimensional mesh. First, we describe the algorithm for labeling the vertices and then we prove that the minimal difference of neighbouring labels in this labeling is matching the upper bound up to the third order term.

The labeling of vertices of M_3 proceeds in two phases. The first phase is the labeling of all $v \in B(r)$ for $r = 0, 2, 4, \dots$, r is even. The second phase continues with $r = 1, 3, 5, \dots$, r is odd. The labeling of one $B(r)$ proceeds as follows. Start at the top of the cut $B(r)$, i.e. in the vertex with maximal z and y -axis. We call this level of a cut a "base level". See the Figure 1.

This vertex obtains the first label. Then continue from the left to right and from the top to bottom with the rest of vertices of $B(r)$. See the example of the labeling in the Table 1.

In the next paragraphs we analyze the labeling algorithm. First, we start with some simple observations. Consider a position of a vertex v_1 in a cut denoted by a triplet r, h, w where r is a distance from the vertex $(0, 0, 0)$, h is a height of vertex v_1 counted from the top of the cut and w is the order in the h -th row of the cut.

Observation 4.1 *Let v_1 and v_2 be two neighbouring vertices in M_3 and let r be even. Then the neighbour of a vertex $v_1 \in B(r)$, with maximal label is a vertex v_2 at position $r-1, h, w-1$.*

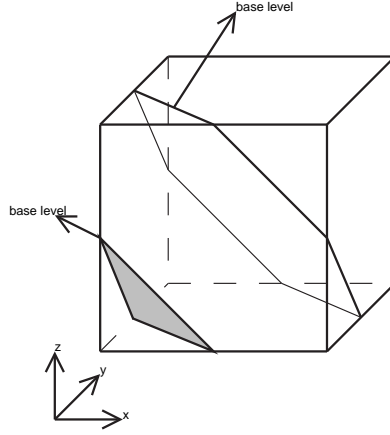


Fig. 1. Cuts in a 3D mesh

42	17	56	29	13	53	27	63	49	24	61	32	20	58	30	64
5	43	18	57	39	14	54	28	10	50	25	62	46	21	59	31
34	6	44	19	3	40	15	55	37	11	51	26	8	47	22	60
1	35	7	45	33	4	41	16	2	38	12	52	36	9	48	23
1st level				2nd level				3rd level				4th level			

Table 1

Labeling of particular levels of $P_4 \times P_4 \times P_4$ mesh

Denote by L_i^r the i th level of a cut $B(r)$ counted from its base level, i.e. the level consisting of vertices with the greatest z -axis. The base level of $B(r)$ is denoted by L_0^r .

Observation 4.2 Let $R = 3n/2$ then

- (i) $|L_i^R| - |L_i^{R-1}| = 1$ for all $i \leq n/2 - 2$
- (ii) $|L_i^R| - |L_i^{R-1}| = -1$ for all $i : n/2 - 2 < i < n$

Theorem 4.1 $ab(M_3) \geq \left\lceil \frac{(4n-3)(n^2-1)}{8} \right\rceil$

Proof. We show an exact proof for the case when n is divisible by four. The rest of cases are similar.

Let r be even and let v_1 and v_2 be two neighbouring vertices, i.e. $v_1 \in B(r)$ and $v_2 \in B(r-1)$. Moreover, let v_2 be the smallest neighbour of v_1 .

If we denote the label assigned to a vertex v by $f(v)$ we get

$$(9) \quad f(v_1) = |B(0)| + |B(2)| + \dots + |B(r-2)| + C(v_1)$$

$$(10) \quad f(v_2) = \frac{n^3}{2} + |B(1)| + |B(3)| + \dots + |B(r-3)| + C(v_2)$$

where $C(v_1)$ and $C(v_2)$ stand for orders of v_1 and v_2 in cuts $B(r)$ and $B(r-1)$ respectively.

$$(11) \quad \begin{aligned} f(v_2) - f(v_1) &= \frac{n^3}{2} + |B(1)| - |B(2)| + \\ &\quad |B(3)| - |B(4)| + \dots + |B(r-3)| - |B(r-2)| \\ &\quad - |B(0)| + C(v_2) - C(v_1) \end{aligned}$$

Using the formula for $|B(k)|$ and after some algebraic manipulations we get

$$(12) \quad |B(k)| - |B(k-1)| = 2k - 3n + 4$$

for $k \geq n - 1$ and

$$(13) \quad |B(k)| - |B(k-1)| = -k - 2$$

for $k < n - 1$.

Substituting (12) and (13) into (11), using $|B(0)| = 1$ we get

$$(14) \quad f(v_2) - f(v_1) = \frac{2n^3 - 3n^2 - 6rn + 2r^2 + 4}{4} + C(v_2) - C(v_1)$$

The Equation (14) consists of two terms: the fraction and the difference $C(v_2) - C(v_1)$. Our aim is to minimize the value of (14). Considering the fraction from (14) as a function of r we get that its value is minimal for $r = 3n/2$.

After substitution we have

$$(15) \quad f(v_2) - f(v_1) \geq \frac{4n^3 - 3n^2}{8} + C(v_2) - C(v_1)$$

Now, we discuss the difference $C(v_2) - C(v_1)$. An easy observation shows that the minimal difference between two cuts is realised in the middle of the mesh. Moreover, this observation is confirmed by the way the inequality (15) is obtained. Therefore, we concentrate our attention to cuts in the middle of the mesh, i.e. around the value $r = 3n/2$. For larger meshes these cuts are of hexagonal shape. See the Figure 2. The difference between two corresponding levels in

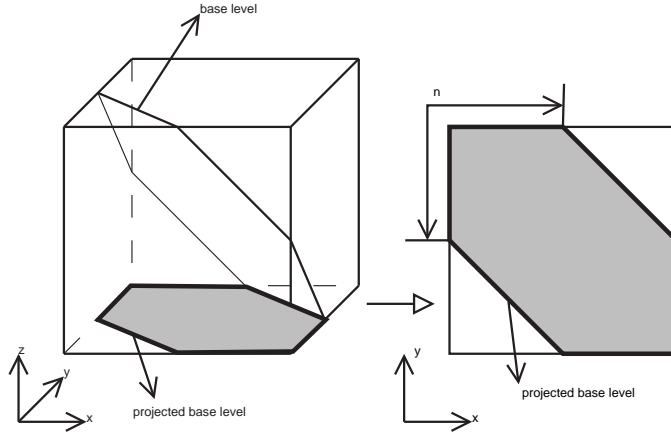


Fig. 2. Projection of a cut in a mesh

these cuts is at most 1. One cut has at most n levels. See the Figure 2 for illustration. So far, we can say that

$$(16) \quad C(v_2) - C(v_1) = O(n)$$

Let $R = 3n/2$. We need to compute the difference (16) for two neighbouring cuts $B(R)$ and $B(R-1)$. Consider the projection of both cuts into the basic plane of the mesh. Note that number of vertices of a projected cut is the same as the original one.

Both cuts $B(R)$ and $B(R-1)$ are labeled starting with their base levels. There is exactly $n/2 - 2$ consecutive levels in $B(R)$ such that each one of them is one vertex longer than appropriate level in $B(R-1)$. Since we are looking for the greatest value of $C(v_1)$ this will be obtained according to Observation 4.2 in level $n/2 - 2$. Then, with use of Observation 4.1 we have

$C(v_2) - C(v_1) = -n/2 + 2 - 1$. And finally

$$f(v_2) - f(v_1) \geq \frac{4n^3 - 3n^2}{8} - n/2 + 1 = \left\lceil \frac{(4n - 3)(n^2 - 1)}{8} \right\rceil$$

□

We conjecture that the lower bound given by Theorem 4.1 is optimal.

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