

CONDITIONAL EQUI-CONCENTRATION OF TYPES

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To Mar, in memoriam

ABSTRACT. Conditional Equi-concentration of Types on I-projections is presented. It provides an extension of Conditioned Weak Law of Large Numbers to the case of several I-projections. Also a multiple I-projections extension of Gibbs Conditioning Principle is developed. A resemblance between the probabilistic equi-concentration phenomenon and thermodynamic coexistence of phases is noted. μ -projection variants of the probabilistic laws are stated. Conditional Equi-concentration of Types on J and γ -projections as well as on OR-projections is mentioned. Implications of the results for Maximum Probability, Relative Entropy Maximization, Maximum Entropy in the Mean and Maximum Rényi-Tsallis Entropy methods are discussed.

1. TERMINOLOGY AND NOTATION

Let $\{X\}_{i=1}^n$ be a sequence of independently and identically distributed random variables with a common law (measure) on a measurable space. Let the measure be concentrated on m atoms from a set $X = \{x_1, x_2, \dots, x_m\}$ called support or alphabet. Hereafter X will be assumed finite. Let q_i denote the probability (measure) of i -th element of X ; q will be called source or generator. Let $P(X)$ be a set of all probability mass functions (pmf's) on X .

A type (also called n -type, empirical measure, frequency distribution or occurrence vector) induced by a sequence $\{X\}_{i=1}^n$ is the pmf $\nu^n \in P(X)$ whose i -th element ν_i^n is defined as: $\nu_i^n = n_i/n$ where $n_i = \sum_{l=1}^n I(X_l = x_i)$; there $I(\cdot)$ is the characteristic function. Multiplicity $\Gamma(\nu^n)$ of type ν^n is: $\Gamma(\nu^n) = n! / \prod_{i=1}^m n_i!$.

Let $\Pi \subseteq P(X)$. Let P_n denote a subset of $P(X)$ which consists of all n -types. Let $\Pi_n = \Pi \cap P_n$.

μ -projection $\hat{\nu}^n$ of q on $\Pi_n \neq \emptyset$ is defined as: $\hat{\nu}^n = \arg \sup_{\nu^n \in \Pi_n} \pi(\nu^n; q)$, where $\pi(\nu^n; q) = \Gamma(\nu^n) \prod_{i=1}^m (q_i)^{n\nu_i^n}$. Alternatively, the μ -projection can be defined as $\hat{\nu}^n = \arg \sup_{\nu^n \in \Pi_n} \pi(\nu^n | \nu^n \in \Pi_n; q)$, where $\pi(\nu^n | \nu^n \in \Pi_n; q)$ denotes the conditional probability that if an n -type belongs to Π_n then it is just the type ν^n . μ -projection can be also equivalently defined as a supremum a-posteriori probability, cf. [15].

I-projection \hat{p} of q on Π is $\hat{p} = \arg \inf_{p \in \Pi} I(p||q)$, where $I(p||q) = \sum_X p_i \log \frac{p_i}{q_i}$ is I-divergence, Kullback-Leibler distance, or minus relative entropy.

$\pi(\nu^n \in A | \nu^n \in B; q \mapsto \nu^n)$ will denote the conditional probability that if a type drawn from $q \in P(X)$ belongs to $B \subseteq \Pi$ then it belongs to $A \subseteq \Pi$.

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2. INTRODUCTION

Consider the following Boltzmann-Jaynes Inverse Problem (BJIP): Let there be a set $\Pi_n \subseteq P(X)$ of n -types, which come from source \mathbf{q} . BJIP amounts to selection of specific type(s) from Π_n , when the information-quadruple $\{X, n, \mathbf{q}, \Pi_n\}$ is supplied.

Under the BJIP setup, the Relative Entropy Maximization (REM/MaxEnt) method is usually applied for selection of type(s). It is done mostly when $n \rightarrow \infty$, so that the set of types effectively turns into a set of probability mass functions.

Moreover, traditionally Π is defined by moment consistency constraints (mcc) of the following form¹: $\Pi_{\text{mcc}} = \{p : \sum_{i=1}^m p_i u_i = \mathbf{a}, \sum_{i=1}^m p_i = 1\}$, where $\mathbf{a} \in \mathbb{R}$ is a given number, \mathbf{u} is a given vector. The feasible set Π_{mcc} which mcc define is convex and closed. I-projection \hat{p} of \mathbf{q} on Π_{mcc} is unique and belongs to the exponential family of distributions; $\hat{p}_i = k(\lambda) q_i e^{-\lambda u_i}$, where $k(\lambda) = 1 / \sum_{i=1}^m q_i e^{-\lambda u_i}$, and λ is such that \hat{p} satisfies mcc.

In the case of BJIP with Π_{mcc} , or in general: for any closed, convex set Π , application of REM/MaxEnt method is justified by Conditioned Weak Law of Large Numbers (CWLLN). CWLLN, in one of its forms, reads:

CWLLN. *Let X be a finite set. Let Π be closed, convex set. Let $n \rightarrow \infty$. Then for $\epsilon > 0$ and $i = 1, 2, \dots, m$:*

$$\lim_{n \rightarrow \infty} \pi(|v_i^n - \hat{p}_i| < \epsilon | v^n \in \Pi; \mathbf{q} \mapsto v^n) = 1.$$

CWLLN says that if types are confined to a convex, closed set Π then they asymptotically conditionally concentrate on the I-projection \hat{p} of the source of types \mathbf{q} on the set Π . Stated, informally, from another side: if a source \mathbf{q} is confined to produce types from convex and closed Π it is asymptotically conditionally 'almost impossible' to find a type other than the one which has the highest/supremal value of relative entropy wrt \mathbf{q} .

Conditioned Weak Law of Large Numbers emerged from a series of works which include [1], [33], [32], [31], [5], [9]. For a new developments see [20].

An information-theoretic proof (see [4]) of CWLLN utilizes so-called Pythagorean theorem (cf. [2]), Pinsker inequality and standard inequalities for factorial. The Pythagorean theorem is known to hold for closed, convex sets. Alternatively, CWLLN can be obtained as a consequence of Sanov's Theorem (ST). The ST-based proof of CWLLN will be recalled here. First, Sanov's Theorem and its proof (adapted from [7], [4]).

Sanov's Theorem. *Let X be finite. Let $A \subseteq \Pi$ be an open set. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \pi(v^n \in A) = -I(\hat{p} \| \mathbf{q}),$$

where \hat{p} is the I-projection of \mathbf{q} on A .

Proof. [4], [7] $\pi(v^n \in A) = \sum_{v^n \in A} \pi(v^n; \mathbf{q})$. Upper and lower bounds on $\pi(v^n; \mathbf{q})$ (recall proof of the Lemma at Appendix, Sect. 10.1):

$$\left(\frac{m}{n}\right)^m \prod_{i=1}^m \left(\frac{q_i}{v_i^n}\right)^{n v_i^n} < \pi(v^n; \mathbf{q}) \leq \prod_{i=1}^m \left(\frac{q_i}{v_i^n}\right)^{n v_i^n}.$$

$\sum_{v^n \in A} \pi(v^n; \mathbf{q}) < N \prod_{i=1}^m \left(\frac{q_i}{\hat{v}_i^n}\right)^{n \hat{v}_i^n}$, where N stands for number of all n -types and \hat{v}_i^n is the I-projection of \mathbf{q} on $A_n = A \cap P_n$ (i.e., the n -type which attains supremal value of $\prod_{i=1}^m \left(\frac{q_i}{v_i^n}\right)^{n v_i^n}$). N is smaller than $(n+1)^m$.

¹In the simplest case of single non-trivial constraint.

Thus

$$\begin{aligned} \frac{1}{n} \left(n \sum_{i=1}^m \hat{v}_i^n \log \frac{q_i}{\hat{v}_i^n} + m(\log m - \log n) \right) &< \frac{1}{n} \log \pi(\mathbf{v}^n \in A) \\ &< \frac{1}{n} \left(n \sum_{i=1}^m \hat{v}_i^n \log \frac{q_i}{\hat{v}_i^n} + m \log(n+1) \right). \end{aligned}$$

Since A is by the assumption open, $\lim_{n \rightarrow \infty} \sum \hat{v}_i^n \log \frac{q_i}{\hat{v}_i^n} = \sum \hat{p}_i \log \frac{q_i}{\hat{p}_i}$, where \hat{p} is the I-projection of q on A . Thus, for $n \rightarrow \infty$ the upper and lower bounds on $\frac{1}{n} \log \pi(\mathbf{v}^n \in A)$ collapse into $\sum_{i=1}^m \hat{p}_i \log \frac{q_i}{\hat{p}_i}$. \square

A proof of CWLLN. [7] Take $A \subseteq \Pi$ to be an open set, such that it does not contain \hat{p} (i.e., the I-projection of q on Π). Then by ST $\lim_{n \rightarrow \infty} \frac{1}{n} \log \pi(\mathbf{v}^n \in A | \mathbf{v}^n \in \Pi; q \mapsto \mathbf{v}^n) = -(I(\hat{p}_A \| q) - I(\hat{p}_\Pi \| q))$. Since $I(\hat{p}_A \| q) - I(\hat{p}_\Pi \| q) > 0$ and since the set Π admits unique I-projection (the uniqueness arises from the fact that the set is convex and closed, and $I(\cdot \| \cdot)$ is convex), the proof is complete. \square

Frequency moment constraints considered by physicists (see for instance [28]) define a non-convex feasible set of probability distributions which in general can admit multiple I-projections. This work builds upon [13], [24], [14], [16], [17] and aims to develop an extension of CWLLN and Gibbs Conditioning Principle to the case of multiple I-projections.

It has also another goal: to introduce concept of μ -projection and to formulate μ -projection variants of the probabilistic laws. They, among other things, allow for a more elementary reading than their I-projection counterparts. At the same time they provide a probabilistic justification to Maximum Probability method [12].

The paper is organized as follows: in the next section Conditional Equi-concentration of Types on I-projections is stated; a sketch of its proof and illustrative examples are relegated to the Appendix. Next, an extension of Gibbs Conditioning Principle - a stronger form of CWLLN - is provided. Asymptotic identity of I-projections and μ -projections is discussed at Section 5; and μ -variants of the probabilistic laws are presented afterwards. Implications of the results for Maximum Entropy, Maximum Probability and Maximum Rényi-Tsallis Entropy methods are drawn at Section 7. Section 8 contains further results: Conditional Equi-concentration of Types on J- and γ -projections as well as r -tuple extension of CWLLN and the equi-concentration. Section 9 summarizes the paper.

3. CONDITIONAL EQUI-CONCENTRATION OF TYPES

What happens when Π admits multiple I-projections? Do the conditional concentration of types happens on them? If yes, do types concentrate on each of them? If yes, what is the proportion? Conditional Equi-concentration of Types on I-projections (ICET) provides an answer to the questions.

Let $d(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^m |a_i - b_i|$ be the total variation metric (or any other equivalent metric) on the set of probability distributions $P(X)$. The space is compact. Let $B(\mathbf{a}, \epsilon)$ denote an ϵ -ball - defined by the metric d - which is centered at $\mathbf{a} \in P(X)$.

An I-projection \hat{p} of q on Π will be called proper if \hat{p} is not isolated point of Π .

ICET. *Let X be a finite set. Let Π be such that it admits k proper I-projections $\hat{p}^1, \hat{p}^2, \dots, \hat{p}^k$ of q . Let $\epsilon > 0$ be such that for $j = 1, 2, \dots, k$ \hat{p}^j is the only proper I-projection of q on Π in the ball $B(\hat{p}^j, \epsilon)$. Let $n \rightarrow \infty$. Then for $j = 1, 2, \dots, k$:*

$$\pi(\mathbf{v}^n \in B(\epsilon, \hat{p}^j) | \mathbf{v}^n \in \Pi; q \mapsto \mathbf{v}^n) = 1/k.$$

Conditional Equi-concentration of Types on I-projections (ICET) states that if a set Π admits several I-projections then for n growing to infinity the conditional probability (measure) becomes split among the proper I-projections equally.

Notes. 1) On an I-projection \hat{p} which is not rational and at the same time it is an isolated point no conditional concentration of types happens. However, if a set Π is such that an I-projection \hat{p} of q on the set is rational, then types can concentrate on it even if it is isolated point; see Example 5 at the Appendix, Sect. 10.3. 2) Since X is finite, k is finite. 3) If Π admits unique proper I-projection then ICET reduces to CWLLN. 4) Weak Law of Large Numbers is special - unconditional - case of CWLLN. CWLLN itself is just a special - unique proper I-projection - case of ICET. 5) Two illustrative examples of ICET, taken from the exploratory study [14] are at the Appendix, Sect. 10.2. At [14] Asymptotic Equiprobability of I-projections - a precursor to ICET - was formulated.

ICET says, informally, that source/generator q , when confined to produce types from a set Π , - as n gets large - hides itself behind any of the proper I-projections equally likely. Expressed in Statistical Physics terminology ICET says that each of equilibrium points (I-projections) is asymptotically conditionally equally possible. Perhaps, it can be also noted here that the Conditional Equi-concentration of Types 'phenomenon' resembles the phase coexistence (i.e., triple point of water, ice and vapor) phenomena of Thermodynamics.

4. GIBBS CONDITIONING PRINCIPLE AND ITS EXTENSION

Gibbs conditioning principle (cf. [5], [8], [26]) - also known as a stronger form of CWLLN - complements CWLLN by stating that:

GCP. *Let X be a finite set. Let Π be closed, convex set. Let $n \rightarrow \infty$. Then for a fixed t*

$$\lim_{n \rightarrow \infty} \pi(X_1 = x_1, \dots, X_t = x_t | v^n \in \Pi; q \mapsto n) = \prod_{l=1}^t \hat{p}_{x_l}.$$

GCP, says, very informally, that if the source q is confined to produce sequences which lead to types in a closed, convex set Π then elements of any such sequence (of fixed length t) behave asymptotically conditionally as if they were drawn identically and independently from the I-projection of q on Π .

GCP was developed at [5] under the name of conditional quasi-independence of outcomes. Later on, it was brought into more abstract form in large deviations literature, where it also obtained the GCP name (cf. [8], [26]). A simple proof of GCP can be found at [7]. GCP is proven also for continuous alphabet (cf. [18], [7], [8]).

The following theorem provides an extension of GCP to the case of multiple I-projections.

EGCP. *Let X be a finite set. Let Π be such that it admits k proper I-projections $\hat{p}^1, \hat{p}^2, \dots, \hat{p}^k$ of q . Then for a fixed t and $n \rightarrow \infty$:*

$$\pi(X_1 = x_1, \dots, X_t = x_t | v^n \in \Pi; q \mapsto n) = \frac{1}{k} \sum_{j=1}^k \prod_{l=1}^t \hat{p}_{x_l}^j.$$

For $t = 1$ Extended Gibbs Conditioning Principle (EGCP) says that the conditional probability of a letter is asymptotically given by the equal-weight mixture of proper I-projection probabilities of the letter. For a general length sequence, EGCP states that the conditional probability of a sequence is asymptotically equal to the

mixture of joint probability distributions. Any (j-th) of the k joint distributions is such as if the sequence was iid distributed according to a (j-th) proper I-projection.

A proof of EGCP is sketched at the Appendix, Sect. 10.4.

5. ASYMPTOTIC IDENTITY OF μ -PROJECTIONS AND I-PROJECTIONS

At ([12], Thm 1 and its Corollary, aka MaxProb/MaxEnt Thm) it was shown that maximum probability type converges to I-projection; provided that Π is defined by a differentiable constraints. A more general result which states asymptotic identity of μ -projections and I-projections for general sets Π was presented at [17].

MaxProb/MaxEnt. [17] *Let X be a finite set. Let M_n be set of all μ -projections of q on Π_n . Let I be set of all I-projections of q on Π . For $n \rightarrow \infty$, $M_n = I$.*

Proof. [17] Necessary and sufficient conditions for \hat{v}^n to be a μ -projection of q on Π_n are: *a)* $\pi(\hat{v}^n; q) \geq \pi(v^n; q)$, $\forall v^n \in \Pi_n$; *b)* whenever \tilde{v}^n has the property *a)* then $\pi(\hat{v}^n; q) \leq \pi(\tilde{v}^n; q)$. Requirement *a)* can be equivalently stated as:

$$(1) \quad \left(\prod \frac{n_i!}{\hat{n}_i!} \right)^{1/n} \geq \left(\prod q_i^{n_i - \hat{n}_i} \right)^{1/n}$$

and *b)* similarly. Standard inequality $(n/e)^n < n! < n(n/e)^n$ (valid for $n > 6$) allows to bind the LHS of (1):

$$(2) \quad \frac{\prod (v_i^n)^{v_i^n}}{n^{m/n} \prod (\hat{v}_i^n)^{\hat{v}_i^n} (\prod \hat{v}_i^n)^{1/n}} < \text{LHS} < \frac{n^{m/n} \prod (v_i^n)^{v_i^n} (\prod v_i^n)^{1/n}}{\prod (\hat{v}_i^n)^{\hat{v}_i^n}}$$

and similar bounds can be stated in the case of the requirement *b)*². Since m is by assumption finite, as $n \rightarrow \infty$ the lower and upper bounds at (2) collapse into $\prod (v_i^n)^{v_i^n} / \prod (\hat{v}_i^n)^{\hat{v}_i^n}$. Consequently, the necessary and sufficient conditions *a)*, *b)* for μ -projection turn as $n \rightarrow \infty$ into (expressed in an equivalent log-form): *i)* $\sum (v_i^n \log v_i^n - \hat{v}_i^n \log \hat{v}_i^n) \geq \sum (v_i^n - \hat{v}_i^n) \log q_i$ for all $v^n \in \Pi_n$; and *ii)* whenever \tilde{v}^n has the property *i)* then $\sum \hat{v}_i^n \log \hat{v}_i^n - \tilde{v}_i^n \log \tilde{v}_i^n \geq \sum (\hat{v}_i^n - \tilde{v}_i^n) \log q_i$.

Necessary and sufficient conditions for \hat{p} to be an I-projection of q on Π are: *I)* $\sum (p_i \log p_i - \hat{p}_i \log \hat{p}_i) \geq \sum (p_i - \hat{p}_i) \log q_i$ for all $p \in \Pi$; and *II)* whenever \tilde{p} has the property *I)* then $\sum \hat{p}_i \log \hat{p}_i - \tilde{p}_i \log \tilde{p}_i \geq \sum (\hat{p}_i - \tilde{p}_i) \log q_i$.

A comparison of *i)*, *ii)* and *I)*, *II)* then completes the proof. \square

Since $\pi(v^n; q)$ is defined for $v^n \in Q^m$, μ -projection can be defined only for Π_n when n is finite. MaxProb/MaxEnt Thm permits to define a μ -projection \hat{v} also on Π : $\hat{v} = \arg \sup_{r \in \Pi} - \sum_{i=1}^m r_i \log \frac{r_i}{q_i}$. Thus, μ -projections and I-projections on Π are identical.

It is worth highlighting that for a finite n , μ -projections and I-projections of q on Π_n are in general different. This explains why μ -form of the probabilistic laws deserves to be stated separately of the I-form; though formally they are undistinguishable. Thus, the MaxProb/MaxEnt Thm (in its new and to a smaller extent also in its old version) permits directly to state μ -projection variants of CWLLN, GCP, ICET and EGCP: μ CWLLN, μ GCP, μ CET and Boltzmann Conditioning Principle (BCP).

²Note that if an i -th component v_i^n of a type is zero then it can be effectively omitted from calculations of $\pi(v^n; q)$. Thus, it is assumed that product operations at (1), (2) are performed on non-zero components only.

6. μ -VARIANTS OF THE PROBABILISTIC LAWS

μ -variant of CWLLN reads:

μ CWLLN. *Let X be a finite set. Let Π be closed, convex set. Let $n \rightarrow \infty$. Then for $\epsilon > 0$ and $i = 1, 2, \dots, m$:*

$$\lim_{n \rightarrow \infty} \pi(|v_i^n - \hat{v}_i| < \epsilon | v^n \in \Pi; q \mapsto v^n) = 1.$$

Core of μ CWLLN can be loosely expressed as: *types, when confined to a convex, closed set Π , conditionally concentrate on the asymptotically most probable type \hat{v} .*

A μ -projection \hat{v} of q on Π will be called proper if \hat{v} is not isolated point of Π .

μ CET. *Let X be a finite set. Let there be k proper μ -projections $\hat{v}^1, \hat{v}^2, \dots, \hat{v}^k$ of q on Π . Let $\epsilon > 0$ be such that for $j = 1, 2, \dots, k$ \hat{v}^j is the only proper μ -projection of q on Π in the ball $B(\hat{v}^j, \epsilon)$. Let $n \rightarrow \infty$. Then for $j = 1, 2, \dots, k$:*

$$\pi(v^n \in B(\epsilon, \hat{v}^j) | v^n \in \Pi; q \mapsto v^n) = 1/k.$$

Core of μ -variant of the Conditional Equi-concentration of Types states, loosely, that types conditionally concentrate on each of the asymptotically most probable types in equal measure.

μ GCP. *Let X be a finite set. Let Π be closed, convex set. Let $n \rightarrow \infty$. Then for a fixed t :*

$$\lim_{n \rightarrow \infty} \pi(X_1 = x_1, \dots, X_t = x_t | v^n \in \Pi; q \mapsto v^n) = \prod_{l=1}^t \hat{v}_{x_l}.$$

μ -variant of EGCP deserves a special name. It will be called Boltzmann Conditioning Principle (BCP).

BCP. *Let X be a finite set. Let there be k proper μ -projections $\hat{v}^1, \hat{v}^2, \dots, \hat{v}^k$ of q on Π . Then for a fixed t and $n \rightarrow \infty$:*

$$\pi(X_1 = x_1, \dots, X_t = x_t | v^n \in \Pi; q \mapsto v^n) = \frac{1}{k} \sum_{j=1}^k \prod_{l=1}^t \hat{v}_{x_l}^j.$$

7. IMPLICATIONS

7.1. I- or μ -Projection? MaxEnt or MaxProb? With μ -projection, the Maximum Probability method (MaxProb, [12]) is associated. Given the BJIP information-quadruple $\{X, n, q, \Pi_n\}$, MaxProb prescribes to select from Π_n type(s) which has the supremal/maximal probability $\pi(v^n; q)$.

μ -projections and I-projections are asymptotically indistinguishable (recall MaxProb/MaxEnt Thm). In plain words: for $n \rightarrow \infty$ the Relative Entropy Maximization method (REM/MaxEnt) (either in its Jaynes' [21], [22] or Csiszár's interpretation [6]) selects the same distribution(s) as MaxProb (in its more general form which instead of the maximum probable types selects supremum-probable μ -projections). This result (in the older form, [12]) was at [12] interpreted as saying that REM/MaxEnt can be viewed as an asymptotic instance of the simple and self-evident Maximum Probability method.

Alternatively, [29] suggests to view REM/MaxEnt as a separate method and hence to read the MaxProb/MaxEnt Thm as claiming that REM/MaxEnt asymptotically coincides with MaxProb. If one adopts this interesting and legitimate view then it is necessary to face the fact that if n is finite, the two methods in general differ. This'd open new questions. First of all: MaxEnt/REM or MaxProb? (i.e., I- or μ -projection?).

7.2. I/ μ - or τ -projection? MaxEnt/MaxProb or maxTent? The previous question (i.e., MaxEnt or MaxProb?) is a problem of drawing an interpretational consequences from two variants of the same probabilistic laws, and in this sense it can be viewed as an 'internal problem' of MaxEnt and MaxProb. From outside, from the point of view of the Maximum Rényi-Tsallis Entropy method (maxTent, [30], [23], [3]) MaxProb and MaxEnt can be viewed as 'twins'.

maxTent is to the best of our knowledge intended by its proponents for selection of probability distribution(s) under the setting of BJIP with Π defined by X -frequency moment constraints (cf. [16]). It is not known whether such a feasible set Π admits unique distribution with maximal value of Rényi-Tsallis entropy (called τ -projection at [16]) as it is also not known whether I-projection on such a set is unique or not. The non-uniqueness makes it difficult to rely upon CWLLN when one wants to draw from an established non-identity of τ and I-projection conclusion that maxTent method violates CWLLN [24]. At [16] this difficulty was avoided by considering an instance of the X -frequency constraints such that the feasible set becomes convex; and hence CWLLN can be invoked. Since τ - and I-projection on such a set are different CWLLN directly implies that maxTent selects asymptotically conditionally improbable distribution. The Example below (taken from [16]) illustrates the point.

Example 1. [16] Let $\Pi = \{p : \sum_{i=1}^3 p_i^2(x_i - b) = 0, \sum_{i=1}^3 p_i - 1 = 0\}$. Let $X = [-2 \ 0 \ 1]$ and let $b = 0$. Then $\Pi = \{p : p_3^2 = 2p_1^2, \sum p_i - 1 = 0\}$ which effectively reduces to $\Pi = \{p : p_2 = 1 - p_1(1 + \sqrt{2}), p_3 = \sqrt{2}p_1\}$. The source q is assumed to be uniform u .

The feasible set Π is convex. Thus I-projection \hat{p} of u on Π is unique, and can be found by direct analytic maximization to be $\hat{p} = [0.2748 \ 0.3366 \ 0.3886]$. Straightforward maximization of Rényi-Tsallis' entropy lead to maxTent pmf $\hat{p}_T = [0.2735 \ 0.3398 \ 0.3867]$, which is different than \hat{p} .

Convexity of the feasible set guarantees uniqueness of the I-projection, and consequently allows to invoke CWLLN to claim that any pmf from Π other than the I-projection has asymptotically zero conditional probability that it will be generated by u . \diamond

Obviously, ICET permits to show the fatal flow of maxTent in a more direct and more general way.

8. FURTHER RESULTS

8.1. Conditional Equi-concentration of Types on Jeffreys Projections.

Within BJIP, it is possible to assume that q is the source of types and at the same time, any of the n -types is supposed to produce q as an n -type. With this, concept of J - and γ -projection is associated.

γ -projection \tilde{v}^n of $q \in Q^m$ on Π_n is $\tilde{v}^n = \arg \sup_{v^n \in \Pi_n} \pi(v^n; q) \pi(q; v^n)$. J -projection (or Jeffreys projection) \tilde{p} of $q \in Q^m$ on Π is $\tilde{p} = \arg \inf_{p \in \Pi} \sum_{i=1}^m p_i \log \frac{p_i}{q_i} + q_i \log \frac{q_i}{p_i}$.

Let $q \in Q^m$. $\pi(v^n \in A | v^n \in B; (q \mapsto v^n) \wedge (v^n \mapsto q))$ will denote the conditional probability that if a type - which was drawn from $q \in P(X)$ and was at the same time used as a source of the type q - belongs to $B \subseteq \Pi$ then it belongs to $A \subseteq \Pi$.

A J -projection \tilde{p} of q on Π will be called proper if \tilde{p} is not isolated point of Π .

Conditional Equi-concentration of types on J -projections (ICET) states:

ICET. Let X be a finite set. Let $q \in Q^m$. Let there be k proper J -projections $\tilde{p}^1, \tilde{p}^2, \dots, \tilde{p}^k$ of q on Π . Let $\epsilon > 0$ be such that for $j = 1, 2, \dots, k$ \tilde{p}^j is the only proper J -projection of q on Π in the ball $B(\tilde{p}^j, \epsilon)$. Let n_0 be denominator of the

smallest common divisor of q_1, q_2, \dots, q_m . Let $n = un_0$, $u \in \mathbb{N}$. Let $u \rightarrow \infty$. Then

$$\pi(\nu^n \in B(\epsilon, \tilde{p}^j) | \nu^n \in \Pi; (q \mapsto \nu^n) \wedge (\nu^n \mapsto q)) = 1/k \text{ for } j = 1, 2, \dots, k.$$

In words, types which were 'emitted' from q and were at the same time used as a source of q types, conditionally equi-concentrate on J-projections of q on Π .

Similarly, a Jeffreys conditioning principle³ (JCP) can be formulated, in analogy with EGCP.

Along the lines of proof MaxProb/MaxEnt Thm it is not difficult to show that J-projections and γ -projections asymptotically coincide when the limiting process is the same as at the JCET, above (cf. [17], and [11] for an illustrative example). Hence, a γ -projection alternative of JCET is valid as well. It says that: types which were 'emitted' from q and were at the same time used as a source of q types, conditionally equi-concentrate on those of them which have the highest/supremal value of $\pi(\nu^n; q) \pi(nq; \nu^n)$. – Similarly, JCP can be considered in its J- or γ -form.

μ -projection is based on the probability $\pi(\nu^n; q)$; thus it can be viewed as a UNI-projection. γ -projection is based on $\pi(\nu^n; q) \pi(nq; \nu^n)$, thus it can be viewed as AND-projection. It is possible to consider also an OR-projection defined as $\hat{\nu}^n = \arg \sup_{\nu^n \in \Pi_n} \pi(\nu^n; q) + \pi(nq; \nu^n)$. However, there seems to be no obvious analytic way how to define its asymptotic form. Despite that, it is possible to expect that OR-type of CWLLN/CET holds.

The μ -, γ -, OR-projection CET can be summarized by a (bold) statement: types conditionally equi-concentrate on those which are asymptotically the most probable.

8.2. r-tuple ICET/CWLLN and MEM/GME methods. Maximum Entropy in the Mean method (MEM), or Generalized Maximum Entropy (GME) - its discrete-case relative - are interesting extensions of the standard REM/MaxEnt method⁴. Though, usually a hierarchical structure of the methods is highlighted, here a different feature of the method(s) will appear to be important.

First, a Golan-Judge-Miller ill-posed inverse problem (GJMIP) has to be introduced. Its simple instance can be described as follows: Let there be two independent sources q^1, q^2 of sequences and hence types. Let X, Y be support of the first, second source, respectively. Let a set C_n comprise *pairs* of the types $[\nu^{n,1}, \nu^{n,2}]$ which were drawn *at the same time*. GJMIP amounts to selection of specific pair(s) of types from C_n when the information $\{X, Y, n, q^1 \perp q^2, C_n\}$ is supplied.

Example 2. An example of GJMIP. Let $X = Y = [1 \ 2 \ 3]$. Let $q^1 = q^2 = [1/3 \ 1/3 \ 1/3]$; $q^1 \perp q^2$; $(q^1 \mapsto \nu^{n,1}) \wedge (q^2 \mapsto \nu^{n,2})$. Let $n = 100$, $C_n = \{[\nu^{n,1}, \nu^{n,2}] : \sum_{i=1}^3 \nu_i^{n,1} x_i + \nu_i^{n,2} y_i = 4; \sum_{i=1}^3 \nu_i^{n,1} = 1; \sum_{i=1}^3 \nu_i^{n,2} = 1\}$. Given this information, we are asked to select a pair (one or more) of types from C_n . \diamond

Since throughout the paper discrete and finite alphabet is assumed, GME will be considered instead of MEM, in what follows. The important feature of GME is that it selects jointly and independently drawn pairs (or r-tuples) of types/pmfs. Thus, it is suitable for application at the GJMIP context. An r-tuple extension of CWLLN (rCWLLN) provides a probabilistic justification to the GME, at the GJMIP.

Given GJMIP information, GME selects from the feasible set of the pairs of pmfs the one $[\hat{p}^1, \hat{p}^2]$ (or more) which maximize sum of the relative entropies wrt q^1, q^2 ; respectively.

³It should not be confused with Jeffrey principle of updating subjective probability.

⁴For a tutorial on MEM see [19]. GME was introduced at [10], see also [27].

($r = 2$)-tuple CWLLN. Assume a GJMIP. Let C be convex, closed set. Let $B([\hat{p}^1, \hat{p}^2], \epsilon)$ be an ϵ -ball centered at the pair

$$[\hat{p}^1, \hat{p}^2] = \arg \sup_{[p^1, p^2] \in C} \sum_{i=1}^{m^1} p_i^1 \log \frac{p_i^1}{q_i^1} + \sum_{j=1}^{m^2} p_j^2 \log \frac{p_j^2}{q_j^2}.$$

Let $n \rightarrow \infty$. Then

$$\begin{aligned} \pi([v^{n,1}, v^{n,2}] \in B([\hat{p}^1, \hat{p}^2], \epsilon) \mid [v^{n,1}, v^{n,2}] \in C; (q^1 \mapsto v^{n,1}) \wedge (q^2 \mapsto v^{n,2}); q^1 \perp q^2) \\ = 1 \end{aligned}$$

Proof of rCWLLN can be constructed along the same lines as the proof of CWLLN; the assumption that the pairs of sequences/types are drawn at the same time and from independent sources is crucial for establishing the result. Similarly, r -generalization of ICET can be formulated and proven; and obviously μ -variants of the results hold true.

rCWLLN permits to rise the same objections to a Rényi entropy based variant of GME in GJMIP as were formulated to the maxTent in BJIP context.

Needless to say, μ -variant of rCWLLN provides a probabilistic justification to MaxProb variant of GME.

9. SUMMARY

Conditional Equi-concentration of Types on I-projections (ICET) – an extension of Conditioned Weak Law of Large Numbers (CWLLN) to the case of non-unique I-projection – was presented. ICET states that the conditional concentration of types happens on each of proper I-projections in equal measure. Also, Gibbs Conditioning Principle (GCP) was enhanced to capture multiple I-projections. Extended GCP says (when $t = 1$) that conditional probability of letter is asymptotically given by an equal-weight mixture of proper I-projection probabilities. The conditional equi-concentration/equi-probability 'phenomenon' is in our view an interesting feature of 'randomness'. It is worth further study, especially in the case of continuous alphabet. The conditional equi-concentration of types might be of some interest also for Statistical Mechanics since it resembles the coexistence of phases phenomena of Thermodynamics (eg. triple point of water, vapor and ice).

An elaboration of MaxProb/MaxEnt Thm, which states asymptotic identity of I- and μ -projections, was recalled. It permits directly to formulate μ -projection variants of CWLLN/GCP/ICET/EGCP. The μ -variants allow for a deeper reading than their I-projection counterparts - since the μ -laws express the asymptotic conditional behavior of types in terms of the most probable types. (For instance, μ -projection variant of CWLLN says that types conditionally concentrate on the asymptotically most probable one. This is more obvious statement than that made by I-variant of CWLLN.)

The main results – the Conditional Equi-concentration of Types (CET) in both its I- and μ -projection form as well as Extended Gibbs Conditioning and Boltzmann Conditioning – were supplemented also by further considerations. They are summarized below.

Though μ -projections and I-projections asymptotically coincide, for a finite n they are, in general, different. In light of this, the asymptotic identity of μ - and I-projections can be viewed in two ways: either as saying that 1) I-projection of q on Π is asymptotic form of μ -projection of q on Π_n or that 2) μ -projections on Π_n and I-projections on Π_n asymptotically coincide. Regardless of the preferred view, μ -variants of the laws provide a probabilistic justification of Maximum Probability method (at least at the area of Boltzmann-Jaynes inverse problem). If the second

view is adopted, then for a finite \mathbf{n} it is necessary to face a challenge of selecting between Relative Entropy Maximization (REM/MaxEnt) and Maximum Probability (MaxProb) method.

The equi-concentration results have a relevance also for Maximum Rényi-Tsallis Entropy method (maxTent), which is over the last years in vogue in Statistical Physics. Since, in general, maxTent distributions (τ -projections on Π) are different than I/μ -projections on Π , ICET implies that the maxTent method selects asymptotically conditionally improbable/impossible distributions.

Conditional Equi-concentration of Types on Jeffreys projections (JCET) was also discussed. JCET provides a probabilistic justification to the method of Jeffreys Entropy Maximization. Obviously, Maximum Probability variant of the method is justified by the μ -variant of JCET.

A straightforward extension of CWLLN/CET for \mathbf{r} -tuples of types was also mentioned. It was noted that the extension provides a justification to the Generalized Maximum Entropy method in the area of Golan-Judge-Miller ill-posed inverse problem.

10. APPENDIX

10.1. A sketch of proof of ICET.

Proof. First, a standard inequality should be recalled:

Lemma. *Let $\mathbf{v}^n, \hat{\mathbf{v}}^n$ be two \mathbf{n} -types. Then*

$$\frac{\pi(\mathbf{v}^n; \mathbf{q})}{\pi(\hat{\mathbf{v}}^n; \mathbf{q})} < \left(\frac{\mathbf{n}}{\mathbf{m}}\right)^m \prod_{i=1}^m \frac{\left(\frac{q_i}{v_i^n}\right)^{n v_i^n}}{\left(\frac{q_i}{\hat{v}_i^n}\right)^{n \hat{v}_i^n}}.$$

Proof. $\pi(\mathbf{v}^n; \mathbf{q}) \leq \prod_{i=1}^m \left(\frac{q_i}{v_i^n}\right)^{n v_i^n}$. Since for $\mathbf{n} > 6$, $(\mathbf{n}/e)^{\mathbf{n}} < \mathbf{n}! < \mathbf{n}(\mathbf{n}/e)^{\mathbf{n}}$, $\pi(\hat{\mathbf{v}}^n; \mathbf{q}) > \frac{1}{\hat{n}_1 \dots \hat{n}_m} \prod_{i=1}^m \left(\frac{q_i}{\hat{v}_i^n}\right)^{n \hat{v}_i^n}$. $\hat{n}_1 \dots \hat{n}_m < \left(\frac{\mathbf{n}}{\mathbf{m}}\right)^m$. \square

$$(3) \quad \pi(\mathbf{v}^n \in B(\epsilon, \hat{\mathbf{p}}^j) | \mathbf{v}^n \in \Pi; \mathbf{q} \mapsto \mathbf{v}^n) \leq \frac{\sum_{\mathbf{v}^n \in B} \pi(\mathbf{v}^n; \mathbf{q})}{\sum_{\mathbf{v}^n \in \Pi} \pi(\mathbf{v}^n; \mathbf{q})}.$$

$B_{\mathbf{n}}(\epsilon, \hat{\mathbf{p}}^j) = B(\epsilon, \hat{\mathbf{p}}^j) \cap \Pi_{\mathbf{n}}$. Let there be $k_{B, \mathbf{n}}$ I-projections $\hat{\mathbf{p}}_{B, \mathbf{n}}^1, \hat{\mathbf{p}}_{B, \mathbf{n}}^2, \dots, \hat{\mathbf{p}}_{B, \mathbf{n}}^{k_{B, \mathbf{n}}}$ of \mathbf{q} on $\bigcup_{j=1}^k B_{\mathbf{n}}(\epsilon, \hat{\mathbf{p}}^j)$. Let $k_{B, \mathbf{n}}^j$ denote the number of I-projections of \mathbf{q} on $B_{\mathbf{n}}(\epsilon, \hat{\mathbf{p}}^j)$. $\hat{\mathbf{p}}_{B, \mathbf{n}}^j$ will stand for any of such I-projections. By $B \setminus k_{B, \mathbf{n}}^j$ denote the set $B_{\mathbf{n}}(\epsilon, \hat{\mathbf{p}}^j) \setminus \bigcup_{i=1}^{k_{B, \mathbf{n}}^j} \{\hat{\mathbf{p}}_{B, \mathbf{n}}^i\}$.

Similarly, let there be $k_{\Pi, \mathbf{n}}$ I-projections $\hat{\mathbf{p}}_{\Pi, \mathbf{n}}^1, \hat{\mathbf{p}}_{\Pi, \mathbf{n}}^2, \dots, \hat{\mathbf{p}}_{\Pi, \mathbf{n}}^{k_{\Pi, \mathbf{n}}}$ of \mathbf{q} on $\Pi_{\mathbf{n}}$. Denote the set $\Pi_{\mathbf{n}} \setminus \bigcup_{i=1}^{k_{\Pi, \mathbf{n}}} \{\hat{\mathbf{p}}_{\Pi, \mathbf{n}}^i\}$ as $\Pi \setminus k_{\Pi, \mathbf{n}}$. MaxProb/MaxEnt Thm implies that for $\mathbf{n} \rightarrow \infty$ the RHS of (3) can be written as:

$$(4) \quad \frac{\pi(\hat{\mathbf{p}}_{B, \mathbf{n}}^j; \mathbf{q}) \left(k_{B, \mathbf{n}}^j + \frac{\sum_{\mathbf{v}^n \in B \setminus k_{B, \mathbf{n}}^j} \pi(\mathbf{v}^n; \mathbf{q})}{\pi(\hat{\mathbf{p}}_{B, \mathbf{n}}^j; \mathbf{q})} \right)}{\pi(\hat{\mathbf{p}}_{\Pi, \mathbf{n}}; \mathbf{q}) \left(k_{\Pi, \mathbf{n}} + \frac{\sum_{\mathbf{v}^n \in \Pi \setminus k_{\Pi, \mathbf{n}}} \pi(\mathbf{v}^n; \mathbf{q})}{\pi(\hat{\mathbf{p}}_{\Pi, \mathbf{n}}; \mathbf{q})} \right)}.$$

The Lemma implies that the ratio in the nominator of (4) converges to zero as $\mathbf{n} \rightarrow \infty$. The same implication holds for the ratio in the denominator.

$\hat{\mathbf{p}}_{B, \mathbf{n}}^j$ converges in the metric to $\hat{\mathbf{p}}^j$, hence $k_{B, \mathbf{n}}^j$ converges to 1 as $\mathbf{n} \rightarrow \infty$. Similarly, $k_{\Pi, \mathbf{n}}$ converges to k and $\frac{\pi(\hat{\mathbf{p}}_{B, \mathbf{n}}^j; \mathbf{q})}{\pi(\hat{\mathbf{p}}_{\Pi, \mathbf{n}}; \mathbf{q})}$ converges to 1 as \mathbf{n} goes to infinity.

This taken together implies that the RHS of (3) converges to $1/k$ as $\mathbf{n} \rightarrow \infty$. The inequality (3) thus turns into equality. \square

10.2. Two illustrative examples of CET. Below, two illustrative examples of the Conditional Equi-concentration of Types on I-projections (ICET) - taken from [14] - are recalled.

Example 3. Let $\Pi = \{p : \sum_{i=1}^m p_i^\alpha - \alpha = 0, \sum_{i=1}^m p_i - 1 = 0\}$, where $\alpha, \mathbf{a} \in \mathbb{R}$. Note that the first constraint, known as frequency constraint, is non-linear in p and Π is for $|\alpha| > 1$ non-convex.

Let $\alpha = 2$, $m = 3$ and $\mathbf{a} = 0.42$ (the value was obtained for $p = [0.5 \ 0.4 \ 0.1]$). Then there are three I-projections of uniform distribution $q = [1/3 \ 1/3 \ 1/3]$ on Π : $\hat{p}_1 = [0.5737 \ 0.2131 \ 0.2131]$, $\hat{p}_2 = [0.2131 \ 0.5737 \ 0.2131]$ and $\hat{p}_3 = [0.2131 \ 0.2131 \ 0.5737]$ (see [16]). Note that they form a group of permutations. As it will become clear later, it suffices to investigate convergence to say \hat{p}_1 .

For $n = 30$ there are only two groups of types in Π : G1 comprises $[0.5666 \ 0.2666 \ 0.1666]$ and five other permutations; G2 consists of $[0.5 \ 0.4 \ 0.1]$ and the other five permutations. So, together there are 12 types.

Value of the square of the Euclidean distance δ between v and \hat{p}_1 attains its minimum $\delta_{G1} = 0.0051$ within G1 group for two types: $[0.5666 \ 0.2666 \ 0.1666]$, $[0.5666 \ 0.1666 \ 0.2666]$. Within G2 the smallest $\delta_{G2} = 0.0532$ is attained by $[0.5 \ 0.4 \ 0.1]$ and $[0.5 \ 0.1 \ 0.4]$.

Thus, if $\epsilon = \epsilon_1$ is chosen so that the ball $B(\hat{p}_1, \epsilon_1)$ contains only the two types from G1 (which at the same time guarantees that \hat{p}_1 is the only I-projection in the ball), then $\pi(v \in B(\hat{p}_1, \epsilon_1) | v \in \Pi) = 2 * 0.1152 = 0.2304$. In words: probability that if q generated a type from Π than the type falls into the ball containing only types which are closest to the I-projection is 0.2304. If $\epsilon = \epsilon_2$ is chosen so that also the two types from G2 are included in the ball and also so that it is the only I-projection in the ball (any $\epsilon_2 \in (\sqrt{0.0532}, \sqrt{0.1253})$ satisfies both the requirements), then $\pi(v^n \in B(\hat{p}_1, \epsilon_2) | v^n \in \Pi) = \frac{1}{3}$.

For $n = 330$ there are four groups of types in Π : G1, G2 and a couple of new one: G3 consists of $[0.4727 \ 0.4333 \ 0.0939]$ and all its permutations; G4 comprises $[0.5727 \ 0.2333 \ 0.1939]$ and its permutations. Hence, the total number of types from Π which are supported by random sequences of size $n = 330$ is 24.

δ_{G3} for the two types from G3 which are closest to \hat{p}_1 is 0.0729. The smallest $\delta_{G4} = 0.00077$ is attained by $[0.5727 \ 0.2333 \ 0.1939]$ and by $[0.5727 \ 0.1939 \ 0.2333]$. Thus, clearly, the two types from G4 have the smallest Euclidean distance to \hat{p}_1 among all types from Π which are based on samples of size $n = 330$. Again, setting ϵ such that the ball $B(\hat{p}_1, \epsilon)$ contains only the two types which are closest to \hat{p}_1 leads to the 0.261 value of the conditional probability. Note the important fact, that the probability has risen, as compared to the corresponding value 0.2304 for $n = 30$.

Moreover, if ϵ is set such that besides the two types from G4 also the second closest types (i.e. the two types from G1) are included in the ball then the conditional probability is indistinguishable from $\frac{1}{3}$. Hence, there is virtually no conditional chance of observing any of the remaining 4 types. The same holds for the types which concentrate around \hat{p}_2 or \hat{p}_3 . Thus, in total, a half of the 24 types is almost impossible to observe.

The Example illustrates, that the conditional probability of finding a type which is close (in the Euclidean distance) to one of the three I-projections goes to $\frac{1}{3}$. \diamond

Example 4. Let $\Pi = \Pi_1 \cup \Pi_2$, where $\Pi_j = \{p : \sum_{i=1}^m p_i x_i = a_j; \sum_{i=1}^m p_i = 1\}$, $j = 1, 2$. Thus Π is union of two sets, each of whose is given by the moment consistency constraint. If q is chosen to be the uniform distribution, then values a_1, a_2 such that there will be two different I-projections of the uniform q on Π with the same value of I-divergence (as well as of the Shannon's entropy) can be easily found. Indeed, for any $a_1 = \mu + \Delta$, $a_2 = \mu - \Delta$, where $\mu = EX$ and $\Delta \in (0, (X_{\max} - X_{\min})/2)$,

\hat{p}_1 is just a permutation of \hat{p}_2 , and as such attains the same value of Shannon's entropy. To see that types which are based on random samples of size n from Π indeed concentrate on the I-projections with equal measure note, that for any n to each type in Π_1 corresponds a unique permutation of the type in Π_2 . Thus, types in ϵ -ball with center at \hat{p}_1 have the same conditional probabilities π as types in the ϵ -ball centered at \hat{p}_2 . This, together with convexity and closed-ness of both Π_j , for which the conditional concentration of types on the respective I-projection is established by CWLLN, directly implies that

$$\lim_{n \rightarrow \infty} \pi(\nu \in B(\hat{p}_j, \epsilon) | \nu^n \in \Pi) = \frac{1}{2} \quad j = 1, 2. \quad \diamond$$

10.3. Rational improper I- and μ -projections. That types can concentrate on rational μ - or I-projection $\hat{p} \in Q^m$ even though the I-projection is isolated point of Π can be seen from the following Example.

Example 5. Consider the following feasible set of types $\Pi = \{p, \dot{p}\}$, where $p = [n_1/n_0, \dots, n_m/n_0]$, $\dot{p} = [\dot{n}_1/n_0, \dots, \dot{n}_m/n_0]$, $n_0 \in N$. For $n \neq kn_0$, $k \in N$ the set Π_n is empty; otherwise it contains p and \dot{p} .

In this case, the concentration of types on μ -projection is direct consequence of the next two Lemmas. The I-variant of the concentration then arises from MaxProb/MaxEnt Thm.

Lemma 1. Let $\nu^n, \dot{\nu}^n$ be two n -types. Let $\delta = \nu^n - \dot{\nu}^n$. Let K denote the non-negative elements of $n\delta$, L the absolute value of negative elements of $n\delta$. Let $c = \prod_- \dot{n}_i^{L_i} / \prod_+ \dot{n}_i^{K_i}$, where the subscript $-$, $+$ indicates that the index i goes through the elements of K , L , respectively. Then $\frac{\Gamma(k\nu^{kn})}{\Gamma(k\dot{\nu}^{kn})} < c^k$, for any $k \in N$.

Proof. $\frac{\Gamma(k\nu^{kn})}{\Gamma(k\dot{\nu}^{kn})} = \frac{\prod_- (k(\dot{n}_i - L_i) + 1) \dots k\dot{n}_i}{\prod_+ (k\dot{n}_i + 1) \dots k(\dot{n}_i + K_i)}$. So $\frac{\Gamma(k\nu^{kn})}{\Gamma(k\dot{\nu}^{kn})} \leq \frac{\prod_- \dot{n}_i^{kL_i}}{\prod_+ \dot{n}_i^{kK_i}}$, which is just c^k . \square

Lemma 2. Let types $\nu^n, \dot{\nu}^n$ be such that $\pi(\dot{\nu}^n; q) < \pi(\nu^n; q)$. Then $\frac{\pi(k\nu^{kn}; q)}{\pi(k\dot{\nu}^{kn}; q)} \rightarrow 0$ as $k \rightarrow \infty$.

Proof. By the assumption, $\prod q_i^{n_i - \dot{n}_i} < \frac{\Gamma(\dot{\nu}^n)}{\Gamma(\nu^n)}$. The gamma-ratio is, by the Lemma 1, smaller or equal to c , as defined at the Lemma. Thus, $\prod q_i^{n_i - \dot{n}_i} = \gamma c$, where $\gamma \in [0, 1)$, $\gamma \in R$. By Lemma 1, for any $k \in Z$, $\frac{\pi(k\nu^{kn}; q)}{\pi(k\dot{\nu}^{kn}; q)} \leq (1/c)^k \prod q_i^{k(n_i - \dot{n}_i)}$. The RHS of the inequality, γ^k , goes for $k \rightarrow \infty$ to zero, which completes the proof. \square

In this case, if Π admits several rational I/ μ -projections, then clearly, types equi-concentrate on them. \diamond

10.4. A sketch of proof of EGCP.

Proof.

$$(5) \quad \pi(X_1 = x_1, \dots, X_t = x_t | \nu^n \in \Pi; q \mapsto \nu^n) \\ = \frac{\sum_{\nu^n \in \Pi} \pi(X_1 = x_1, \dots, X_t = x_t, \nu^n)}{\sum_{\nu^n \in \Pi} \pi(\nu^n; q)}.$$

Partition Π_n into $\Pi \setminus k_{\Pi, n}$ and the rest, which will be denoted by $\bigcup \hat{p}_{\Pi, n}$.

MaxProb/MaxEnt Thm implies that for $n \rightarrow \infty$ the RHS of (5) can be written as:

$$(6) \quad \frac{\sum_{\nu^n \in \cup \hat{\Pi}_{\Pi, n}} \pi(X_1 = x_1, \dots, X_t = x_t, \nu^n)}{\pi(\hat{\Pi}_{\Pi, n}; \mathbf{q})(k_{\Pi, n} + \frac{\sum_{\nu^n \in \Pi \setminus k_{\Pi, n}} \pi(\nu^n; \mathbf{q})}{\pi(\hat{\Pi}_{\Pi, n}; \mathbf{q})})} + \frac{\sum_{\nu^n \in \Pi \setminus k_{\Pi, n}} \pi(X_1 = x_1, \dots, X_t = x_t, \nu^n)}{\pi(\hat{\Pi}_{\Pi, n}; \mathbf{q})(k_{\Pi, n} + \frac{\sum_{\nu^n \in \Pi \setminus k_{\Pi, n}} \pi(\nu^n; \mathbf{q})}{\pi(\hat{\Pi}_{\Pi, n}; \mathbf{q})})}.$$

By the Lemma, the ratio in the denominators (6) converges to zero as n goes to infinity. The second nominator as well goes to zero as $n \rightarrow \infty$ (to see this, express the joint probability $\pi(X_1 = x_1, \dots, X_t = x_t, \nu^n)$ as $\pi(X_1 = x_1, \dots, X_t = x_t | \nu^n) \pi(\nu^n; \mathbf{q})$ and employ the Lemma). Thus, for $n \rightarrow \infty$ the RHS of (5) becomes $1/k \sum_{j=1}^k \pi(X_1 = x_1, \dots, X_t = x_t | \hat{\rho}^j)$. Finally, invoke Csiszár's 'urn argument' (cf. [7], p. 2510) to conclude that the asymptotic form of the RHS of (6) is $1/k \sum_{j=1}^k \prod_{l=1}^t \hat{\rho}_{X_l}^j$. \square

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11. CHANGES WRT THE ARXIV:/MATH.PR/0407102, VERSION 3

The two major changes: 1) Definition of proper I-projection has been changed.
2) An argument preceding Eq. (7) at the proof of ICET (and similarly Eq. (9) at the proof of EGCP) is now correctly stated. – Other changes are minor.

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