# Fundamental Groups of Prime 3-Manifolds of Genus at Most Two

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(joint work with )

Present paper deals with fundamental groups of 3-manifolds represented by a certain family $S$ of bipartite 4-edge-coloured graphs. List of fundamental groups of prime 3-manifolds of genus two represented by graphs in $S$ with at most 42 vertices is produced.

The aim of the present paper is to deal with fundamental groups of 3-manifolds represented by a certain family $\mathcal{S}$ of bipartite 4-edge-coloured graphs. The complexity of such a graph representation is half of the number of vertices of the graph. The list of fundamental and homology groups is derived for the set of 3-manifolds of complexity [2] at most 21. Fundamental groups are given in terms of presentations of rank 2. The presentations are derived from the graphs in $\mathcal{S}$ using the algorithm of Gagliardi [5]. The isomorphism problem is solved using some group theoretical techniques in combination with the software GAP (Groups, Algorithms and Programming [6]).

Due to Pezzana [10] a closed connected 3-manifold can be represented by a particular 4-valent 4-edge-coloured graph called a crystallisation. The concept of crystallisations plays a crucial role in our study of 3-manifolds. We emphasise two approaches. First one is based on investigation of the structure of a crystallisation using some techniques based on dipole-moves [2, 4, 7, 9]. This approach can be considered as graph-theoretic. Using dipole moves one can recognise whether two 3-manifolds are homeomorphic not using any other knowledge about represented manifold.

The other approach is based on determining some topological invariants such as the fundamental or homology group from the structure of a crystallisation. Next, group theory techniques to identify the isomorphism class of fundamental group of represented 3-manifold are used. Note that the knowledge about isomorphism of fundamental groups is not sufficient to claim that

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two 3-manifold are homeomorphic. On the other hand, the non-isomorphism of the fundamental groups guarantees that two respective 3-manifolds are not homeomorphic.

All considered 3-manifolds are assumed to be connected, compact and orientable with empty boundary. A 3-manifold is prime if it cannot be expressed as a connected sum of two non-trivial 3-manifolds. By Kneser Theorem (see [8, Theorem 3.15, 3.21]) an orientable compact connected 3-manifold has a unique decomposition into prime factors with respect to the connected sum. The corresponding operation on the respective fundamental groups is the free product of groups. In fact, combining some known non-trivial results (such as Thurston's Symmetrisation Theorem), one can prove that prime 3-manifolds of Heegaard genus two are characterised by the fact that the fundamental group cannot be expressed as a free product of two cyclic groups. List of isomorphism classes of fundamental groups of prime 3-manifolds of genus two represented by graphs in $\mathcal{S}$ with at most 42 vertices is produced in present note.

**Definition 1 (4-coloured graph).** *[9] Let $\Gamma$ be a finite bipartite, connected graph wihout loops. The* ***edge colouring*** *on a graph $\Gamma$ is a map $\gamma : E(\Gamma) \to C_4 = \{0, 1, 2, 3\}$ such that for every distinct $e, f \in E(\Gamma) : \gamma(e) \neq \gamma(f)$, if the edges $e$ and $f$ are adjacent. The graph $(\Gamma, \gamma)$ is called a* ***4-coloured graph*** *if and only if all vertices of $\Gamma$ are of degree four. The notation $(\Gamma, \gamma)$ is used for such graphs.*

Since the colouring of $(\Gamma, \gamma)$ is regular, a factor induced by two colours is a disjoint union of bicoloured cycles. Let $\mathcal{I}$ denotes the set of 2-cell embeddings of $(\Gamma, \gamma)$ into a closed orientable surface such that the local rotation of colours induced by the embedding in "black" vertices is the same, say $\rho$, while the local rotation of colours in "white" vertices is $\rho^{-1}$. Note that there are six possibilities for choosing $\rho$. It follows that faces of such embedding are bounded by bicoloured cycles. Out of these six possibilities for $\rho$ we choose such $\rho$ that the genus of the underlying surface is minimal in $\mathcal{I}$. The integer $g$ is an invariant of a 3-manifold $\mathcal{M}$ represented by $(\Gamma, \gamma)$ and it is called the *regular genus* of $\mathcal{M}$ (or shortly the genus of $\mathcal{M}$). It is known that regular genus of $\mathcal{M}$ is equal to the Heegaard genus of $\mathcal{M}$ [1].

Let $\Gamma_{\hat{B}}$ be a subgraph of $\Gamma$ such that $V(\Gamma) = V(\Gamma_{\hat{B}})$ and every edge in $\Gamma_{\hat{B}}$ is coloured by a colour from the set $C_4 \setminus B$. The *residual graph* $\Gamma_{\hat{B}}$ can be simply obtained from $\Gamma$ by deleting the edges coloured by colours from the set $B$. In particular, the graph $\Gamma_{\hat{c}}$ is the residual graph created by deleting the edges coloured by the colour $c$.

**Definition 2 (Contracted graph).** *[1] The graph $(\Gamma, \gamma)$ is* ***contracted****, if for all colours $c \in C_4$ the residuals $\Gamma_{\hat{c}}$ are connected graphs.*

Simple method of reconstruction of the underlying 3-dimensional simplicial complex given by a 4-coloured graph is introduced in [9]. Note that a general 4-coloured graph may determine a more general space of dimension 3 called a compact pseudomanifold of dimension 3.

**Definition 3 (Crystallisation $(\Gamma, \gamma)$).** *[1] A 4-coloured graph $(\Gamma, \gamma)$ will be called* ***crystallisation*** *if and only if it is contracted and admits a 3-manifold.*

A basis of the theory of crystallisations was found by Mario Pezzana [1]. In particular, we have the following theorem.

**Theorem 1 (Pezzana's Existence Theorem).** *[1] Every closed connected 3-manifold admits a crystallisation.*

It follows from [3] that each 3-manifold of genus at most two which can be represented by a crystallisation which structure can be coded by a 6-tuple of integers satisfying the following conditions. Let $\widetilde{\mathcal{F}_2}$ be a set of 6-tuples:

$$f = (h_0, h_1, h_2; q_0, q_1, q_2), \;\; h_i, q_i \in \mathbb{N}.$$

satisfying

(i) $\forall i \in \mathbb{Z}_3 : h_i > 0$,

(ii) all $h_i$ has the same parity,

(iii) $\forall i \in \mathbb{Z}_3 : 0 \leq q_i < h_{i-1} + h_i = 2l_i$,

(iv) all $q_i$ has the same parity.

*Remark.* From here all operations with numbers $q_i$ will be considered modulo $2l_i$, and according to (iii), $q_i$ will be allways the least non negative integer of the class.
Let us define the set $V(f)$ for a 6-tuple $f \in \widetilde{\mathcal{F}_2}$ to be

$$V(f) = \bigcup_{i \in \mathbb{Z}_3} \{i\} \times \mathbb{Z}_{2l_i}$$

and the following permutations acting on $V(f)$:

$$\alpha_0(i,j) = (i, j + (-1)^j),$$
$$\alpha_1(i,j) = (i, j - (-1)^j),$$
$$\alpha_2(i,j) = \begin{cases} (i+1, 2l_{i+1} - j - 1); & 0 < j < h_i \\ (i-1, 2l_i \quad - j - 1); & h_i \leq j < 2l_i \end{cases},$$
$$\alpha_3(i,j) = \rho \circ \alpha_2 \circ \rho^{-1},$$

where $\rho : V(f) \to V(f)$ is a bijection defined by the rule

$$\rho(i,j) = (i, j + q_i).$$

Now let $f \in \widetilde{\mathcal{F}_2}$ and satisfy the following conditions:

(v) $\forall i \in \mathbb{Z}_3 : h_i + q_i$ is odd, $h_i$ and $q_i$ have different parity,

(vi) the group $\langle \alpha_2, \alpha_3 \rangle$ has exactly three orbits.

**Definition 4 (Admissible 6-tuple).** *The elements of the set $\mathcal{F}_2 \subset \widetilde{\mathcal{F}_2}$ satisfying conditions (i) – (vi) will be called **admissible 6-tuples**.*

Given admissible 6-tuple $f$ there is an associated 4-coloured graph $\Gamma(f)$, called the derived graph, which vertices are elements of $V(f)$ and $xy$ is an edge coloured by a colour $i \in \{0,1,2,3\}$ if and only of $y = \alpha_i(x)$.

**Theorem 2.** *[3] For every orientable compact connected 3-manifold $\mathcal{M}$ of genus 2 there exists $f \in \mathcal{F}_2$ such that $\Gamma(f)$ represents $\mathcal{M}$.*

**Definition 5 (Complexity of 6-tuple).** *Let $f$ be an admissible 6-tuple. The number $z(f) = h_0 + h_1 + h_2$ will be called a **complexity of** $f$.*

In what follows we describe an algorithm constructing a presentation of the fundamental group of a 3-manifold represented by $\Gamma(f)$, $f \in \mathcal{F}_2$ (see [5]).

Let $f$ be an admissible 6-tuple and $\Gamma(f)$ be the derived graph. The fundamental group $\pi_1(f)$ of the 3-manifold given by $\Gamma(f)$ has a presentation of the form $\pi_1(f) = \langle a, b, c \mid R_0, R_1, R_2 \rangle$. Set $a, b, c$ to be the three cycles of the 2-factor induced by edges coloured by $\{0,1\}$. Hence we have a mapping $\beta : V(\Gamma) \to \{a, b, c\}$. By the definition, the $\{2,3\}$-factor has 3-cycles $C_0, C_1, C_2$.

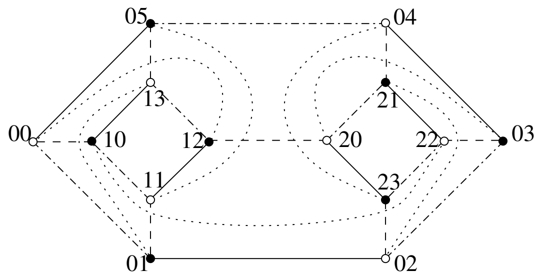


Figure 1: The graph $\Gamma(f)$ represented by 6-tuple $f = (3, 1, 3; 2, 2, 0)$

The relator $R_i$ is given by $C_i = (v_{0,i}v_{1,i}v_{2,i} \dots v_{k_i,i})$, $i = 0, 1, 2$. We assume that the edge $v_{0,i}v_{1,i}$ is coloured by the colour 2. We set

$$R_i = \prod_{j=0}^{(k_i-1)/2} \beta(v_{2j,i})\beta(v_{2j+1,i})^{-1}, \quad i = 0, 1, 2.$$

Add the relator $c = 1$ into the presentation of the group after obtaining the relators. Thus the group can be considered as a group of rank 2.

*Example.* Let us determine the fundamental group given by the 6-tuple `(3,1,3,2,2,0)`(see Fig. 1). Define the mapping $\beta : V(\Gamma) \to \{a, b, c\}$ as follows: $\beta(v) = a, b, c$ depending on whether $v$ belongs to $C_0$, $C_1$ or $C_2$, respectively. The $\{2, 3\}$-factor is a union of the cycles:

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C1 = ([0,0][1,0][2,2][0,3][2,0][1,2])
C2 = ([0,1][1,1][0,5][1,3])
C3 = ([0,2][2,3][0,4][2,1]).
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These cycles transform to relators $R_1 := ab^{-1}ca^{-1}cb^{-1} = 1$, $R_2 := ab^{-1}ab^{-1} = 1$ and $R_3 := ac^{-1}ac^{-1} = 1$. Hence

$$\pi_1((3, 1, 3, 2, 2, 0)) = \langle a, b \mid ab^{-1}a^{-1}b^{-1} = 1, (ab^{-1})^2 = 1, a^2 = 1\rangle.$$

Next step of processing the presentation of the fundamental group is to apply Tieze transformations to get the relators as short as possible. It is easy to derive the first homology group $H_1(f)$ from the presentation of the fundamental group. Simply add the commutator of the two generators into its presentation. Homology groups splits the set of 6-tuples into homology classes. Of course if $H_1(f) \ncong H_1(e)$ then $\pi_1(f) \ncong \pi_1(e)$.

The finiteness of homology groups is tested by solving the system of equations over the ring $\mathbb{Z}$ given by relators of the homology group. The computation is done using GAP [6] in the finite case. In the infinite case the homotopy group has a form of a direct product $\mathbb{Z} \times \mathbb{Z}_n$. We have an exact formula determining $n$.

Having homology classes we analyse fundamental groups in each homology class first determining some group invariants such as centers, low index subgroups, normal low index subgroups and the respective quotients. If the invariants of given two groups in the same homology class coincide, we test the isomorphism of the groups. As a result we have got the following theorem.

**Theorem 3.** *Prime 3-manifolds of genus at most two, represented by admissible 6-tuples of complexity* $\leq 21$, *have fundamental groups of one of the following types:*

- *trivial group,*
- *cyclic groups* $\mathbb{Z}_n, 2 \leq n \leq 29$,
- *infinite cyclic group* $\mathbb{Z}$,
- *acyclic groups.*

*The list includes **78** isomorphism classes of acyclic fundamental groups of prime 3-manifolds of genus two. Among them, there are **39** elliptic manifolds with finite groups, **4** Euclidean manifolds and **35** other manifolds with infinite fundamental groups.*

Note that there are also non-prime 3-manifolds of genus two, which fundamental groups are connected sums of two lens spaces, or of a lens space and $S^1 \times S^2$.

Since the classification of 3-manifolds of genus less than two is known, cyclic groups are "less interesting" for further study. The list of all acyclic fundamental groups of 3-manifolds of genus two follows. The list is divided into three parts including at first elliptic 3-manifolds, at second Euclidean 3-manifolds and at the end, 3-manifolds with other geometries according to [12].

**Fundamental groups of prime elliptic 3-manifolds of genus two with finite fundamental group (Classes F)**

| | Presentation | Abelianisation |
|---|---|---|
| F.1 | $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab)^2 \rangle$ | $H_1 = 1$ |
| F.2 | $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab)^2 \rangle$ | $H_1 = \mathbb{Z}_2$ |
| F.3 | $\pi_1(f) = \langle a, b \mid a^2 = b^2 = (ab)^2 \rangle$ | $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ |
| F.4 | $\pi_1(f) = \langle a, b \mid a^4 = b^2 = (ab)^2 \rangle$ | $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ |
| F.5 | $\pi_1(f) = \langle a, b \mid a^6 = b^2 = (ab)^2 \rangle$ | $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ |
| F.6 | $\pi_1(f) = \langle a, b \mid a^8 = b^2 = (ab)^2 \rangle$ | $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ |
| F.7 | $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^2 \rangle$ | $H_1 = \mathbb{Z}_3$ |
| F.8 | $\pi_1(f) = \langle a, b \mid a^3 = b^2 = (ab)^2 \rangle$ | $H_1 = \mathbb{Z}_4$ |
| F.9 | $\pi_1(f) = \langle a, b \mid a^5 = b^2 = (ab)^2 \rangle$ | $H_1 = \mathbb{Z}_4$ |
| F.10 | $\pi_1(f) = \langle a, b \mid a^7 = b^2 = (ab)^2 \rangle$ | $H_1 = \mathbb{Z}_4$ |
| F.11 | $\pi_1(f) = \langle a, b \mid a^2 = b^2 = (ab)^{-2} \rangle$ | $H_1 = \mathbb{Z}_6 \times \mathbb{Z}_2$ |
| F.12 | $\pi_1(f) = \langle a, b \mid a^4 = b^2 = (ab^{-1})^2 \rangle$ | $H_1 = \mathbb{Z}_6 \times \mathbb{Z}_2$ |
| F.13 | $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (a^2b)^2 \rangle$ | $H_1 = \mathbb{Z}_7$ |
| F.14 | $\pi_1(f) = \langle a, b \mid a^3 = b^2 = (a^2b)^2 \rangle$ | $H_1 = \mathbb{Z}_8$ |
| F.15 | $\pi_1(f) = \langle a, b \mid a^5 = b^2 = (a^2b)^2 \rangle$ | $H_1 = \mathbb{Z}_8$ |
| F.16 | $\pi_1(f) = \langle a, b \mid a^7 = (a^4b)^2, b^2 = (a^2b)^2 \rangle$ | $H_1 = \mathbb{Z}_8$ |
| F.17 | $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (a^{-1}b)^2 \rangle$ | $H_1 = \mathbb{Z}_9$ |
| F.18 | $\pi_1(f) = \langle a, b \mid a^4 = b^2 = (ab)^3 \rangle$ | $H_1 = \mathbb{Z}_{10}$ |
| F.19 | $\pi_1(f) = \langle a, b \mid a^2 = b^2 = (a^3b^3)^2 \rangle$ | $H_1 = \mathbb{Z}_{10} \times \mathbb{Z}_2$ |
| F.20 | $\pi_1(f) = \langle a, b \mid a^4 = b^2 = (ab)^{-2} \rangle$ | $H_1 = \mathbb{Z}_{10} \times \mathbb{Z}_2$ |
| F.21 | $\pi_1(f) = \langle a, b \mid a^6 = b^2 = (ab^{-1})^2 \rangle$ | $H_1 = \mathbb{Z}_{10} \times \mathbb{Z}_2$ |
| F.22 | $\pi_1(f) = \langle a, b \mid a^5 = b^{-2} = (ab^{-1})^3 \rangle$ | $H_1 = \mathbb{Z}_{11}$ |
| F.23 | $\pi_1(f) = \langle a, b \mid a^5 = b^2 = (a^2b^{-1})^2 \rangle$ | $H_1 = \mathbb{Z}_{12}$ |
| F.24 | $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (a^2b^{-1})^2 \rangle$ | $H_1 = \mathbb{Z}_{13}$ |
| F.25 | $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab^{-1})^2 \rangle$ | $H_1 = \mathbb{Z}_{14}$ |
| F.26 | $\pi_1(f) = \langle a, b \mid a^6 = b^2 = (ab)^{-2} \rangle$ | $H_1 = \mathbb{Z}_{14} \times \mathbb{Z}_2$ |
| F.27 | $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (a^{-2}b)^2 \rangle$ | $H_1 = \mathbb{Z}_{15}$ |
| F.28 | $\pi_1(f) = \langle a, b \mid a^3 = b^2 = (ab)^{-2} \rangle$ | $H_1 = \mathbb{Z}_{16}$ |
| F.29 | $\pi_1(f) = \langle a, b \mid a^5 = b^2 = (ab^{-1})^2 \rangle$ | $H_1 = \mathbb{Z}_{16}$ |
| F.30 | $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (a^2b^2)^3 \rangle$ | $H_1 = \mathbb{Z}_{17}$ |
| F.31 | $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab^{-1})^2 \rangle$ | $H_1 = \mathbb{Z}_{19}$ |
| F.32 | $\pi_1(f) = \langle a, b \mid a^3 = b^2 = (a^2b)^{-2} \rangle$ | $H_1 = \mathbb{Z}_{20}$ |
| F.33 | $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^{-2} \rangle$ | $H_1 = \mathbb{Z}_{21}$ |

| | | |
|---|---|---|
| F.34 | $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab^{-2})^2 \rangle$ | $H_1 = \mathbb{Z}_{22}$ |
| F.35 | $\pi_1(f) = \langle a, b \mid a^5 = b^2 = (ab)^{-2} \rangle$ | $H_1 = \mathbb{Z}_{24}$ |
| F.36 | $\pi_1(f) = \langle a, b \mid a^7 = b^2 = (ab^{-1})^2 \rangle$ | $H_1 = \mathbb{Z}_{24}$ |
| F.37 | $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab)^{-2} \rangle$ | $H_1 = \mathbb{Z}_{26}$ |
| F.38 | $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (a^2b)^{-2} \rangle$ | $H_1 = \mathbb{Z}_{27}$ |
| F.39 | $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab)^{-2} \rangle$ | $H_1 = \mathbb{Z}_{31}$ |

**Fundamental groups of prime Eucleidean 3-manifolds of genus two with infinite fundamental group (Classes E)**

| | Presentation | Abelianisation |
|---|---|---|
| E.1 | $\pi_1(f) = \langle a, b \mid a^2 = (ab^2)^2, b^2 = (a^2b)^2 \rangle$ | $H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$ |
| E.2 | $\pi_1(f) = \langle a, b \mid a^6 = b^3 = (ab)^2 \rangle$ | $H_1 = \mathbb{Z}$ |
| E.3 | $\pi_1(f) = \langle a, b \mid a^4 = b^4 = (ab)^2 \rangle$ | $H_1 = \mathbb{Z} \times \mathbb{Z}_2$ |
| E.4 | $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^3 \rangle$ | $H_1 = \mathbb{Z} \times \mathbb{Z}_3$ |

**Fundamental groups of prime 3-manifolds of genus two with infinite fundamental group (Classes I)**

| | Presentation | Abelianisation |
|---|---|---|
| I.1 | $\pi_1(f) = \langle a, b \mid a^7 = b^3 = (ab)^2 \rangle$ | $H_1 = 1$ |
| I.2 | $\pi_1(f) = \langle a, b \mid a^5 = b^4 = (ab)^2 \rangle$ | $H_1 = \mathbb{Z}_2$ |
| I.3 | $\pi_1(f) = \langle a, b \mid a^8 = b^3 = (ab)^2 \rangle$ | $H_1 = \mathbb{Z}_2$ |
| I.4 | $\pi_1(f) = \langle a, b \mid a^6 = b^4 = (ab)^2 \rangle$ | $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ |
| I.5 | $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab^{-2})^{-3} \rangle$ | $H_1 = \mathbb{Z}_3$ |
| I.6 | $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^3 \rangle$ | $H_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$ |
| I.7 | $\pi_1(f) = \langle a, b \mid a^4 = b^2a^{-1}b^2, a^3 = b^3a^{-3}b \rangle$ | $H_1 = \mathbb{Z}_4$ |
| I.8 | $\pi_1(f) = \langle a, b \mid a^2 = (a^{-1}b^2)^2, b^2 = (a^2b)^2 \rangle$ | $H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$ |
| I.9 | $\pi_1(f) = \langle a, b \mid a^5 = b^5 = (ab)^2 \rangle$ | $H_1 = \mathbb{Z}_5$ |
| I.10 | $\pi_1(f) = \langle a, b \mid a^7 = b^2 = (a^{-2}b)^3 \rangle$ | $H_1 = \mathbb{Z}_5$ |
| I.11 | $\pi_1(f) = \langle a, b \mid a^3b^{-1}a = ba^{-1}b^3 = (a^2b^2)^2 \rangle$ | $H_1 = \mathbb{Z}_5 \times \mathbb{Z}_5$ |
| I.12 | $\pi_1(f) = \langle a, b \mid a^5 = b^4 = (a^2b)^2 \rangle$ | $H_1 = \mathbb{Z}_6$ |
| I.13 | $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab^{-2})^{-3} \rangle$ | $H_1 = \mathbb{Z}_6$ |
| I.14 | $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (a^2b)^3 \rangle$ | $H_1 = \mathbb{Z}_6 \times \mathbb{Z}_3$ |
| I.15 | $\pi_1(f) = \langle a, b \mid a^4 = b^4 = (ab)^3 \rangle$ | $H_1 = \mathbb{Z}_8$ |
| I.16 | $\pi_1(f) = \langle a, b \mid a^4 = b^4 = (a^{-1}b)^2 \rangle$ | $H_1 = \mathbb{Z}_8 \times \mathbb{Z}_2$ |

| | | |
|---|---|---|
| I.17 | $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab)^3 \rangle$ | $H_1 = \mathbb{Z}_9$ |
| I.18 | $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab)^3 \rangle$ | $H_1 = \mathbb{Z}_9$ |
| I.19 | $\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^{-3} \rangle$ | $H_1 = \mathbb{Z}_9 \times \mathbb{Z}_3$ |
| I.20 | $\pi_1(f) = \langle a, b \mid a^6 = b^3 = (a^{-1}b)^2 \rangle$ | $H_1 = \mathbb{Z}_{12}$ |
| I.21 | $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab^{-1})^3 \rangle$ | $H_1 = \mathbb{Z}_{15}$ |
| I.22 | $\pi_1(f) = \langle a, b \mid a^4 = b^4 = (a^{-1}b)^3 \rangle$ | $H_1 = \mathbb{Z}_{16}$ |
| I.23 | $\pi_1(f) = \langle a, b \mid a^4 = b^4 = (ab)^{-2} \rangle$ | $H_1 = \mathbb{Z}_{16} \times \mathbb{Z}_2$ |
| I.24 | $\pi_1(f) = \langle a, b \mid a^5 = b^4 = (a^{-1}b)^2 \rangle$ | $H_1 = \mathbb{Z}_{18}$ |
| I.25 | $\pi_1(f) = \langle a, b \mid a^2 = b^2ab^2, a^3 = ba^{-3}b^3 \rangle$ | $H_1 = \mathbb{Z}_{20}$ |
| I.26 | $\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab^{-1})^3 \rangle$ | $H_1 = \mathbb{Z}_{21}$ |
| I.27 | $\pi_1(f) = \langle a, b \mid a^5 = b^4 = (ab^{-1})^2 \rangle$ | $H_1 = \mathbb{Z}_{22}$ |
| I.28 | $\pi_1(f) = \langle a, b \mid a^6 = b^3 = (ab^{-1})^2 \rangle$ | $H_1 = \mathbb{Z}_{24}$ |
| I.29 | $\pi_1(f) = \langle a, b \mid a = b^2a^2b^2, b = a^3ba^3 \rangle$ | $H_1 = \mathbb{Z}_{24}$ |
| I.30 | $\pi_1(f) = \langle a, b \mid a^{-3} = ba^3b^{-1}, a^4 = b^2a^{-1}b^2 \rangle$ | $H_1 = \mathbb{Z}_{24}$ |
| I.31 | $\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab)^{-3} \rangle$ | $H_1 = \mathbb{Z}_{33}$ |
| I.32 | $\pi_1(f) = \langle a, b \mid a^2ba^{-1} = b^{-1}ab^2, b^{-1}a^2b^{-1} = ab^{-2}a \rangle$ | $H_1 = \mathbb{Z}$ |
| I.33 | $\pi_1(f) = \langle a, b \mid a^3 = b^{-1}ab^2ab^{-1}, ab^{-1}ab = b^{-1}a^{-1}ba^{-1} \rangle$ | $H_1 = \mathbb{Z}$ |
| I.34 | $\pi_1(f) = \langle a, b \mid a^3 = b^2a^{-1}b^2, b^3 = a^2b^{-1}a^2 \rangle$ | $H_1 = \mathbb{Z} \times \mathbb{Z}_4$ |
| I.35 | $\pi_1(f) = \langle a, b \mid ab^{-2}a^2 = b^2a^{-2}b, ba^{-2}b = a^{-1}b^2a^{-1} \rangle$ | $H_1 = \mathbb{Z} \times \mathbb{Z}_5$ |

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