

Upper Bound for Queue Number of Shuffle-Exchange Graph (A Note)

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Abstract

Shuffle-Exchange network has some good properties in parallel data processing. Its graph abstraction called Shuffle-Exchange graph is well known in the area of VLSI design.

In this paper we prove that Shuffle-Exchange graph has a 3-queue layout, while it is known that at least 2 queues are necessary. This value provides upper bound for queue number of Shuffle-Exchange graph.

Keywords. Shuffle-Exchange graph, queue layout, queue number.

1 Introduction

1.1 Shuffle-Exchange Graph

Definition 1 *The d -dimensional shuffle-exchange graph (SE_d) has 2^d nodes. Each vertex is numbered by unique binary string of length d . The edges are defined as follows. Vertex represented by binary string αa , where $\alpha \in \{0, 1\}^{(d-1)}$ and $a \in \{0, 1\}$, is connected with vertex $\alpha \neg a$ and $a\alpha$ (where $\neg a$ is negation of a). Edges directions, multiple edges and loops are ignored.*

The edges between vertices αa and $\alpha \neg a$ are called *exchange* edges and the edges between vertices αa and $a\alpha$ are called *shuffle* edges.

The Shuffle-Exchange network provides suitable interconnection patterns for implementation of parallel algorithms like : *polynomial evaluation, fast Fourier transform, sorting and matrix transposition.*

1.2 Linear Layout of a Graph

The linear layout of a graph is such a layout in which the vertices are drawn on a horizontal line in some order (designated by σ in this paper). Although the graph is undirected, we consider the edges orientation given by the ordering of vertex set.

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1.3 K-Queue Layout and Queue Number

A *k*-queue layout of an undirected graph $G = (V, E)$ has two aspects. The first aspect is linear order of V (which we think of as being on a horizontal line). The second aspect is an assignment of each edge in E into one of k queues in such a way that the set of edges assigned to each queue obeys a first-in/first-out discipline. Each queue q_j operates as follows. The vertices of V are scanned in left-to-right ascending order. When vertex i is encountered, any edges assigned to q_j that have vertex i as their right endpoint must be at the front of that queue; they are removed (dequeued). Any edges assigned to q_j that have vertex i as their left endpoint are placed on the back of that queue (enqueued). Queue number (qn) is smallest k such that G has *k*-queue layout. [1]

This layout problem abstracts design problem of fault-tolerant processor arrays, a problem of sorting with parallel queues, and a problem of scheduling parallel processors.

The question of queue number of Shuffle-Exchange graph is still open, although it is known for deBruijn graph (close relative of Shuffle-Exchange). Heath and Rosenberg made the characterization of 1-queue graphs as arched leveled-planar graphs [1]. Queue numbers of some typical graphs are also in [1].

Leighton in [3] showed that crossing number of Shuffle-Exchange graph is $\Theta(N^2/\log^2 N)$, what means that it is not a planar graph. Since SE_d is not planar, it cannot have queue number 1 and therefore $qn(SE_d) \geq 2$.

1.4 K-Rainbow set of edges

Definition 2 Suppose we have a linear graph layout (all vertices are on the horizontal line) with some vertices ordering σ . Then a *k*-rainbow is a set of k edges $e_i = (a_i, b_i), 1 \leq i \leq k$ such that

$$a_1 <_{\sigma} a_2 <_{\sigma} \dots <_{\sigma} a_k <_{\sigma} b_k <_{\sigma} b_{k-1} <_{\sigma} \dots <_{\sigma} b_2 <_{\sigma} b_1.$$

In other words, a rainbow is a *nested* matching. A rainbow is an obstacle for a queue layout because no two nested edges can be assigned to the same queue. [1]

2 Results

2.1 Alternative definition of SE_d

Definition 3 Let $G(V, E)$ be the graph with 2^d vertices. Label the vertices with numbers $0, 1, 2, \dots, 2^d - 1$. The edges are defined as follows. Vertex with number n will be connected

- with vertex $n + 1$ in case of even n or $n = 0$,
- with vertex $\frac{n}{2}$ in case of even n ,
- with vertex $\frac{n-1}{2} + 2^{d-1}$ in case of odd n .

This definition is a modification of SE_d definition from [2].

2.2 Upper bound

Lemma 1 *The queue number $qn(G)$ of a graph G is a minimum, taken over all vertex orderings σ of G , of a maximum size of a rainbow in σ . [1]*

Lemma 2 *Let $p(n), n \in N$ be the non-descending sequence and let G be the graph $G = (V, E)$, where $V = 0, 1, 2, \dots, n$ and all edges from E are of type $(v, p(v)), v \in V$. Then G has one-queue layout with natural $(0, 1, \dots, n)$ ordering of vertices.*

Proof. Let $m_1, m_2, \dots, m_n; m_i \in N$ be the vertex indexes. From non-descending sequence $p(m_i)$ we have

$$m_1 < m_2 < \dots < m_n \Rightarrow p(m_1) \leq p(m_2) \leq \dots \leq p(m_n).$$

Comparing this property with definition of k -rainbow set we see that in this type of graph the k -rainbow set can not exist with $k > 1$. According to Lemma 1 we have ordering with maximum rainbow size of 1, and therefore with queue number 1. \triangle

Theorem 1 *The SE_d graph has 3-queue layout with natural vertices order $0, 1, \dots, 2^d - 1$. The edges will be assigned to queues as follows.*

1. queue : edges of type $(m, m + 1)$ for even m or $m = 0$.
2. queue : edges of type $(m, \frac{m}{2})$ for even m .
3. queue : edges of type $(m, \frac{m-1}{2} + 2^{d-1})$ for odd m .

Proof. By assigning the edges into three queues we have three subgraphs of SE_d . It is sufficient to prove that these subgraphs have 1-queue layout with vertices ordering $0, 1, 2, \dots, 2^d - 1$.

1. queue : edges of type $(m, m + 1)$ for even m or $m = 0$.
Edges of this subgraph are generated by the formula $p(m) = m + 1$. It is ascending sequence and according to lemma 2 graph with such edges has 1-queue layout.
2. queue : edges of type $(m, \frac{m}{2})$ for even m .
Edges generating formula is $p(m) = \frac{m}{2}$. Again, it is ascending sequence and according to lemma 2 graph with such edges has 1-queue layout.
3. queue : edges of type $(m, \frac{m-1}{2} + 2^{d-1})$ for odd m .
Edges generating formula is $p(m) = \frac{m-1}{2} + 2^{d-1}$. Analogue to previous points graph with such edges has 1-queue layout. \triangle

Corollary 1 *The SE_d graph layed out on horizontal line with natural vertices ordering has k -rainbow set of edges with $k = 3$.*

Proof. From our alternative definition of SE_d we have three types of edges. From proof of *theorem 1* only edges of different types can nest. It means that maximal nested matching can have degree 3 (with respect to natural ordering of vertices). It is trivial to find such rainbow. For example one exists in SE_4 and consists of edges $\{(8, 9), (6, 12), (7, 11)\}$. In SE_d where $d > 4$ can be found for example this rainbow: $\{(8, 9), (6, 12), (1, 2^{d-1})\}$. \triangle

Theorem 1 gives upper bound for queue number of Shuffle-Exchange graph. The final value can be 3(from *Theorem 1*) or 2 as a lower bound, since Shuffle Exchange can not have queue number 1 due to its non-planarity.

3-queue layout from *Theorem 1* is according to *corollary 1* minimal. In other words, there exists no 2-queue layout of Shuffle-Exchange graph with natural order of vertices because of existence of 3-rainbow set of edges.

Open problem. To prove that $qn(SE_d) = 2$ or $qn(SE_d) > 2$.

References

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