EXPONENTS OF MAPS

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1. Introduction

In this thesis we deal with embeddings of graphs on closed surfaces, focusing mainly on regular maps on nonorientable surfaces. We introduce the concept of unoriented exponent and we show that it can be useful in classification of regular maps with a prescribed underlying graph.

A map is a 2-cell embedding of a connected graph into some closed surface. A map is said to be *orientable* if the supporting surface is orientable, and is *oriented* if one of the two possible orientations of the surface has been determined; otherwise a map is *unoriented*. We mostly deal with unoriented surfaces and we generalize results that are already known for the orientable ones. Along with maps, we also study map homomorphisms, i. e. branched coverings of supporting surfaces that preserve the underlying graphs.

Among all graph embeddings, regular maps attract an exceptional attention. Their automorphism groups have maximum size, that is, regular maps are endowed with extremal symmetry. Historically, regular maps are a natural generalization of the five Platonic solids, and numerous relationships between regular maps and other mathematical disciplines (e.g. Riemann surfaces, Dyck's triangle groups and Galois theory) have been discovered. A nice survey paper together with many relevant references can be found in [J2]. These connections have also initiated an intensive study of the problem of classification of regular maps.

In study of maps, it is often useful to replace topological objects by their algebraic (or combinatorial) counterparts. Maps on orientable surfaces can be completely described by means of their rotation systems which cyclically permute arcs (directed edges) with the same initial vertices [JS]. Hence, we can consider an oriented map M to be a pair $M = (K, R)$, where K is the underlying graph of M and R is the rotation of M induced by the chosen orientation of the ambient surface.

It is also well known that every map on an orientable or nonorientable surface can be completely described in terms of three involutory permutations acting on flags (or 'blades') of the embedded graph [JT]. A flag can be viewed as a couple consisting of an arc and one of the two (possibly equal) faces incident to the arc.

Thus we can refer to a map as a quadruple $M = (F(K); \lambda, \varrho, \tau)$, where $F(K)$ is the set of flags, K is the underlying graph of M, and λ, ϱ and τ are fixed-point free involutory permutations of $F(K)$. Note that every arc of K gives rise to two flags in $F(K)$. Roughly speaking, the longitudinal involution λ interchanges the flags along one 'side' of an edge, the rotary involution ϱ alternates the flags within an 'angle' of the embedded graph, and the transversal involution τ interchanges the two flags incident to an arc. The three involutions generate a permutation group $\mathrm{Mon}\,(M)$ called the monodromy group of M. This 3-involutory description of a map M also allows map automorphisms to be simply defined as permutations of flags commuting with λ, ϱ and τ [V1].

Along with maps, operations defining their external symmetries are also intensively studied. Generally, an operation Φ is a function associating a map M another map $\Phi(M)$. An operation Φ is typically required to preserve some important properties of maps such as the underlying surface, the underlying graph, or the monodromy group. In the category of unoriented maps the duality operation and the Petrie operation are one of the most familiar operations that preserve morphisms between maps. We shall deal with the operation of congruence which is associated with exponents.

We introduce the unoriented exponent as a generalization of the concept of exponent defined for oriented maps by Nedela and Škoviera in [NS1]: an integer e is an oriented exponent of a map $M = (K, R)$, if the map $M^e = (K, R^e)$ is isomorphic to M. The oriented exponent is so a natural generalization of the reflexivity of oriented maps. However, in contrast to mirror reflections and map automorphisms, the map isomorphism respective to the exponent does not preserve the supporting surface, that is, M and M^e may be embeddings of the graph K into different surfaces.

Let $M = (F(K); \lambda, \varrho, \tau)$ be an embedding of the graph K. If the map $M^t = (F(K); \lambda\tau^j, (\varrho\tau)^{e-1}\varrho, \tau)$ is isomorphic to M, for some integer e coprime with the valency of the graph K, and for some $j \in \mathbb{Z}_2$, we say the expression $t = ep^j$ to be an *unoriented exponent* of M. So the unoriented exponent also includes the Petrie operation P that preserves, as well as the map isomorphisms associated with exponents, both the underlying graph K and the automorphism group of M.

Note that -1 is an exponent of an oriented map $M = (K, R)$ if and only if M is reflexible, i. e., isomorphic to its mirror image (K, R^{-1}). On the other hand, -1 is an unoriented exponent of every 3-involutory map (see Lemma 4.3). The unoriented exponents of a map M (reduced modulo the valency n of M) form an Abelian group $\mathrm{Ex}\,(M)$ which is a subgroup of $\mathbb{Z}_n^* \times \mathbb{Z}_2$, where $\mathbb{Z}_n^*$ is the multiplicative group of invertible elements of the ring $\mathbb{Z}_n$ of integers modulo n.

The mapping which realizes the isomorphism $M \to M^t$ will be called an *exomorphism* (an external morphism) of M associated with the exponent t. The exomorphisms of M form a group $\mathrm{Exo}\,(M)$, which is an extension of the automorphism group $\mathrm{Aut}\,(M)$. The exponent and exomorphism groups are of special importance for regular maps since they provide a measure of their external symmetry. However, many results of this theory can be applied to arbitrary unoriented maps.

The importance of exponents lies in the fact that they allow us to classify regular embeddings of a fixed graph K with a prescribed map automorphism group $G \leq \mathrm{Aut}\,(K)$. The classification of regular maps by the underlying graph was initiated by Hefter more than a century ago [H]. Although the classification of regular maps of the complete graphs and several other infinite classes of graphs is already known, there is a lot of work to be done in this area. Nedela and Škoviera suggested in [NS1] a new graph-theoretical approach to the classification problem in which exponents play an important role.

As already mentioned, the concept of an exponent is closely connected with the relation of congruence between maps. In orientable case, two maps $M = (K, R)$ and $M' = (K, R')$ of the same graph K are called congruent, if $R' = R^e$ for some integer e coprime with the valency of K. One can easily verify that the congruent maps M and M' have equal automorphism groups, that is, $\mathrm{Aut}\,(M) = \mathrm{Aut}\,(M')$. Moreover, the converse theorem holds for regular oriented maps.

Theorem 1.1. [NS1] *Let M and M' be oriented regular maps with the same underlying graph. Then M and M' are congruent whenever* $\mathrm{Aut}\,(M) = \mathrm{Aut}\,(M')$ *or* $\mathrm{Mon}\,(M) = \mathrm{Mon}\,(M')$.

We shall define the concept of congruence for unoriented maps, and we show that a similar result can be proved for unoriented regular maps as well.

The use of the exponent theory to the classification of regular maps can be illustrated by the example of canonical double covering graphs examined in [NS2]. For a given oriented map $M = (K, Q)$ of a connected non-bipartite graph K, we can define the derived map $M_e = (\tilde{K}, \tilde{Q})$ of its canonical double covering graph $\tilde{K}$ by setting

$$\tilde{Q}(x_i) = Q^{e^i}(x)_i,$$

for any arc x of K and $i = 0, 1$.

Nedela and Škoviera proved that it is possible to 'lift' the classification of regular embeddings of a graph K to the classification of regular embeddings of $\tilde{K}$ in case that K is stable, i. e., $\mathrm{Aut}\,(\tilde{K}) \cong \mathrm{Aut}\,(K) \times \mathbb{Z}_2$. Their results are expressed with the following statements.

Theorem 1.2. [NS2] *Let $M = (K, Q)$ be an orientably regular map with underlying connected non-bipartite graph K, and let e be an involutory exponent of M. Then the derived map $M_e = (\tilde{K}, \tilde{Q})$ is regular.*

Theorem 1.3. [NS2] *Let K be a connected stable graph and let $M = (\tilde{K}, Q)$ be an orientably regular map of $\tilde{K}$. Then there exists a regular map N of K and an involutory exponent e of N such that $M \cong N_e$.*

Theorem 1.4. [NS2] *Let M and N be orientably regular maps of a stable n-valent graph K and let e and f be exponents of M and N, respectively. Then M_e is isomorphic to N_f if and only if M is isomorphic to N and $e \equiv f \pmod n$.*

The main goals of our doctoral dissertation are
(a) to define the concept of unoriented exponent and study its properties,
(b) to apply the defined exponent in classification of regular maps with a prescribed underlying graph (particularly, to prove a theorem analogous to Theorem 1.1 for unoriented maps),
(c) to state and prove analogies of Theorems 1.2-1.4 for the classification of regular maps of canonical double covering graphs on unoriented surfaces.

The thesis is organized as follows.

Section 2 briefly reviews the terminology concerning graphs and maps that is necessary for further considerations.

Section 3 points out the importance of the Petrie operation P for the study of regular maps and summarizes the basic properties of Petrie maps that will be used in later sections as well.

In Section 4 we introduce the central concept of an unoriented exponent and we study its properties.

In Section 5 we define the relation of congruence between 3-involutory maps and we prove the analogy of Theorem 1.1 for unoriented regular maps. We also show an application of this result to classification of regular maps with a given underlying graph and automorphism group.

Section 6 describes the properties of canonical double covering graphs and their 3-involutory regular maps.

Section 7 contains main results concerning the problem of classifying 3-involutory regular maps of the canonical double covering graph $\tilde{K}$ in terms of regular maps of the base graph K. We prove an analogy of Theorem 1.2 showing how the automorphisms and exomorphisms of a regular embedding of the graph K can be 'lifted' to automorphisms of an embedding of $\tilde{K}$ to ensure its regularity (Theorem 7.1). Further we prove an analogy of Theorem 1.3 showing that every regular map of the canonical double covering $\tilde{K}$ that meet natural additional conditions can be obtained from some regular map of the graph K with the described construction (Theorem 7.7). Finally we discuss the conditions under which two regular maps of $\tilde{K}$ constructed from regular maps of K are isomorphic and in Theorem 7.9 we prove an analogy of Theorem 1.4.

In Section 8 we show an application of these theoretical results on unoriented regular embeddings of the coctail-party graphs.

2. Graphs and maps

We shall deal with finite connected graphs that can have edges of three types: *links*, *loops*, and *semiedges*. Multiple adjacences are permitted. A link or a loop defines two oppositely directed arcs that are said to be *reverse* to each other. A semiedge gives rise to a single arc that is reverse to itself. We shall denote by $L(x)$ the reverse to the arc x. The set of all arcs of a graph K will be denoted by $D(K)$. Thus, L is an involutory permutation of $D(K)$.

For our purposes it will be useful to view a graph as a quadruple $K = (D, V, I, L)$, where $D = D(K)$ and $V = V(K)$ are non-empty disjoint finite sets, $I\colon D \to V$ is a surjective mapping, and L is an involution on D. The elements of D and V are *arcs* and *vertices*, respectively, I is the *incidence function* assigning to any arc its *initial vertex*, and L is the *arc-reversing involution*. The cycles of L correspond to edges of K.

For these graphs, we can define the usual graph-theoretical concepts. For example, the *valency* of a vertex v, denoted by $\mathrm{val}\,(v)$, is the number of arcs having v as their initial vertex.

Let $K = (D, V, I, L)$ and $K' = (D', V', I', L')$ be two graphs. An *isomorphism* $\alpha\colon K \to K'$ is a bijective mapping $\alpha\colon D \cup V \to D' \cup V'$ with $\alpha(D) = D'$ and $\alpha(V) = V'$ such that $\alpha I = I'\alpha$ and $\alpha L = L'\alpha$. An *automorphism* of K, as usual, is an isomorphism mapping K onto itself. The group of graph automorphisms of K will be denoted by $\mathrm{Aut}\,(K)$.

A *map* M is a connected topological graph cellularly embedded into a closed orientable or nonorientable surface. The respective graph is also said to be the *underlying graph* of the map M. As with graphs, we shall use an algebraic description of a topological map. This description is based on a triple of involutory permutations acting on the *flags* of the embedded graph K, that is, on the pairs consisting of an arc $x \in D(K)$ and one or the other face adjacent to x. It is convenient to bypass the notion of faces and to define formally the set of flags to be $F(K) = \{(x, i); x \in D(K), i \in \{+1, -1\}\}$. We now define three permutations λ, ϱ, and τ acting on $F(K)$ which we shall use to describe graph embeddings.

For each flag $(x, i) \in F(K)$, we define the *longitudinal* involution λ by setting

$$\lambda(x, i) = (L(x), i'),$$

where $i' = \pm i$ has to be chosen in such a way that flags (x, i) and $(L(x), i')$ correspond to the face located on the same side of the edge underlying the arc x.

Similarly, we define the *transversal* involution τ, interchanging the two flags incident to x, by

$$\tau(x,i) = (x,-i).$$

Finally, let u be the initial vertex of x and R_u be a local rotation of arcs emanating from u. (Although the graph may be embedded into a nonorientable surface, we can always restrict to such a small neighbourhood of u which is homeomorphic to 2-cell, and choose there one of the two possible orientations to determine R_u.) Then we set

$$\varrho(x,i) = (R_u^i(x), -i).$$

The *rotary* involution ϱ interchanges the flags (x,i) and $(y,-i)$ for which the first coordinates x and y start at the same vertex u, and the second coordinates correspond to the same face so that the edges underlying x and y are consecutive on the boundary of that face separated only by the vertex u. A short insight on the above definitions suffices to check the following properties.

Lemma 2.1. *Permutations λ, ϱ and τ are fixed point free involutions on the set $F(K)$ such that $\lambda\tau = \tau\lambda$.*

The permutation group $\langle \lambda, \varrho, \tau \rangle$ is called the *unoriented monodromy group* of M and we denote it by $\mathrm{Mon}\,(M)$. Since K is a connected graph, the group $\mathrm{Mon}\,(M)$ acts transitively on $F(K)$. It is known ([JT]) that the triple (λ, ϱ, τ) provides a complete description of the embedding of K on some surface, either orientable or nonorientable. We can therefore refer to the *3-involutory map* $M = (F(K); \lambda, \varrho, \tau)$.

Conversely, if we are given a quadruple $(F; \lambda, \varrho, \tau)$, where F is a finite nonempty set, λ, ϱ and τ are fixed point free involutory permutations of F such that $\lambda\tau = \tau\lambda$, and the group $\langle \lambda, \varrho, \tau \rangle$ acts transitively on F, we are able to construct the corresponding map M and its underlying graph K. We define the vertices of K to be the orbits of the subgroup $\langle \varrho, \tau \rangle$, the edges to be the orbits of $\langle \lambda, \tau \rangle$ and the faces of M to be the orbits of $\langle \lambda, \varrho \rangle$ under the action on F. The incidence is defined by non-trivial set intersection.

A *homomorphism* of 3-involutory maps $M = (F; \lambda, \varrho, \tau)$ and $M' = (F'; \lambda', \varrho', \tau')$ is a mapping $\alpha\colon F \to F'$ such that

$$\alpha\lambda = \lambda'\alpha, \qquad \alpha\varrho = \varrho'\alpha, \quad \text{and} \quad \alpha\tau = \tau'\alpha\ .$$

In particular, a permutation α of F is said to be an *automorphism* of the map $M = (F; \lambda, \varrho, \tau)$ if

$$\alpha\lambda = \lambda\alpha, \qquad \alpha\varrho = \varrho\alpha, \quad \text{and} \quad \alpha\tau = \tau\alpha\ .$$

The automorphism group of M will be denoted by $\mathrm{Aut}\,(M)$. Although its elements are formally permutations of the flag set F, they commute with τ and the cycles of τ correspond to the arcs of K. Therefore each permutation of $\mathrm{Aut}\,(M)$ induces a permutation of $D(K)$, and hence an automorphism of K.

Let $\alpha \in \mathrm{Aut}\,(M)$ be an automorphism of M such that $\alpha(f) = g$ for some flags $f, g \in F$. Let $h \in F$ be any flag. The transitive action of the group $\mathrm{Mon}\,(M)$ on F implies that there is a permutation $w = w(\lambda, \varrho, \tau) \in \mathrm{Mon}\,(M)$ such that $h = w(f)$. Then

$$\alpha(h) = \alpha w(f) = w\alpha(f) = w(g),$$

and the following statement holds.

Lemma 2.2. *Automorphism group* Aut (M) *acts semi-regularly on the flag set* F.

A map $M = (F; \lambda, \varrho, \tau)$ is called *regular* if Aut (M) acts transitively on the set F. In other words, M is regular if and only if $|\operatorname{Aut}(M)| = |F|$. The monodromy group Mon (M) acts on the same set of flags F as the automorphism group Aut (M), and in case that M is regular, Mon $(M) \cong$ Aut (M). However, in general, these two permutation groups are not equal.

Note that a connected graph of valency 1 has a 2-cell embedding only in the sphere, and the resulting map is (orientably) regular. A 2-cell embedding of a connected graph of valency 2 has at most 2 faces. If the embedding has only one face, it is on the projective plane, otherwise it is on the sphere. In both these cases the obtained maps are regular. That is why we can restrict attention to graphs of valency at least 3.

Gardiner, Nedela, Širáň and Škoviera at the theoretical level solved the problem of classification of the finite graphs that underlie regular maps. They give necessary and sufficient conditions for a graph to underlie an oriented or an unoriented regular map. Their classification is based on the properties of the group of graph automorphisms. We shall frequently refer to the statement for general case that follows.

Theorem 2.3. [GNSS] *A connected graph K of valency at least 3 is the underlying graph of a regular map on some closed surface if and only if its automorphism group contains a subgroup G such that*

(1) *G acts transitively on the set of arcs of K,*
(2) *the edge stabilizer G_e of an edge e is dihedral of order 4,*
(3) *the stabilizer G_v of a vertex v is dihedral with a cyclic subgroup of index two acting regularly on arcs incident to v.*

3. Petrie duality

The *Petrie operation* is one of the most familiar functors in the category of unoriented maps which is especially important for the study of maps with high degree of symmetry [CM]. Since it also plays a significant role in our considerations, we devote it this special section to recall some of its basic properties that will be used later.

The Petrie operation P can be applied to an arbitrary unoriented map M. The *Petrie map* (also called the *Petrie dual*) $P(M)$ of M is an unoriented map with the same underlying graph such that the boundaries of its faces are formed by *Petrie polygons* of M, that is, by the 'zig-zag' walks along the arcs in M where two but not three consecutive arcs of the walk share the same face of M. Every edge of M occurs twice in its Petrie polygons, so by filling the Petrie polygons with a 2-cell and joining the two occurences of an edge we obtain the Petrie map $P(M)$.

In general, the Petrie operation does not provide an operator for the category of oriented maps, since its topological application on a map on an orientable surface may result in a map on a nonorientable surface. However, if M is an orientable bipartite map, then $P(M)$ is orientable and bipartite as well [NSZ].

For our purposes the following definition of the Petrie duality will be helpful. Let $M = (F(K); \lambda, \varrho, \tau)$ be a map with underlying graph K. Then the map $P(M) = (F(K); \lambda\tau, \varrho, \tau)$ is the *Petrie map* of the map M.

It is a straightforward matter to see that each automorphism of M provides also an automorphism of $P(M)$ and vice versa, so $\mathrm{Aut}\,(P(M)) = \mathrm{Aut}\,(M)$. Consequently, $P(M)$ is regular if and only if M is regular, too. It can be also easily seen that two maps M and N with the same underlying graph K are isomorphic if and only if their Petrie maps $P(M)$ and $P(N)$ are isomorphic. Moreover, any permutation φ of $F(K)$ is a map isomorphism $\varphi\colon M \to N$ if and only if it provides a map isomorphism $\varphi\colon P(M) \to P(N)$ as well. This also implies that isomorphic maps have Petrie polygons of the same length.

Clearly,

$$P(P(M)) = (F(K);\, (\lambda\tau)\tau, \varrho, \tau) = (F(K);\, \lambda, \varrho, \tau) = M\ ,$$

so Petrie operation gives birth to couples of maps with the same underlying graph which are Petrie duals of each other. In the special case when M is isomorphic to $P(M)$, the map M is said to be *self-Petrie*.

Summing up, we can state the following.

Lemma 3.1. *Let $P(M)$ and $P(N)$ be the Petrie maps of maps M and N, respectively. Then*

(1) *M and $P(M)$ have the same underlying graph,*
(2) *M is isomorphic to N if and only if $P(M)$ is isomorphic to $P(N)$,*
(3) *P(P(M))=M,*
(4) *M is an orientable bipartite map if and only if P(M) is orientable and bipartite as well.*

Examples.

1. The Petrie dual to the spherical embedding (tetrahedron) of the complete graph K_4 is the 3-face embedding of K_4 into the projective plane.

2. The 4-gonal spherical map of Q_3 and the 6-gonal regular embedding of Q_3 in torus are Petrie duals.

3. The famous 5-gonal embedding of Petersen graph in the projective plane is a self-Petri map.

A generalization of the concept of the Petrie duality for oriented bipartite maps can be found in [NSZ].

4. Exponents of graphs and maps

Let $M = (F(K);\, \lambda, \varrho, \tau)$ be a map with underlying graph K. Let v be an arbitrary vertex of K. Denote by F_v the set of flags of M incident to v. Notice that $(\varrho\tau)_v = \varrho\tau|F_v$ is a 2-cycle permutation acting on F_v. Each of the two cycles rotates all flags with equal second coordinates, and induces the rotation of arcs emanating from v that form the first coordinates of these flags. The induced rotations coincide with the two inverse local rotations R_v and R_v^{-1} that can be observed in a small 2-cell neighbourhood of v in M.

Formally, we define the local rotation R_v of arcs incident to v by setting

$$R_v(x) = \eta(\varrho\tau)(x, +1),$$

where $\eta\colon F(K) \to D(K)$ is the mapping that erases the second coordinates of flags, i. e., $\eta(x, i) = x$, for any $x \in D(K)$ and $i \in \{+1, -1\}$.

An integer e will be called an *exponent* of K if $(R_v)^e$ is also a local rotation of arcs emanating from v, for any vertex $v \in V(K)$.

However, $(R_v)^e$ is a rotation if and only if it consists of a single cycle. Hence, for any v, e must be coprime with the valency of v, that is, $\gcd(e, \mathrm{val}\,(v)) = 1$. This is equivalent to the condition that e is coprime with the least common multiple of the valencies of vertices of K. We will denote this value by $\mathrm{val}\,(K)$ and call it the *virtual valency* of K. In case that the valencies of all vertices of K are equal to n, $\mathrm{val}\,(K) = n$, too, and K is said to be *n-valent graph*. The virtual valency of a map M is defined to be equal to the virtual valency of its underlying graph.

Using this terminology, we can conclude:

Proposition 4.1. [NS1] *An integer e is an exponent of a connected graph K if and only if e is coprime with the virtual valency* $\mathrm{val}\,(K)$.

This means that the residue classes modulo $n = \mathrm{val}\,(K)$ of exponents of K form a multiplicative group which coincides with the group $\mathbb{Z}_n^*$ of invertible elements of $\mathbb{Z}_n$. We shall denote this group by $\mathrm{Ex}\,(K)$ and call it the *exponent group* of K.

It is easy to see that we can use an exponent of a graph and the Petrie operation to construct a new 3-involutory map from a given one with the same underlying graph.

Lemma 4.2. *Let e be an exponent of a connected graph K and $M = (F(K);\, \lambda, \varrho, \tau)$ be a map of K. Then $M' = (F(K);\, \lambda', \varrho', \tau')$ such that*

$$\lambda'(f) = \lambda\tau^j(f), \qquad \varrho'(f) = (\varrho\tau)^{e-1}\varrho(f), \qquad \tau'(f) = \tau(f)\ ,$$

for any flag $f \in F(K)$ and $j \in \mathbb{Z}_2$, is a 3-involutory map of the graph K, too.

Now we are prepared to introduce the key concept of this thesis. The definition follows.

Let e be an exponent of a connected graph K and let $M = (F(K);\, \lambda, \varrho, \tau)$ be a map of K. The expression $t = ep^j$, $j \in \mathbb{Z}_2$, will be called an *unoriented exponent* of M if the map $M^t = (F(K);\, \lambda\tau^j, (\varrho\tau)^{e-1}\varrho, \tau)$ is isomorphic to M.

Notice that the unoriented exponent defined above consists of two parts. The integer part e is a natural generalization of the exponent defined for oriented maps in [NS1]. The Petrie part p^j turns out to be a fruitful extension of the idea of exponents for unoriented maps since the combination of these two operations can be really useful in the classification of regular maps on nonorientable surfaces.

If $j = 0$, then the exponent $t = e$ is said to be an *unoriented integer exponent* of M. In the special interesting case when $e \neq 1$ and $j = 1$, the exponent $t = ep$ is called a *mixed exponent* of M.

From the definition it directly follows that t is an unoriented exponent of M if and only if there exists a permutation φ of $F(K)$ which provides a map isomorphism $M \to M^t$. The isomorphism φ is then said to be *associated* with the unoriented exponent t of M.

Since we further deal mainly with unoriented exponents, we will usually omit the adjective 'unoriented' whenever no ambiguity can occur.

Lemma 4.3. *Let $M = (F(K);\, \lambda, \varrho, \tau)$ be a map. Then*

(1) ± 1 *are exponents of* M*,*

(2) p *is an exponent of* M *if and only if* M *is self-Petrie.*

Proof. (1) Obviously, 1 is an exponent of every map. Also, -1 is an exponent of every (non-trivial) connected graph, and we can construct the map M^{-1}:

$$M^{-1} = (F(K);\, \lambda, (\varrho\tau)^{-2}\varrho, \tau) = (F(K);\, \lambda, \tau\varrho\tau, \tau),$$

because

$$(\varrho\tau)^{-2}\varrho = ((\varrho\tau)^{-1})^2\varrho = (\tau\varrho)^2\varrho = \tau\varrho\tau\ .$$

We need to find a permutation φ of $F(K)$ such that φ is an isomorphism $M \to M^{-1}$, i. e.,

$$\varphi\lambda = \lambda\varphi, \qquad \varphi\varrho = \tau\varrho\tau\varphi, \qquad \varphi\tau = \tau\varphi\ .$$

However, we can easily check that $\varphi = \tau$ meets the required conditions and thus it provides the desired map isomorphism.

(2) In accordance with the definition, p is an exponent of M if and only if $M \cong M^p$. But $M^p = (F(K);\, \lambda\tau, \varrho, \tau) = P(M)$, that is, M is a self-Petrie map. □

Let M be any map, and let $s = ep^j$ and $t = fp^k$ be two exponents of M. Then we can define their product st by setting

$$st = (ef)p^{j+k}\ .$$

Clearly, $ts = (fe)p^{k+j} = (ef)p^{j+k} = st$, and we shall show that exponents of M (modulo virtual valency of M) with the operation defined above form an Abelian group.

Lemma 4.4. *Let $s = ep^j$ be an exponent of a map $M = (F(K);\, \lambda, \varrho, \tau)$, and n be the virtual valency of the underlying graph K of M. Let $f \equiv e \pmod{n}$. Then $t = fp^j$ is an exponent of M as well.*

Proof. Since $s \in \mathrm{Ex}\,(M)$, M must be isomorphic to $M^s = (F(K);\, \lambda\tau^j, (\varrho\tau)^{e-1}\varrho, \tau)$. But $M^t = (F(K);\, \lambda\tau^j, (\varrho\tau)^{f-1}\varrho, \tau)$ clearly equals to M^s. Thus, $M \cong M^t$, and t is an exponent of M. □

Lemma 4.5. *Let s and t be exponents of a map $M = (F(K);\, \lambda, \varrho, \tau)$. Then st, too, is an exponent of M.*

Proof. Let $s = ep^j$ and $t = fp^k$, where $e, f \in \mathrm{Ex}\,(K)$ and $j, k \in \mathbb{Z}_2$. Let $\varphi_s\colon M \to M^s$ and $\varphi_t\colon M \to M^t$ be map isomorphisms associated with s and t, respectively. Hence,

$$\varphi_s\lambda = \lambda\tau^j\varphi_s, \qquad \varphi_s\varrho = (\varrho\tau)^{e-1}\varrho\varphi_s, \qquad \varphi_s\tau = \tau\varphi_s\ ,$$

and

$$\varphi_t\lambda = \lambda\tau^k\varphi_t, \qquad \varphi_t\varrho = (\varrho\tau)^{f-1}\varrho\varphi_t, \qquad \varphi_t\tau = \tau\varphi_t\ .$$

Clearly, $\varphi_s\varphi_t$ is a permutation of $F(K)$ for which we obtain

$$\varphi_s\varphi_t\lambda = \varphi_s\lambda\tau^k\varphi_t = \lambda\tau^j\varphi_s\tau^k\varphi_t = \lambda\tau^{j+k}\varphi_s\varphi_t\ ,$$

$$\varphi_s\varphi_t\varrho = \varphi_s(\varrho\tau)^{f-1}\varrho\varphi_t = ((\varrho\tau)^{e-1}\varrho\tau)^{f-1}(\varrho\tau)^{e-1}\varrho\varphi_s\varphi_t = (\varrho\tau)^{ef-1}\varrho\varphi_s\varphi_t \ ,$$

$$\varphi_s\varphi_t\tau = \varphi_s\tau\varphi_t = \tau\varphi_s\varphi_t \ .$$

Thus, $\varphi_s\varphi_t$ isomorphically maps M to $M^{st} = (F(K);\, \lambda\tau^{j+k}, (\varrho\tau)^{ef-1}\varrho, \tau)$, and consequently, st is an exponent of M. □

If M is an unoriented map with underlying connected graph K, the propositions stated above imply that the residue classes of the integer parts of exponents of M modulo $n = \mathrm{val}\,(K)$ along with the exponent p^j form a multiplicative group. We will denote this group by $\mathrm{Ex}\,(M)$ and call it the *unoriented exponent group* of M. For the sake of simplicity, we shall usually omit the word 'unoriented'. Also, we shall not draw a distinct in notation between an exponent and its residue class in $\mathrm{Ex}\,(M)$.

The integer exponents of an unoriented map M obviously form a group that is a subgroup of $\mathrm{Ex}\,(M)$. We denote the subgroup consisting of all unoriented integer exponents of M by $\mathrm{Ex}_e\,(M)$.

From previous considerations it follows, too, that $\mathrm{Ex}\,(M)$ is always a subgroup of $\mathrm{Ex}\,(K) \times \mathbb{Z}_2$. Moreover, we can precise this relationship in the following way.

Proposition 4.6. *Let M be an unoriented map and n be the virtual valency of its underlying graph K. Then $\mathrm{Ex}\,(M)$ is a subgroup of $\mathbb{Z}_n^* \times \mathbb{Z}_2$.*

An exponent t of a map M is said to be *involutory* if its order in $\mathrm{Ex}\,(M)$ is ≤ 2. Clearly, $t = ep^j$ is an involutory exponent of M if and only if e is an involutory exponent of the graph K which underlies M. In other words, $t = ep^j$ is an involutory exponent of $M = (F;\, \lambda, \varrho, \tau)$ if and only if

$$(\varrho\tau)^{e^2} = \varrho\tau \ .$$

For instance, ± 1 are involutory exponents of every unoriented map and $\pm p$ are involutory exponents of all self-Petrie maps. We denote the subgroup of $\mathrm{Ex}\,(M)$ consisting of all involutory exponents of M by $\mathrm{Ex}_2\,(M)$, and the subgroup of all integer involutory exponents of M by $\mathrm{Ex}_{e^2}\,(M)$.

Now we show that unoriented exponent groups of Petrie maps coincide.

Proposition 4.7. *Let M be a 3-involutory map and $P(M)$ be the Petrie map of M. Then,*

$$\mathrm{Ex}\,(M) = \mathrm{Ex}\,(P(M)) \ .$$

Proof. Let $M = (F;\, \lambda, \varrho, \tau)$. Then $P(M) = (F;\, \lambda\tau, \varrho, \tau)$. First take any exponent $t \in \mathrm{Ex}\,(M)$, $t = ep^j$. By the definition of unoriented exponent,

$$M^t = (F;\, \lambda\tau^j, (\varrho\tau)^{e-1}\varrho, \tau) \cong M \ .$$

Then also

$$P(M^t) \cong P(M) \ .$$

But,

$$P(M^t) = (F;\, \lambda\tau^{j+1}, (\varrho\tau)^{e-1}\varrho, \tau) = P(M)^t \ .$$

Hence,

$$P(M)^t \cong P(M) \ ,$$

that is, $t \in \mathrm{Ex}\,(P(M))$.

Conversely, suppose that $t \in \mathrm{Ex}\ P(M)$. It means that $P(M)^t \cong P(M)$. Since $P(M)^t = P(M^t)$, we have

$$P(M^t) \cong P(M) \ .$$

Consequently,

$$P(P(M^t)) \cong P(P(M)) \ ,$$

which is equivalent to

$$M^t \cong M \ .$$

This implies $t \in \mathrm{Ex}\,(M)$, and the claim follows. □

Examples.

1. Regular embedding M_1 of the complete graph K_4 into the projective plane. Since $\mathrm{val}\,(K_4) = 3$ and the map is not self-Petrie, $\mathrm{Ex}\,(M_1) = \{\pm 1\} = \{1, 2\}$.

2. Regular embedding M_2 of the graph $C_4^{(2)}$ (cycle of length 4 with multiplicity of edges 2) in torus. The map M_2 is self-Petrie with virtual valency 4, thus $\mathrm{Ex}\,(M_2) = \{\pm 1, \pm p\} = \{1, 3, p, 3p\}$.

3. Regular embedding M_3 of the bouquet B_d of d circles in the projective plane. It is not self-Petrie, thus $\mathrm{Ex}\,(M_3) = \mathrm{Ex}\,(B_d) = \mathbb{Z}_{2d}^*$ [NS1].

Let M be a map with underlying graph K. Let s and t be exponents of M, and let φ and ψ be permutations of $F(K)$ associated with s and t, respectively. Then $\varphi\psi$ is a permutation of $F(K)$ associated with the exponent st. Thus, the permutations associated with the exponents of M form a group which we call the *exomorphism group* of M and denote by $\mathrm{Exo}\,(M)$. The elements of $\mathrm{Exo}\,(M)$ will be called *exomorphisms* of M.

Clearly, $\mathrm{Aut}\,(M)$ consists of all exomorphisms of M associated with the exponent 1, so $\mathrm{Aut}\,(M) \leq \mathrm{Exo}\,(M)$. The mapping $\Phi\colon\ \mathrm{Exo}\,(M) \to \mathrm{Ex}\,(M)$, which assigns to an exomorphism φ the exponent it is associated with, is a group epimorphism, and its kernel is $\mathrm{Aut}\,(M)$. Thus, $\mathrm{Aut}\,(M)$ is a normal subgroup of $\mathrm{Exo}\,(M)$.

We have already mentioned that any automorphism φ of a map M induces an automorphism of its underlying graph K. But, in general, this correspondence is not one-to-one. If $\tau \in \mathrm{Aut}\,(M)$, then the automorphisms φ and $\varphi\tau$ give rise to the same automorphism of K. However, this can occur only when the valency of K is ≤ 2, and regular maps of valency ≤ 2 are trivial. Therefore we can restrict our attention to maps with valency ≥ 3 for which $\mathrm{Aut}\,(M)$ induces a subgroup G of $\mathrm{Aut}\,(K)$ acting on $D(K)$ faithfully. Thus we obtain $\mathrm{Aut}\,(M) \cong G \leq \mathrm{Aut}\,(K)$.

On the other hand, the transversal involution τ is an exomorphism of every map M (see proof of Lemma 4.3), and it generates the group $\langle\tau\rangle$ which is a normal subgroup of $\mathrm{Exo}\,(M)$. Thus, $\mathrm{Exo}\,(M)/\langle\tau\rangle \cong H \leq \mathrm{Aut}\,(K)$.

We know that the algebraic description based on the three involutions can be applied to any 2-cell embedding on a closed orientable or nonorientable surface. Consider a 3-involutory map $M = (F(K); \lambda, \varrho, \tau)$ on an unoriented but orientable surface. Then we can choose one of the two possible orientations of the ambient surface and define the corresponding oriented map in a natural way.

To do this first observe that the arcs of the underlying graph K correspond to the orbits of $\langle\tau\rangle$ on the set $F(K)$. We represent an arbitrary arc $x \in D(K)$ by the orbit of $\langle\tau\rangle$ consisting of the two flags f and $\tau(f)$ incident to the arc x. To simplify the notation we denote the orbit $\{f, \tau(f)\}$ by f^τ. Hence $D(K) = \{f^\tau; f \in F(K)\}$ and we can define the permutations R and L of $D(K)$ in terms of the involutions of $F(K)$ by setting

$$R(f^\tau) = (\varrho\tau(f))^\tau \qquad \text{and} \qquad L(f^\tau) = (\lambda\tau(f))^\tau \ .$$

It is easy to see that R provides a rotation of arcs of K radiating from the same initial vertex and L is the arc-reversing involution. Thus we have constructed an oriented embedding $M^+ = (D(K), R, L)$ of the graph K.

Note that for any exponent e of K the permutation R^e of the arcs of K such that $R^e(f^\tau) = ((\varrho\tau)^e(f))^\tau$ also defines a rotation system on $D(K)$, therefore $(M^+)^e = (D(K), R^e, L)$ is an oriented map of the graph K,

The following statement answers the natural question about the mutual correspondence between the unoriented integer exponents of the map M and oriented exponents of the map M^+.

Theorem 4.8. *Let $M = (F(K); \lambda, \varrho, \tau)$ be a 3-involutory map on an orientable surface. Let M^+ be the oriented map created from M by choosing one of the two possible orientations of the underlying surface and let $\mathrm{Ex}^+(M^+)$ be the oriented exponent group of M^+. Then*

(1) $\mathrm{Ex}_e(M) = \mathrm{Ex}^+(M^+)$ *if M^+ is reflexible,*
(2) $\mathrm{Ex}_e(M) = \mathrm{Ex}^+(M^+) \times \langle -1\rangle$ *otherwise.*

Proof. Let $e \in \mathrm{Ex}_e(M)$ be any integer exponent of M and let $\varphi_e \in \mathrm{Exo}(M)$ be the exomorphism associated with e. Then

$$\varphi_e\lambda = \lambda\varphi_e, \quad \varphi_e\varrho = (\varrho\tau)^{e-1}\varrho\varphi_e \quad \text{and} \quad \varphi_e\tau = \tau\varphi_e \ ,$$

whence

$$\varphi_e(\varrho\tau) = (\varrho\tau)^{e-1}\varrho\varphi_e\tau = (\varrho\tau)^e\varphi_e \ .$$

Define a mapping $\varphi_e^+ \colon D(K) \to D(K)$ by setting

$$\varphi_e^+(f^\tau) = (\varphi_e(f))^\tau, \qquad \text{for any } f \in F(K) \ .$$

Since

$$\begin{aligned}\varphi_e^+ R(f^\tau) &= \varphi_e^+((\varrho\tau(f))^\tau) = (\varphi_e\varrho\tau(f))^\tau \\ &= ((\varrho\tau)^e\varphi_e(f))^\tau = R^e((\varphi_e(f))^\tau) = R^e\varphi_e^+(f^\tau)\end{aligned}$$

and

$$\begin{aligned}\varphi_e^+ L(f^\tau) &= \varphi_e^+((\lambda\tau(f))^\tau) = (\varphi_e\lambda\tau(f))^\tau \\ &= (\lambda\tau\varphi_e(f))^\tau = L((\varphi_e(f))^\tau) = L\varphi_e^+(f^\tau) \ ,\end{aligned}$$

the permutation $\varphi_e^+ \colon M^+ \to (M^+)^e$ is a map isomorphism. Consequently, φ_e^+ is an exomorphism of the oriented map M^+ associated with the oriented exponent e.

Let $\mathrm{Exo}^+(M^+)$ be the oriented exomorphism group of M^+. It can be easily seen that the assignment $\Sigma\colon \varphi_e \mapsto \varphi_e^+$ is a group homomorphism $\mathrm{Exo}(M) \to \mathrm{Exo}^+(M^+)$ that naturally induces a group homomorphism $\Theta\colon \mathrm{Ex}_e(M) \to \mathrm{Ex}^+(M^+)$. However, these mappings are not one-to-one. From Lemma 4.3 we know that -1 is an unoriented exponent of M and τ is the exomorphism associated with -1. Thus $\varphi_{-1} = \tau$ and we obtain

$$\varphi_{-1}^+(f^\tau) = (\tau(f))^\tau = \{\tau(f), \tau^2(f)\} = \{f, \tau(f)\} = f^\tau = \varphi_1^+(f^\tau)\ .$$

Nevertheless, τ along with the identity are the only permutations of $F(K)$ mapping the set f^τ onto itself. It follows that the kernel of Σ coincides with $\{\varphi_1, \varphi_{-1}\} = \{1, \tau\}$ and the kernel of Θ is equal to $\{+1, -1\}$. On the other hand, if the oriented map M^+ is chiral, Θ obviously provides a group epimorphism $\mathrm{Ex}_e(M) \to \mathrm{Ex}^+(M^+)$. Hence $\mathrm{Ex}_e(M) = \mathrm{Ex}^+(M^+) \times \langle -1 \rangle \cong \mathrm{Ex}^+(M^+) \times \mathbb{Z}_2$. But in case that the oriented map M^+ is reflexible, -1 is an oriented exponent of M^+ and $\mathrm{Ex}^+(M^+) = \Theta(\mathrm{Ex}_e(M)) \times \langle -1 \rangle = \mathrm{Ex}_e(M)$. □

From the previous proof it follows that the 3-involutory description of an orientable map does not contain enough information to reveal the reflexibility or the chirality of the oriented map. So the 3-involutory descriptions are in fact meaningful for embeddings of graphs on nonorientable surfaces.

5. Congruence of unoriented maps

Two maps $M = (F(K);\, \lambda, \varrho, \tau)$ and $N = (F(K);\, \lambda', \varrho', \tau')$ with the same underlying graph K are *congruent* if there is a permutation φ of $F(K)$ such that

$$\varphi\lambda' = \lambda\tau^j\varphi, \qquad \varphi\tau' = \tau\varphi, \qquad \varphi\varrho' = (\varrho\tau)^{e-1}\varrho\varphi,$$

for some exponent e of the graph K and $j \in \mathbb{Z}_2$. In this case, the permutation φ is called a *congruence*.

Notice that the congruence φ provides a map isomorphism $N \to M^t$, where $t = ep^j$ and e is any integer coprime with $\mathrm{val}(K)$. Of course, t need not be an unoriented exponent of the map M.

One can easily verify that the relation of congruence is reflexive, symmetric, and transitive on the set of maps of the same graph.

Proposition 5.1. *The relation of congruence is an equivalence on the set of all 3-involutory maps with the same underlying graph K.*

Now we state and prove the main result of this section.

Theorem 5.2. *Let M and N be two unoriented regular maps with the same underlying graph. If* $\mathrm{Aut}(M) = \mathrm{Aut}(N)$ *or* $\mathrm{Mon}(M) = \mathrm{Mon}(N)$*, then the maps M and N are congruent.*

Proof. Let $M = (F(K);\, \lambda, \varrho, \tau)$; $N = (F(K);\, \lambda', \varrho', \tau')$. Since the underlying graph K is the same, the definition of the transversal involution of a map forces $\tau' = \tau$.

Firstly let us consider the case $\mathrm{Aut}(M) = \mathrm{Aut}(N) = G$. Theorem 2.3 states that the stabilizer G_d of a fixed edge d under the action of G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Let $F_d = \{f_0, \tau(f_0), \lambda(f_0), \lambda\tau(f_0)\} = \{f_0, \tau'(f_0), \lambda'(f_0), \lambda'\tau'(f_0)\}$ be the set of four flags incident to d. The regularity of the action of G implies that

$$\langle \lambda, \tau \rangle | F_d = G_d | F_d = \langle \lambda', \tau' \rangle | F_d \ .$$

Hence we have $\lambda' | F_d = \lambda\tau^j | F_d$, for some $j \in \{0, 1\}$. We need to show that this equality extends to the whole set of flags $F(K)$. Indeed, let g be any flag in $F(K)$ and let φ_g be the unique automorphism in G mapping f_0 to g, that is, $\varphi_g(f_0) = g$. Then we have

$$\lambda'(g) = \lambda'\varphi_g(f_0) = \varphi_g\lambda'(f_0) = \varphi_g\lambda\tau^j(f_0) = \lambda\tau^j\varphi_g(f_0) = \lambda\tau^j(g),$$

for $j \in \mathbb{Z}_2$.

Furthermore, from Theorem 2.3 we also know that the stabilizer G_u of a fixed vertex u in the action of G is dihedral of order $2n$, where $n = \mathrm{val}\,(K)$. Let

$$\begin{aligned} F_u &= \{h_0, \varrho(h_0), \tau\varrho(h_0), \varrho\tau\varrho(h_0), \ldots, (\varrho\tau)^{2n-1}\varrho(h_0)\} \\ &= \{h_0, \varrho'(h_0), \tau'\varrho'(h_0), \varrho'\tau'\varrho'(h_0), \ldots, (\varrho'\tau')^{2n-1}\varrho'(h_0)\} \end{aligned}$$

be the set of $2n$ flags incident to the vertex u.

G acts regularly on $F(K)$, thus we obtain

$$\langle \varrho, \tau \rangle | F_u = G_u | F_u = \langle \varrho', \tau' \rangle | F_u \ .$$

However, $\tau' = \tau$, and taking different $e \in \mathrm{Ex}\,(K)$ we obtain the complete set of pairs of involutions $\langle \varrho', \tau \rangle | F_u$ generating a dihedral group of order $2n$. Therefore, $\varrho' | F_u = (\varrho\tau)^{e-1}\varrho | F_u$ for some integer e coprime with n.

We show that $\varrho' = (\varrho\tau)^{e-1}\varrho$ holds for all flags in $F(K)$. Actually, let us take any flag g and denote by ψ_g the unique automorphism in G for which $\psi_g(h_0) = g$. Then we obtain

$$\begin{aligned} \varrho'(g) &= \varrho'\psi_g(h_0) = \psi_g\varrho'(h_0) = \psi_g(\varrho\tau)^{e-1}\varrho(h_0) \\ &= (\varrho\tau)^{e-1}\varrho\psi_g(h_0) = (\varrho\tau)^{e-1}\varrho(g), \end{aligned}$$

that completes the proof for the case $\mathrm{Aut}\,(M) = \mathrm{Aut}\,(N)$.

In case that $\mathrm{Mon}\,(M) = \mathrm{Mon}\,(N)$, the statement follows from the fact that $\mathrm{Mon}\,(M) = \mathrm{Mon}\,(N)$ if and only if $\mathrm{Aut}\,(M) = \mathrm{Aut}\,(N)$. $\square$

From the proof of Theorem 5.2 it follows that we can take the identity mapping for the congruence φ. Thus, we have obtained even a stronger result.

Corollary 5.3. *Let $M = (F; \lambda, \varrho, \tau)$ and $N = (F; \lambda', \varrho', \tau')$ be regular maps with the same underlying graph K. Then the following statements are equivalent:*

(1) $\mathrm{Aut}\,(M) = \mathrm{Aut}\,(N)$,
(2) $\mathrm{Mon}\,(M) = \mathrm{Mon}\,(N)$,
(3) *M is congruent with N,*
(4) *there exist $e \in \mathrm{Ex}\,(K)$ and $j \in \mathbb{Z}_2$ such that*

$$\lambda' = \lambda\tau^j, \qquad \tau' = \tau, \quad \text{and} \quad \varrho' = (\varrho\tau)^{e-1}\varrho \ .$$

From the proof of Theorem 2.3 [GNSS] it directly follows that the number of regular maps with given underlying graph K and fixed group of map automorphisms equals $2.|\,\mathrm{Ex}\,(K)| = 2.\phi(n)$, where n is the valency of (any vertex of) the graph K and ϕ is Euler's integer function. Of course, some of these regular maps may be isomorphic.

Let M be a fixed regular map of K and let N be any regular map of K with $\mathrm{Aut}\,(N) = \mathrm{Aut}\,(M)$. From Corollary 5.3 we have $N = M^t$, where $t = ep^j$, $e \in \mathrm{Ex}\,(K)$ and $j \in \{0, 1\}$. Since $M \cong M^t$ for $t \in \mathrm{Ex}\,(M)$, there is a one-to-one correspondence between the isomorphism classes of the maps congruent with M and the elements of the quotient group $\mathbb{Z}_n^* \times \mathbb{Z}_2 / \mathrm{Ex}\,(M)$. Thus we have

Corollary 5.4. *Let K be a connected n-valent graph and let M be a regular map of K. Then the number of non-isomorphic regular maps N of K with* $\mathrm{Aut}\,(N) = \mathrm{Aut}\,(M)$ *equals* $2\phi(n)/|\,\mathrm{Ex}\,(M)|$.

6. Canonical double coverings of graphs and their maps

Consider a graph $K = (D, V, I, L)$. The *canonical double covering* of K is the graph $\tilde{K} = (\tilde{D}, \tilde{V}, \tilde{I}, \tilde{L})$ with the arc-set $\tilde{D} = D(\tilde{K}) = D(K) \times \mathbb{Z}_2$, the vertex-set $\tilde{V} = V(\tilde{K}) = V(K) \times \mathbb{Z}_2$, the incidence function

$$\tilde{I}(x, i) = (I(x), i), \qquad i \in \mathbb{Z}_2,$$

and the arc-reversing involution

$$\tilde{L}(x, i) = (L(x), i + 1), \qquad i \in \mathbb{Z}_2,$$

where $x \in D(K)$. The graph K is then said to be the *base graph* of $\tilde{K}$.

The mapping $\pi\colon \tilde{K} \to K$ which erases the second coordinate is a two-fold covering *projection*. In order to simplify the notation, we shall often write x_i for the couple (x, i), where x can be an arc or a vertex of K.

One can easily see that $\tilde{K}$ is a connected graph if and only if K is not bipartite. However, $\tilde{K}$ is always a bipartite graph, and we can view any of its vertices $v_i \in \tilde{V}$ as being coloured by the colour $i \in \mathbb{Z}_2$. Let $\tilde{K}$ has bipartition $\tilde{V} = V_0 \cup V_1$. An automorphism $\tilde{\varphi} \in \mathrm{Aut}\,(\tilde{K})$ will be called *colour-preserving* if $\tilde{\varphi}(V_i) = V_i$, and *colour-reversing* if $\tilde{\varphi}(V_i) = V_{i+1}$, for $i \in \mathbb{Z}_2$. Obviously, all colour-preserving automorphisms of $\tilde{K}$ form a group which is a subgroup of $\mathrm{Aut}\,(\tilde{K})$.

Next lemma says that in case K is not a bipartite graph, every automorphism of $\tilde{K}$ is either colour-preserving or colour-reversing, that is, $\mathrm{Aut}\,(\tilde{K})$ does not contain 'mixed' automorphisms.

Lemma 6.1. [NS2] *Every automorphism of a connected bipartite graph is either colour-preserving or colour-reversing.*

As also shown in [NS2], we are able to characterize the automorphisms of canonical double coverings in more details. Namely, for an arbitrary graph K, the automorphism group $\mathrm{Aut}\,(\tilde{K})$ contains a subgroup isomorphic to the Cartesian product $\mathrm{Aut}\,(K) \times \mathbb{Z}_2$. Indeed, for any automorphism $\psi \in \mathrm{Aut}\,(K)$ the mapping $\tilde{\psi}\colon \tilde{K} \to \tilde{K}$, defined by

$$\tilde{\psi}(x_i) = \psi(x)_i, \qquad x_i \in D(\tilde{K})\ ,$$

is a colour-preserving automorphism of $\tilde{K}$, and the assignment $\Theta\colon \psi \mapsto \tilde{\psi}$ is a monomorphism $\mathrm{Aut}\,(K) \to \mathrm{Aut}\,(\tilde{K})$. Besides that, $\mathrm{Aut}\,(\tilde{K})$ also contains a colour-reversing automorphism $\beta \notin \Theta(\mathrm{Aut}\,(K))$ given by

$$\beta(x_i) = x_{i+1}, \qquad x_i \in D(\tilde{K})$$

and $\tilde{\psi}\beta = \beta\tilde{\psi}$ for any $\psi \in \mathrm{Aut}\,(K)$.

Although $\mathrm{Aut}\,(\tilde{K})$ may contain additional automorphisms, the important special case happens when it does not. We call a graph K *stable* if $\mathrm{Aut}\,(\tilde{K})$ is isomorphic to $\mathrm{Aut}\,(K) \times \mathbb{Z}_2$; otherwise we call it *unstable*.

If K is a bipartite graph, then $\tilde{K}$ is disconnected and hence K is unstable. Thus a stable graph must be non-bipartite. The complete graph K_n provides an example of a symmetrical stable graph while the bouquet of n circles B_n is symmetrical, but unstable [NS2].

In later discussions we shall frequently refer to the following technical result.

Theorem 6.2. [NS2] *Let K be a stable graph and let $\pi\colon \tilde{K} \to K$ be the natural projection. Then,*

$$\pi\psi(x_0) = \pi\psi(x_1)\ ,$$

for every $\psi \in \mathrm{Aut}\ \tilde{K}$ and every $x \in D(K)$.

This allows us to define a mapping $\psi'\colon K \to K$ by setting

$$\psi'(x) = \pi\psi(x_0)\ ,$$

for $x \in D(K)$. The mapping ψ' is an automorphism of K [NS2].

Take a connected non-bipartite graph K and its 3-involutory (not necessarily regular) map $M = (F(K);\, \lambda, \varrho, \tau)$, where $F(K) = D(K) \times \{+1, -1\}$. Let $\tilde{K}$ be the canonical double covering of the graph K. Then each arc x of K is covered by a couple of arcs $x_0, x_1 \in D(\tilde{K})$, and consequently, for each flag $f = (x, k) \in F(K)$, $k \in \{+1, -1\}$, we have two corresponding flags $f_0 = (x_0, k)$ and $f_1 = (x_1, k)$ in $F(\tilde{K})$. Thus, $F(\tilde{K}) = F(K) \times \mathbb{Z}_2$.

Let $\pi\colon \tilde{K} \to K$ be the natural projection. For each arc (edge) x of K the two elements of $\pi^{-1}(x)$ of $\tilde{K}$ will be called *twin arcs* (*twin edges*).

Lemma 6.3. *Let K be a stable graph and G be the automorphism group of a regular embedding of $\tilde{K}$. Then the stabilizer of the eight flags incident to every twin edges under the action of G is isomorphic to $\mathbb{Z}_2^3$ or D_8 or $\mathbb{Z}_4 \times \mathbb{Z}_2$.*

Proof. Let $x \in D(K)$ be a fixed arc and $L(x) \in D(K)$ be the reverse arc to the arc x. Let F_x be the set of flags of the graph $\tilde{K}$ incident to twin arcs x_i and $L(x)_i$ that cover the arcs x and $L(x)$, respectively. Clearly, $F_x = \{(x_i, \pm 1), (L(x)_i, \pm 1)\}$, $i \in \{0, 1\}$, and the cardinality of F_x is 8. Let G_x be the subgroup of the automorphism group G that stabilizes F_x. Since the embedding $\tilde{M} = (F(\tilde{K}); \tilde{\lambda}, \tilde{\varrho}, \tilde{\tau})$ of the graph $\tilde{K}$ is regular, the order of G_x equals 8.

Notice that G_x always contains a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The embedding $\tilde{M}$ under consideration is regular, and from Theorem 2.3 we know that the stabilizer of any edge incident to F_x must be dihedral of order 4.

See also that the transversal involution $\tilde{\tau}$ of the embedding of $\tilde{K}$ coincides on F_x with one of the elements of G_x. In fact, let $\varphi_{\tilde{\tau}} \in G_x$ be the unique automorphism mapping the flag $f_0 = (x_0, +1)$ to the flag $\tilde{\tau}(f_0) = (x_0, -1)$, and vice versa. By Theorem 6.2, the stability of K implies that $\varphi_{\tilde{\tau}}$ maps the set of two flags incident to the arc x_1 onto itself. Furthermore, from the definition of the graph automorphism induced by $\varphi_{\tilde{\tau}}$ it follows that $\varphi_{\tilde{\tau}}$ stabilizes the set of flags incident to the arc $L(x)_i$ for both $i \in \{0, 1\}$. But besides the identity, the involution $\tilde{\tau}$ is the only permutation of F_x that meets all these conditions, thus $\varphi_{\tilde{\tau}}|F_x = \tilde{\tau}|F_x$.

For the flag $f_0 = (x_0, +1)$, denote ψ the colour-preserving element of G_x mapping f_0 to the flag $L(f)_0 = (L(x)_0, +1)$, and θ the colour-reversing element of G_x mapping f_0 to the flag $f_1 = (x_1, +1)$. The graph K is stable by the assumption, so

$$\psi^2(f_0) = f_0 \qquad \text{or} \qquad \psi^2(f_0) = \tilde{\tau}(f_0)$$

and

$$\theta^2(f_0) = f_0 \qquad \text{or} \qquad \theta^2(f_0) = \tilde{\tau}(f_0)\ .$$

In order to determine the stabilizer G_x we examine all the cases for ψ^2 and θ^2 that may happen.

(a) First consider $\psi^2(f_0) = \theta^2(f_0) = f_0$, that is, both ψ^2 and θ^2 must be identity mappings. Then we get

$$(x_0, +1) \xrightarrow{\psi} (L(x)_0, +1) \xrightarrow{\tilde{\tau}} (L(x)_0, -1) \xrightarrow{\psi} (x_0, -1)$$

and, for some $j \in \{+1, -1\}$,

$$(x_0, +1) \xrightarrow{\theta} (x_1, +1) \xrightarrow{\tilde{\tau}} (x_1, -1) \xrightarrow{\psi} (L(x)_1, j) \xrightarrow{\tilde{\tau}} (L(x)_1, -j)\ .$$

We see that in this case

$$G_x = \langle \psi,\ \theta,\ \tilde{\tau} \mid \psi^2 = \theta^2 = \tilde{\tau}^2 = 1 \rangle \cong \mathbb{Z}_2^3\ ,$$

where 1 is of course the identity permutation of $F(\tilde{K})$.

(b) Suppose that $\psi^2(f_0) = \tilde{\tau}(f_0)$ and $\theta^2(f_0) = f_0$. Then obviously ψ^4 and θ^2 are identity permutations whereas ψ^2 is not. So we obtain

$$(x_0, +1) \xrightarrow{\psi} (L(x)_0, +1) \xrightarrow{\psi} (x_0, -1) \xrightarrow{\psi} (L(x)_0, -1)$$

and

$$(x_0, +1) \xrightarrow{\theta} (x_1, +1) \xrightarrow{\psi} (L(x)_1, j) \xrightarrow{\psi} (x_1, -1) \xrightarrow{\psi} (L(x)_1, -j)\ .$$

Observe also that $\theta\psi(f_0) = \tilde{\lambda}\tilde{\tau}^i(f_0)$, for some $i \in \{0, 1\}$, whence $(\theta\psi)^2(f_0) = (\tilde{\lambda}\tilde{\tau}^i)^2(f_0) = f_0$. Therefore $(\theta\psi)^2 = 1$ and we have

$$G_x = \langle \psi,\ \theta \mid \psi^4 = \theta^2 = (\theta\psi)^2 = 1 \rangle \cong D_8\ .$$

(c) If $\psi^2(f_0) = (f_0)$ and $\theta^2(f_0) = \tilde{\tau}(f_0)$, we can analogously show that

$$G_x = \langle \psi,\ \theta \mid \psi^2 = \theta^4 = (\psi\theta)^2 = 1 \rangle \cong D_8\ .$$

(d) Assume that $\psi^2(f_0) = \theta^2(f_0) = \tilde{\tau}(f_0)$. Hence both ψ^4 and θ^4 are identity permutations while ψ^2 and θ^2 are not. Consequently,

$$(x_0, +1) \xrightarrow{\psi} (L(x)_0, +1) \xrightarrow{\psi} (x_0, -1) \xrightarrow{\psi} (L(x)_0, -1)$$

and, for some $j \in \{+1, -1\}$,

$$(x_0, +1) \xrightarrow{\theta} (x_1, +1) \xrightarrow{\psi} (L(x)_1, j) \xrightarrow{\psi} (x_1, -1) \xrightarrow{\psi} (L(x)_1, -j) \ .$$

Moreover, for some $i \in \{0, 1\}$ it obviously holds

$$(\psi\theta)^2 = (\tilde{\lambda}\tilde{\tau}^i)^2 = 1 = (\tilde{\tau})^2 = \psi^2\theta^2 \ .$$

Multiplying the equality $(\psi\theta)^2 = \psi^2\theta^2$ by $(\psi)^{-1}$ on the left and by $(\theta)^{-1}$ on the right we obtain $\theta\psi = \psi\theta$. Hence

$$G_x = \langle \psi, \ \theta \mid \psi^4 = \theta^4 = 1, \ \theta\psi = \psi\theta \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \ .$$

To complete the proof we need to show that $G_x \cong G_y$ for any $x, y \in D(K)$. Let $g \in F_y$ be an arbitrary fixed flag. Since the embedding of $\tilde{K}$ is regular, there is a unique automorphism $\psi_g \in \mathrm{Aut}\,(\tilde{M})$ taking $g \to f_0$. The stability of K implies that ψ_g is a one-to-one and onto mapping $F_y \to F_x$. Furthermore, for any $\sigma \in G_x$, the automorphism $\psi_g^{-1}\sigma\psi_g \in \mathrm{Aut}\,(\tilde{M})$ is obviously a stabilizer of F_y; thus the mapping $\Theta\colon \sigma \mapsto \psi_g^{-1}\sigma\psi_g$ provides the desired group isomorphism $G_x \to G_y$. □

Let K be a connected non-bipartite graph and $M = (F(K); \lambda, \varrho, \tau)$ be a map with the underlying graph K. Let $t = ep^j$ be an arbitrary unoriented exponent of M. Then we can construct the following permutations of the set $F(K) \times \mathbb{Z}_2$ by setting

$$\begin{aligned}
&\tilde{\lambda}(f_i) = \lambda\tau^j(f)_{i+1} \ , \\
&\tilde{\tau}(f_i) = \tau(f)_i \ , \\
&\tilde{\varrho}(f_i) = \begin{cases} \varrho(f)_0 \ , & \text{for } i = 0 \\ (\varrho\tau)^{e-1}\varrho(f)_1 \ , & \text{for } i = 1 \end{cases} \ ,
\end{aligned}$$

for any $f \in F(K)$ and $i \in \mathbb{Z}_2$.

We shall verify that the *derived* map $M_t = (F(K) \times \mathbb{Z}_2; \tilde{\lambda}, \tilde{\varrho}, \tilde{\tau})$ is a 3-involutory map with underlying graph $\tilde{K}$, a canonical double covering of the graph K. Since by assumption λ, ϱ and τ are involutory permutations of $F(K)$ and $\lambda\tau = \tau\lambda$, for all $f \in F(K)$ and $i \in \mathbb{Z}_2$ we obtain

$$\begin{aligned}
&\tilde{\lambda}^2(f_i) = \tilde{\lambda}(\lambda\tau^j(f)_{i+1}) = \lambda\tau^j(\lambda\tau^j(f))_i = \lambda^2\tau^{2j}(f)_i = f_i \ , \\
&\tilde{\tau}^2(f_i) = \tilde{\tau}(\tau(f)_i) = \tau(\tau(f))_i = \tau^2(f)_i = f_i \ , \\
&\tilde{\varrho}^2(f_0) = \tilde{\varrho}(\varrho(f)_0) = \varrho(\varrho(f))_0 = \varrho^2(f)_0 = f_0 \ , \\
&\tilde{\varrho}^2(f_1) = \tilde{\varrho}((\varrho\tau)^{e-1}\varrho(f)_1) = (\varrho\tau)^{e-1}\varrho(\varrho\tau)^{e-1}\varrho(f)_1 = (\varrho\tau)^{e-1}(\tau\varrho)^{e-1}(f)_1 = f_1 \ , \\
&\tilde{\lambda}\tilde{\tau}(f_i) = \tilde{\lambda}(\tau(f)_i) = \lambda\tau^j(\tau(f))_{i+1} = \tau(\lambda\tau^j(f))_{i+1} = \tilde{\tau}(\lambda\tau^j(f)_{i+1}) = \tilde{\tau}\tilde{\lambda}(f_i) \ .
\end{aligned}$$

We see that $\tilde{\lambda}$, $\tilde{\varrho}$ and $\tilde{\tau}$ are involutions of the flag set $F(\tilde{K}) = F(K) \times \mathbb{Z}_2$ and $\tilde{\lambda}\tilde{\tau} = \tilde{\tau}\tilde{\lambda}$. A short reflection shows that the underlying graph coincides with $\tilde{K}$, that is, the orbits of $\langle\tilde{\lambda}, \tilde{\tau}\rangle$ and $\langle\tilde{\varrho}, \tilde{\tau}\rangle$ correspond to edges and vertices of $\tilde{K}$, respectively. Thus the derived map M_t is in fact an embedding of $\tilde{K}$.

From Proposition 4.7 it follows that an arbitrary unoriented exponent t of M is an unoriented exponent of the Petrie map $P(M)$ as well. So applying the given technique we can build a derived map $P(M)_t$ with the same underlying graph $\tilde{K}$. The following statement shows a close relationship between maps of $\tilde{K}$ derived from Petrie duals M and $P(M)$ of the graph K.

Proposition 6.4. *Let M be a 3-involutory map and $t = ep^j$ be an unoriented exponent of M. Then,*

$$M_t \cong P(M)_{-t} \ .$$

Proof. Let $M = (F;\, \lambda, \varrho, \tau)$. Hence $P(M) = (F;\, \lambda\tau, \varrho, \tau)$. Since $t \in \mathrm{Ex}\,(M) = \mathrm{Ex}\,(P(M))$, then $-t \in \mathrm{Ex}\,(P(M))$, too, and we can define derived maps M_t and $P(M)_{-t}$.

Denote $M_t = (F \times \mathbb{Z}_2;\, \lambda', \varrho', \tau')$ where

$$\begin{aligned} \lambda'(f_i) &= \lambda\tau^j(f)_{i+1} \ , \\ \tau'(f_i) &= \tau(f)_i \ , \\ \varrho'(f_i) &= \begin{cases} \varrho(f)_0, & \text{for } i = 0 \\ (\varrho\tau)^{e-1}\varrho(f)_1, & \text{for } i = 1 \end{cases} \ , \end{aligned}$$

and $P(M)_{-t} = (F \times \mathbb{Z}_2;\, \bar{\lambda}, \bar{\varrho}, \bar{\tau})$ where

$$\begin{aligned} \bar{\lambda}(f_i) &= \lambda\tau^{j+1}(f)_{i+1} \ , \\ \bar{\tau}(f_i) &= \tau(f)_i \ , \\ \bar{\varrho}(f_i) &= \begin{cases} \varrho(f)_0, & \text{for } i = 0 \\ (\varrho\tau)^{-e-1}\varrho(f)_1, & \text{for } i = 1 \end{cases} \ , \end{aligned}$$

for all $f \in F$ and $i \in \mathbb{Z}_2$. Observe that

$$(\varrho\tau)^{-e-1}\varrho = \left((\varrho\tau)^{-1}\right)^{e+1}\varrho = \left(\tau^{-1}\varrho^{-1}\right)^{e+1}\varrho = (\tau\varrho)^{e+1}\varrho = (\tau\varrho)^e\tau \ ,$$

therefore $\bar{\varrho}(f_1) = (\tau\varrho)^e\tau(f)_1$.

Now we define a mapping $\varphi\colon F \times \mathbb{Z}_2 \to F \times \mathbb{Z}_2$ by setting

$$\varphi(f_i) = \tau^i(f)_i \ ,$$

for all $f_i \in F \times \mathbb{Z}_2$. We claim that φ is a map isomorphism $M_t \to P(M)_{-t}$. Indeed,

$$\begin{aligned} \varphi\lambda'(f_i) &= \varphi(\lambda\tau^j(f)_{i+1}) = \tau^{i+1}\lambda\tau^j(f)_{i+1} = \lambda\tau^{j+1}(\tau^i(f))_{i+1} = \bar{\lambda}(\tau^i(f)_i) = \bar{\lambda}\varphi(f_i), \\ \varphi\tau'(f_i) &= \varphi(\tau(f)_i) = \tau^i\tau(f)_i = \tau(\tau^i(f))_i = \bar{\tau}(\tau^i(f)_i) = \bar{\tau}\varphi(f_i) \ , \\ \varphi\varrho'(f_0) &= \varphi(\varrho(f)_0) = \tau^0\varrho(f)_0 = \varrho(f)_0 = \bar{\varrho}(f_0) = \bar{\varrho}(\tau^0(f)_0) = \bar{\varrho}\varphi(f_0) \ , \\ \varphi\varrho'(f_1) &= \varphi((\varrho\tau)^{e-1}\varrho(f)_1) = \tau^1(\varrho\tau)^{e-1}\varrho(f)_1 = (\tau\varrho)^e(f)_1 = (\tau\varrho)^e\tau(\tau(f))_1 \\ &= \bar{\varrho}(\tau^1(f)_1) = \bar{\varrho}\varphi(f_1) \ . \end{aligned}$$

We see that

$$\varphi\lambda' = \bar\lambda\varphi, \qquad \varphi\tau' = \bar\tau\varphi, \quad \text{and} \quad \varphi\varrho' = \bar\varrho\varphi \ ,$$

so φ provides the desired map isomorphism that proves our statement. □

From the definition of derived maps it directly follows that $P(M_t) = P(M)_t$, for any $t \in \mathrm{Ex}\,(M)$. By Proposition 6.4, $P(M)_t$ is isomorphic to M_{-t}, and we have the following result.

Corollary 6.5. *Let t be an exponent of a map M. Then,*

$$P(M_t) \cong M_{-t} \ .$$

As far as the involutory exponents are concerned, we can say a little bit more about the relationship between the maps derived from a couple of Petrie duals M and $P(M)$.

Proposition 6.6. *Let $e \in \mathbb{Z}$ and ep be involutory exponents of a map M. Then,*

$$M_e \cong M_{ep} \ .$$

Proof. Since e is an exponent of M, the map $M = (F;\, \lambda, \varrho, \tau)$ is isomorphic to the map $M^e = (F;\, \lambda, (\varrho\tau)^{e-1}\varrho, \tau)$. Let $\varphi\colon F \to F$ be the exomorphism associated with e, that is,

$$\varphi\lambda = \lambda\varphi, \qquad \varphi\tau = \tau\varphi, \qquad \varphi\varrho = (\varrho\tau)^{e-1}\varrho\varphi \ .$$

Analogously, let $\psi\colon F \to F$ be the exomorphism associated with the exponent ep. Then ψ provides a map isomorphism $M \to M^{ep}$, where $M^{ep} = (F;\, \lambda\tau, (\varrho\tau)^{e-1}\varrho, \tau)$, therefore

$$\psi\lambda = \lambda\tau\psi, \qquad \psi\tau = \tau\psi, \qquad \psi\varrho = (\varrho\tau)^{e-1}\varrho\psi \ .$$

In order to show that the derived map $M_e = (F \times \mathbb{Z}_2;\, \lambda', \varrho', \tau')$ where

$$\begin{aligned}
\lambda'(f_i) &= \lambda(f)_{i+1} \ , \\
\tau'(f_i) &= \tau(f)_i \ , \\
\varrho'(f_i) &= \begin{cases} \varrho(f)_0, & \text{for } i = 0 \\ (\varrho\tau)^{e-1}\varrho(f)_1, & \text{for } i = 1 \end{cases} \ ,
\end{aligned}$$

is isomorphic to the derived map $M_{ep} = (F \times \mathbb{Z}_2;\, \lambda'', \varrho'', \tau'')$ such that

$$\begin{aligned}
\lambda''(f_i) &= \lambda\tau(f)_{i+1} \ , \\
\tau''(f_i) &= \tau(f)_i \ , \\
\varrho''(f_i) &= \begin{cases} \varrho(f)_0, & \text{for } i = 0 \\ (\varrho\tau)^{e-1}\varrho(f)_1, & \text{for } i = 1 \end{cases} \ ,
\end{aligned}$$

we define a permutation $\theta\colon F \times \mathbb{Z}_2 \to F \times \mathbb{Z}_2$ by setting

$$\theta(f_i) = \varphi\psi(f)_i \ ,$$

for every $f \in F$ and $i \in \mathbb{Z}_2$. Using the fact that e is involutory along with the equalities stated above we obtain

$$
\begin{aligned}
\theta\lambda'(f_i) &= \theta(\lambda(f)_{i+1}) = \varphi\psi\lambda(f)_{i+1} = \varphi\lambda\tau\psi(f)_{i+1} \\
&= \lambda\tau\varphi\psi(f)_{i+1} = \lambda''(\varphi\psi(f)_i) = \lambda''\theta(f_i) \ , \\
\theta\tau'(f_i) &= \theta(\tau(f)_i) = \varphi\psi\tau(f)_i = \tau\varphi\psi(f)_i = \tau''(\varphi\psi(f)_i) = \tau''\theta(f_i) \ , \\
\theta\varrho'(f_0) &= \theta(\varrho(f)_0) = \varphi\psi\varrho(f)_0 = \varphi(\varrho\tau)^{e-1}\varrho\psi(f)_0 \\
&= ((\varrho\tau)^{e-1}\varrho\tau)^{e-1}(\varrho\tau)^{e-1}\varrho\varphi\psi(f)_0 = (\varrho\tau)^{e^2-1}\varrho\varphi\psi(f)_0 \\
&= \varrho\varphi\psi(f)_0 = \varrho''(\varphi\psi(f)_0) = \varrho''\theta(f_0) \ , \\
\theta\varrho'(f_1) &= \theta((\varrho\tau)^{e-1}\varrho(f)_1) = \varphi\psi(\varrho\tau)^{e-1}\varrho(f)_1 \\
&= \varphi((\varrho\tau)^{e-1}\varrho\tau)^{e-1}(\varrho\tau)^{e-1}\varrho\psi(f)_1 = \varphi(\varrho\tau)^{e^2-1}\varrho\psi(f)_1 \\
&= \varphi\varrho\psi(f)_1 = (\varrho\tau)^{e-1}\varrho\varphi\psi(f)_1 = \varrho''(\varphi\psi(f)_1) = \varrho''\theta(f_1) \ .
\end{aligned}
$$

Thus,

$$\theta\lambda' = \lambda''\theta, \qquad \theta\tau' = \tau''\theta, \quad \text{and} \quad \theta\varrho' = \varrho''\theta \ ,$$

whence θ is the map isomorphism taking $M_e \to M_{ep}$ and completing the proof. □

Note that the derived map M_{ep} clearly coincides with the map $P(M)_e$. Moreover, since e is involutory, $e^{-1} = e$. This fact leads to the following conclusion.

Corollary 6.7. *Let $e \in \mathbb{Z}$ and ep be involutory exponents of a map M. Then,*

$$M_{e^{-1}} = M_e \cong P(M)_e \cong M_{-e} \ .$$

7. Classification theorems for regular maps of canonical double coverings

The following theorem covers the first step in the classification. We show that the map M_t of the canonical double covering $\tilde{K}$ which is derived from a given regular map M of the base graph K with use of unoriented exponents meeting specified criteria is also regular. In order to prove the regularity, we 'lift' every automorphism of the base map M to a colour-preserving automorphism of M_t, and the exomorphism of M associated with a suitable exponent to a colour-reversing automorphism of M_t.

Theorem 7.1. *Let $M = (F; \lambda, \varrho, \tau)$ be a regular map with underlying connected non-bipartite graph K. Let t be an exponent of M such that*

(a) *$t = e$ where $e^2 \equiv 1$ (modulo the valency of K), or*
(b) *$t = ep$ where $e^2 \equiv -1$ (modulo the valency of K).*

Then the derived map $M_t = (F \times \mathbb{Z}_2; \tilde{\lambda}, \tilde{\varrho}, \tilde{\tau})$ defined above is a regular map of the canonical double covering $\tilde{K}$.

Proof. It is sufficient to show that $|\operatorname{Aut}(M_t)| \geq 2|\operatorname{Aut}(M)|$. First, we shall construct a monomorphism mapping $\operatorname{Aut}(M)$ into $\operatorname{Aut}(M_t)$. To do this, we use the fact that every automorphism $\xi \in \mathrm{Aut}\,(M)$ naturally lifts to a colour-preserving automorphism $\tilde{\xi} \in \mathrm{Aut}\,(M_t)$ by setting

$$\tilde{\xi}(f_i) = \xi(f)_i, \qquad \text{for any } f \in F \text{ and } i = 0, 1.$$

Employing the equalities

$$\xi\lambda = \lambda\xi, \qquad \xi\varrho = \varrho\xi, \qquad \xi\tau = \tau\xi \ ,$$

we obtain

$$\begin{aligned}
&\tilde{\xi}\tilde{\lambda}(f_i) = \tilde{\xi}(\lambda\tau^j(f)_{i+1}) = \xi\lambda\tau^j(f)_{i+1} = \lambda\tau^j\xi(f)_{i+1} = \tilde{\lambda}(\xi(f)_i) = \tilde{\lambda}\tilde{\xi}(f_i) \ ,\\
&\tilde{\xi}\tilde{\tau}(f_i) = \tilde{\xi}(\tau(f)_i) = \xi\tau(f)_i = \tau\xi(f)_i = \tilde{\tau}(\xi(f)_i) = \tilde{\tau}\tilde{\xi}(f_i) \ ,\\
&\tilde{\xi}\tilde{\varrho}(f_0) = \tilde{\xi}(\varrho(f)_0) = \xi\varrho(f)_0 = \varrho\xi(f)_0 = \tilde{\varrho}(\xi(f)_0) = \tilde{\varrho}\tilde{\xi}(f_0) \ ,\\
&\tilde{\xi}\tilde{\varrho}(f_1) = \tilde{\xi}((\varrho\tau)^{e-1}\varrho(f)_1) = \xi(\varrho\tau)^{e-1}\varrho(f)_1 = (\varrho\tau)^{e-1}\varrho\xi(f)_1 = \tilde{\varrho}(\xi(f)_1) = \tilde{\varrho}\tilde{\xi}(f_1) \ ,
\end{aligned}$$

for every flag $f \in F$ and every $i \in \mathbb{Z}_2$.

Clearly, the assignment $\Theta\colon \xi \mapsto \tilde{\xi}$ is a monomorphism mapping $\mathrm{Aut}\,(M)$ into $\mathrm{Aut}\,(M_t)$, and we have $|\,\mathrm{Aut}\,(M_t)| \geq |\,\mathrm{Aut}\,(M)|$. But, $\mathrm{Im}\,\Theta$ is a subgroup of $\mathrm{Aut}\,(M_t)$. Hence, by Lagrange's subgroup theorem, $|\,\mathrm{Aut}\,(M)|$ divides $|\,\mathrm{Aut}\,(M_t)|$.

To finish the proof of regularity, it is sufficient to find another automorphism of M_t which does not belong to $\mathrm{Im}\,\Theta$. To accomplish that, we use the exomorphism θ of the base map M associated with the exponent t. We separately discuss the two possible cases for t.

(a) First assume that $t = e$ is an involutory integer exponent of M. Then

$$\theta\lambda = \lambda\theta, \qquad \theta\tau = \tau\theta, \qquad \theta\varrho = (\varrho\tau)^{e-1}\varrho\theta \ .$$

and the derived map $M_t = M_e$ is defined by involutions

$$\tilde{\lambda}(f_i) = \lambda(f)_{i+1}, \qquad \tilde{\tau}(f_i) = \tau(f)_i, \qquad \tilde{\varrho}(f_i) = (\varrho\tau)^{e^i-1}\varrho(f)_i \ .$$

Let $\tilde{\theta}\colon F \times \mathbb{Z}_2 \to F \times \mathbb{Z}_2$ be the mapping defined by

$$\tilde{\theta}(f_i) = \theta(f)_{i+1} \ .$$

We show that $\tilde{\theta}$ is an automorphism of M_t. For any $f \in F$ and $i \in \mathbb{Z}_2$ we have

$$\begin{aligned}
&\tilde{\theta}\tilde{\lambda}(f_i) = \tilde{\theta}(\lambda(f)_{i+1}) = \theta\lambda(f)_i = \lambda\theta(f)_i = \tilde{\lambda}(\theta(f)_{i+1}) = \tilde{\lambda}\tilde{\theta}(f_i) \ ,\\
&\tilde{\theta}\tilde{\tau}(f_i) = \tilde{\theta}(\tau(f)_i) = \theta\tau(f)_{i+1} = \tau\theta(f)_{i+1} = \tilde{\tau}(\theta(f)_{i+1}) = \tilde{\tau}\tilde{\theta}(f_i) \ ,\\
&\tilde{\theta}\tilde{\varrho}(f_0) = \tilde{\theta}(\varrho(f)_0) = \theta\varrho(f)_1 = (\varrho\tau)^{e-1}\varrho\theta(f)_1 = \tilde{\varrho}(\theta(f)_1) = \tilde{\varrho}\tilde{\theta}(f_0) \ ,
\end{aligned}$$

and, using the fact that e is involutory, we get

$$\begin{aligned}
\tilde{\theta}\tilde{\varrho}(f_1) &= \tilde{\theta}((\varrho\tau)^{e-1}\varrho(f)_1) = \theta(\varrho\tau)^{e-1}\varrho(f)_0 = ((\varrho\tau)^{e-1}\varrho\tau)^{e-1}(\varrho\tau)^{e-1}\varrho\theta(f)_0 =\\
&= (\varrho\tau)^{e^2-1}\varrho\theta(f)_0 = \varrho\theta(f)_0 = \tilde{\varrho}(\theta(f)_0) = \tilde{\varrho}\tilde{\theta}(f_1) \ .
\end{aligned}$$

(b) Now assume that $t = ep$ is a mixed exponent of M such that $e^2 \equiv -1$ modulo the valency of K. But θ is the exomorphism of M associated with ep, so we have

$$\theta\lambda = \lambda\tau\theta, \qquad \theta\tau = \tau\theta, \qquad \theta\varrho = (\varrho\tau)^{e-1}\varrho\theta \ .$$

and the derived map $M_t = M_{ep}$ is defined as follows:

$$\tilde{\lambda}(f_i) = \lambda\tau(f)_{i+1}, \qquad \tilde{\tau}(f_i) = \tau(f)_i, \qquad \tilde{\varrho}(f_i) = (\varrho\tau)^{e^i-1}\varrho(f)_i \ .$$

Let $\tilde{\theta}\colon F \times \mathbb{Z}_2 \to F \times \mathbb{Z}_2$ be a colour-reversing permutation defined by setting

$$\tilde{\theta}(f_i) = \tau^i\theta(f)_{i+1} \ ,$$

for every $f \in F$ and $i \in \mathbb{Z}_2$. We show that $\tilde{\theta}$ is an automorphism of the map M_t. Indeed,

$$\begin{aligned}
\tilde{\theta}\tilde{\lambda}(f_i) &= \tilde{\theta}(\lambda\tau(f)_{i+1}) = \tau^{i+1}\theta\lambda\tau(f)_i = \lambda\tau(\tau^i\theta(f))_i = \tilde{\lambda}(\tau^i\theta(f)_{i+1}) = \tilde{\lambda}\tilde{\theta}(f_i) \ ,\\
\tilde{\theta}\tilde{\tau}(f_i) &= \tilde{\theta}(\tau(f)_i) = \tau^i\theta\tau(f)_{i+1} = \tau(\tau^i\theta(f))_{i+1} = \tilde{\tau}(\tau^i\theta(f)_{i+1}) = \tilde{\tau}\tilde{\theta}(f_i) \ ,\\
\tilde{\theta}\tilde{\varrho}(f_0) &= \tilde{\theta}(\varrho(f)_0) = \theta\varrho(f)_1 = (\varrho\tau)^{e-1}\varrho\theta(f)_1 = \tilde{\varrho}(\theta(f)_1) = \tilde{\varrho}\tilde{\theta}(f_0) \ ,
\end{aligned}$$

and from the fact that $e^2 \equiv -1$ (modulo $\mathrm{val}\,(K)$) we finally get

$$\begin{aligned}
\tilde{\theta}\tilde{\varrho}(f_1) &= \tilde{\theta}((\varrho\tau)^{e-1}\varrho(f)_1) = \tau\theta(\varrho\tau)^{e-1}\varrho(f)_0 = \tau((\varrho\tau)^{e-1}\varrho\tau)^{e-1}(\varrho\tau)^{e-1}\varrho\theta(f)_0\\
&= \tau(\varrho\tau)^{e^2-1}\varrho\theta(f)_0 = \tau(\varrho\tau)^{-2}\varrho\theta(f)_0 = \tau(\tau\varrho)^2\varrho\theta(f)_0\\
&= \varrho(\tau\theta(f))_0 = \tilde{\varrho}(\tau\theta(f)_0) = \tilde{\varrho}\tilde{\theta}(f_1) \ .
\end{aligned}$$

We proved that, in both cases, $\tilde{\theta} \in \mathrm{Aut}\,(M_t)$. Since $\tilde{\theta}$ is a colour-reversing automorphism, $\tilde{\theta} \notin \mathrm{Im}\ \Theta$. Hence, $|\,\mathrm{Aut}\,(M_t)| > |\,\mathrm{Aut}\,(M)|$. However, $|\,\mathrm{Aut}\,(M)|$ divides $|\,\mathrm{Aut}\,(M_t)|$. Thus we obtain the inequality $|\,\mathrm{Aut}\,(M_t)| \geq 2|\,\mathrm{Aut}\,(M)|$ which implies the regularity of M_t and completes the proof. □

Now we have a technique which can be applied to any regular map M of a connected non-bipartite graph K and any exponent t of M meeting the stated conditions to produce a regular map M_t of the canonical double covering $\tilde{K}$. Next step in our classification of regular maps of the canonical double covering graphs consists in reversing the Theorem 7.1. In other words, we need to find conditions under which a regular map $\tilde{M}$ of $\tilde{K}$ can be projected to a regular map N of K such that $\tilde{M}$ is isomorphic to N_t for some exponent t of N. Recall that it is the stability of the graph K that provides the additional information sufficient to do that for orientable maps.

First of all we need to analyze in more details the properties of the automorphism groups of regular maps of the canonical double coverings. Let K be a stable graph and $\tilde{M} = (F(\tilde{K});\, \tilde{\lambda}, \tilde{\varrho}, \tilde{\tau})$ be a regular map of the graph $\tilde{K}$. Since $\tilde{K}$ is the canonical double covering of K, each arc $u \in D(K)$ is covered by twin arcs $u_0, u_1 \in D(\tilde{K})$. We can also write

$$F(\tilde{K}) = \tilde{F}_0 \cup \tilde{F}_1 \ ,$$

where the set $\tilde{F}_i$ contains all flags of the graph $\tilde{K}$ incident to some arc u_i, for $u \in D(K)$ and $i = 0, 1$. Clearly, $\tilde{F}_0$ and $\tilde{F}_1$ are disjoint sets, with $|\tilde{F}_0| = |\tilde{F}_1|$. Recall that the involutory permutations $\tilde{\varrho}$ and $\tilde{\tau}$ map each of the sets $\tilde{F}_i$ onto itself, while the involution $\tilde{\lambda}$ provides mappings $\tilde{F}_0 \to \tilde{F}_1$ and $\tilde{F}_1 \to \tilde{F}_0$.

Take a fixed flag $f_0 = (x_0, +1)$ of $\tilde{K}$, $x \in D(K)$. Let θ denote the colour-reversing automorphism of $\tilde{M}$ mapping $f_0 \mapsto f_1$, where $f_1 = (x_1, +1)$. The stability of K implies that either

$$\theta^2(f_0) = (x_0, +1) = f_0 \qquad \text{or} \qquad \theta^2(f_0) = (x_0, -1) = \tilde{\tau}(f_0) \ .$$

Let $g_0 = (y_0, +1) \in F(\tilde{K})$ be an arbitrary flag, $y \in D(K)$. The map $\tilde{M}$ is regular, so there is a unique colour-preserving automorphism $\psi_g \in \mathrm{Aut}\,(\tilde{M})$ such that $\psi_g(g_0) = f_0$. We define a mapping $\xi_g \colon F(\tilde{K}) \to F(\tilde{K})$ by setting

$$\xi_g = \psi_g^{-1} \theta \psi_g \ .$$

Clearly, ξ_g is a colour-reversing automorphism of $\tilde{M}$. We assume that K is stable, hence from Theorem 6.2 it follows that the automorphism of the underlying graph $\tilde{K}$ induced by ψ_g maps the arc $y_1 \in D(\tilde{K})$ to the arc $x_1 \in D(\tilde{K})$. Hence,

$$\xi_g(g_0) = (y_1, +1) = g_1 \qquad \text{or} \qquad \xi_g(g_0) = (y_1, -1) = \tilde{\tau}(g_1) \ .$$

However, we can always rename the flags in $\tilde{F}_1$ so that

$$\xi_g(g_0) = \psi_g^{-1} \theta \psi_g(g_0) = g_1 \ ,$$

for any $g_0 \in F(\tilde{K})$. It means that

$$\psi_g(g_i) = f_i \ .$$

For every $g \in F(K)$ we also obtain

$$\xi_g(g_1) = \psi_g^{-1} \theta \psi_g(g_1) = \psi_g^{-1} \theta (\psi_g \psi_g^{-1}) \theta \psi_g(g_0) = \psi_g^{-1} \theta^2 \psi_g(g_0) = \psi_g^{-1} \theta^2 (f_0) \ ,$$

so

$$\xi_g(g_1) = \psi_g^{-1}(f_0) = g_0 \qquad \text{or} \qquad \xi_g(g_0) = \psi_g^{-1} \tilde{\tau}(f_0) = \tilde{\tau}(g_0) \ .$$

For the fixed flag $f_0 = (x_0, +1)$ of $\tilde{K}$ denote ψ the colour-preserving automorphism of $\tilde{M}$ mapping $f_0 \mapsto L(f)_0$, where $L(f)_0 = (L(x)_0, +1)$. When we apply the index-erasing projection on the graph $\tilde{K}$, the two arcs x_0 and $L(x)_0$ are projected to the reverse arcs x and $L(x)$ of the same edge of K. The quadruple of flags incident to the arcs x_0 and $L(x)_0$ are of course projected to the corresponding flags incident to x and $L(x)$. However, by Theorem 2.3, the stabilizer of this edge in the automorphism group of an arbitrary regular map of the graph K is dihedral of order 4. We intend to construct a regular map of K from the given regular map $\tilde{M}$ of $\tilde{K}$ using the projection, so we need the stabilizer of the two arcs x_0 and $L(x)_0$ to be dihedral of order 4 as well. But ψ is one element of the stabilizer in concern, therefore we naturally require $\psi^2(f_0) = f_0$, that is, $\psi^2 = 1$. This idea leads us to define the concept of splittability for regular maps of the canonical double coverings in the following way.

Let $\tilde{M}$ be a regular map of $\tilde{K}$. We say that $\tilde{M}$ is a *splittable* regular map of $\tilde{K}$ if the stabilizer G_{x_0} of any pair of arcs x_0 and $L(x)_0$ in $\mathrm{Aut}\,(\tilde{M})$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The definition of splittability given above is clearly equivalent to the following one: a regular map $\tilde{M}$ of $\tilde{K}$ is splittable if for any fixed flags $f_0 = (x_0, +1)$ and $L(f)_0 = (L(x)_0, +1)$ of $\tilde{K}$ there is a unique automorphism $\psi \in \mathrm{Aut}\,(\tilde{M})$ such that $\psi(f_0) = L(f)_0$ and $\psi^2 = 1$.

Lemma 7.2. *Map M_e derived from a regular map M of a connected non-bipartite graph K is a regular splittable map of $\tilde{K}$.*

Proof. M_e is regular by Theorem 7.1, so we only need to prove its splittability. Let $f_0 = (x_0, +1)$ and $L(f)_0 = (L(x)_0, +1)$ be fixed flags of $\tilde{K}$, $x \in D(K)$. Then $f = (x, +1)$ and $L(f) = (L(x), +1)$ are flags of K corresponding to f_0 and $L(f)_0$, respectively. Notice that flags f and $L(f)$ are incident to reverse arcs, i. e., to the same edge of the base graph K. Denote by d the edge of K that corresponds to arcs x and $L(x)$.

Let $\varphi \in \mathrm{Aut}\,(M)$ be the unique automorphism mapping f to $L(f)$, i. e., $\varphi(f) = L(f)$. The regularity of M also implies that the stabilizer G_d of the edge d under the action of $\mathrm{Aut}\,(M)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (Theorem 2.3). Hence, $\varphi^2 = 1$, that is, $\varphi(L(f)) = f$.

Consider now the colour-preserving automorphism $\tilde{\varphi}$ of M_e defined by $\tilde{\varphi}(h_i) = \varphi(h)_i$, $h \in F(K)$, $i \in \mathbb{Z}_2$. We have

$$\tilde{\varphi}(f_0) = \varphi(f)_0 = L(f)_0 \qquad \text{and} \qquad \tilde{\varphi}(L(f)_0) = \varphi(L(f))_0 = f_0 \ .$$

Since f_0 was taken to be arbitrary, $\tilde{\varphi}^2 = 1$, and the proof is complete. $\square$

Further we shall study the properties of the automorphism group $\mathrm{Aut}\,(\tilde{M})$ of a splittable regular map $\tilde{M}$. Note that from Lemma 6.3 it follows that the twin edges stabilizer G_x in $\mathrm{Aut}\,(\tilde{M})$ is isomorphic either to $\mathbb{Z}_2^3$ or to D_8. Using the notation defined above we have

$$\theta^2(f_0) = \theta(f_1) = \begin{cases} f_0, & \text{if } G_x \text{ is isomorphic to } \mathbb{Z}_2^3 \\ \tilde{\tau}(f_0), & \text{if } G_x \text{ is isomorphic to } D_8 \end{cases} ,$$

and consequently

$$\xi_g(g_1) = \tilde{\tau}^k(g_0) \ ,$$

where $k = 0$ if $G_x \cong \mathbb{Z}_2^3$, and $k = 1$ if $G_x \cong D_8$.

Recall that we denote by ψ the unique colour-preserving automorphism of $\tilde{M}$ that maps the flag $f_0 = (x_0, +1)$ to the flag $L(f)_0 = (L(x)_0, +1)$. Let $g_i \in F(\tilde{K})$ be an arbitrary flag, $i \in \{0, 1\}$. Then we can define a mapping $\lambda\colon F(\tilde{K}) \to F(\tilde{K})$ by setting

$$\lambda(g_i) = \psi_g^{-1}\psi\psi_g(g_i) \ .$$

Since $\tilde{M}$ is splittable, $\psi^2 = 1$, and we have

$$\lambda^2(g_i) = \psi_g^{-1}\psi(\psi_g\psi_g^{-1})\psi\psi_g(g_i) = \psi_g^{-1}\psi^2\psi_g(g_i) = g_i \ .$$

Thus $\lambda^2 = 1$. Moreover, we shall show that λ follows the behaviour of the involutory permutation $\tilde{\lambda}$ within the whole regular splittable map $\tilde{M}$.

Lemma 7.3. *Let* $\tilde{M} = (F(\tilde{K}); \tilde{\lambda}, \tilde{\varrho}, \tilde{\tau})$ *be a regular splittable map of the graph* $\tilde{K}$*. Let* λ *be the automorphism of* $\tilde{M}$ *defined above. Then,*

$$\lambda(g_i) = \tilde{\lambda}(g_{i+1}) \qquad or \qquad \lambda(g_i) = \tilde{\lambda}\tilde{\tau}(g_{i+1}) \ ,$$

for any $g \in F(K)$ *and* $i \in \mathbb{Z}_2$*.*

Proof. Consider a fixed flag $f_0 = (x_0, +1)$ of $\tilde{K}$, $x \in D(K)$. Note that

$$\lambda(f_i) = \psi(f_i) = (L(x)_i, +1) \ .$$

From the definition of the canonical double covering graph we also have

$$\tilde{\lambda}(x_0, +1) = (L(x)_1, +1) \qquad \text{or} \qquad \tilde{\lambda}(x_0, +1) = (L(x)_1, -1) \ ,$$

whence

$$\lambda(f_1) = \tilde{\lambda}(f_0) \qquad \text{or} \qquad \lambda(f_1) = \tilde{\lambda}\tilde{\tau}(f_0) \ .$$

We distinguish these two cases in the considerations that follow.

(a) First assume that $\lambda(f_1) = \tilde{\lambda}(f_0)$. Then

$$\tilde{\lambda}(f_0) = (L(x)_1, +1) = \psi^{-1}\theta\psi(L(x)_0, +1) = \psi^{-1}\theta(x_0, +1) = \psi^{-1}\theta(f_0) \ ,$$

that is

$$\theta(f_0) = \psi\tilde{\lambda}(f_0) \ .$$

Then

$$f_1 = \theta(f_0) = \psi\tilde{\lambda}(f_0) = \psi\tilde{\lambda}\tilde{\tau}^k\theta(f_1)$$

and from the splittability of $\tilde{M}$ it follows that

$$\theta(f_1) = \psi\tilde{\lambda}\tilde{\tau}^k(f_1) \ .$$

Consequently,

$$\lambda(f_0) = \psi(f_0) = \psi\tilde{\tau}^k\theta(f_1) = \tilde{\tau}^k\psi\theta(f_1) = \tilde{\tau}^k\psi^2\tilde{\lambda}\tilde{\tau}^k(f_1) = \tilde{\lambda}(f_1) \ .$$

Let now g be any flag of K. Then,

$$g_1 = \xi_g(g_0) = \psi_g^{-1}\theta\psi_g(g_0) = \psi_g^{-1}\psi\tilde{\lambda}\psi_g(g_0) = \tilde{\lambda}\psi_g^{-1}\psi\psi_g(g_0) = \tilde{\lambda}\lambda(g_0) \ ,$$

from which it directly follows that $\lambda(g_0) = \tilde{\lambda}(g_1)$. But we also have

$$g_0 = \tilde{\tau}^k\xi_g(g_1) = \tilde{\tau}^k\psi_g^{-1}\theta\psi_g(g_1) = \tilde{\tau}^k\psi_g^{-1}\psi\tilde{\lambda}\tilde{\tau}^k\psi_g(g_1) = \tilde{\lambda}\psi_g^{-1}\psi\psi_g(g_1) = \tilde{\lambda}\lambda(g_1) \ ,$$

that is, $\lambda(g_1) = \tilde{\lambda}(g_0)$. It means that in this case $\lambda(g_i) = \tilde{\lambda}(g_{i+1})$ for any $i \in \mathbb{Z}_2$.

(b) Now assume that $\lambda(f_1) = \tilde{\lambda}\tilde{\tau}(f_0)$. Then we have

$$\begin{aligned}\tilde{\lambda}(f_0) &= (L(x)_1, -1) = \psi^{-1}\theta\psi(L(x)_0, -1) = \psi^{-1}\theta(x_0, -1) \\ &= \psi^{-1}\theta\tilde{\tau}(x_0, +1) = \psi^{-1}\theta\tilde{\tau}(f_0) \ ,\end{aligned}$$

that is

$$\theta(f_0) = \psi\tilde{\lambda}\tilde{\tau}(f_0) \ .$$

Hence

$$f_1 = \theta(f_0) = \psi\tilde{\lambda}\tilde{\tau}(f_0) = \psi\tilde{\lambda}\tilde{\tau}^{k+1}\theta(f_1) \ ,$$

and the splittability of $\tilde{M}$ implies that

$$\theta(f_1) = \psi\tilde{\lambda}\tilde{\tau}^{k+1}(f_1) \ ,$$

whence

$$\lambda(f_0) = \psi(f_0) = \psi\tilde{\tau}^k\theta(f_1) = \psi\tilde{\tau}^k\psi\tilde{\lambda}\tilde{\tau}^{k+1}(f_1) = \psi^2\tilde{\lambda}\tilde{\tau}(f_1) = \tilde{\lambda}\tilde{\tau}(f_1) \ .$$

Therefore, for any $g \in F(K)$, we get

$$g_1 = \xi_g(g_0) = \psi_g^{-1}\theta\psi_g(g_0) = \psi_g^{-1}\psi\tilde{\lambda}\tilde{\tau}\psi_g(g_0) = \tilde{\lambda}\tilde{\tau}\psi_g^{-1}\psi\psi_g(g_0) = \tilde{\lambda}\tilde{\tau}\lambda(g_0) \ ,$$

and

$$\begin{aligned} g_0 = \tilde{\tau}^k\xi_g(g_1) &= \tilde{\tau}^k\psi_g^{-1}\theta\psi_g(g_1) = \tilde{\tau}^k\psi_g^{-1}\psi\tilde{\lambda}\tilde{\tau}^{k+1}\psi_g(g_1) \\ &= \tilde{\lambda}\tilde{\tau}\psi_g^{-1}\psi\psi_g(g_1) = \tilde{\lambda}\tilde{\tau}\lambda(g_1) \ . \end{aligned}$$

Consequently, $\lambda(g_i) = \tilde{\lambda}\tilde{\tau}(g_{i+1})$ and the statement is proved. □

Lemma 7.4. *Let $\tilde{M} = (F(\tilde{K}); \tilde{\lambda}, \tilde{\varrho}, \tilde{\tau})$ be a regular splittable map of the graph $\tilde{K}$. Then it is possible to assign the second coordinates of the flags in $F(\tilde{K})$ so that for any $g \in F(K)$*

$$\pi\tilde{\lambda}(g_0) = \pi\tilde{\lambda}(g_1) \ .$$

Proof. Let $f_0 = (x_0, +1)$ be a fixed flag of $\tilde{K}$, $x \in D(K)$. Recall that ψ denotes the colour-preserving automorphism mapping $f_i \mapsto L(f)_i$, where $L(f)_i = (L(x)_i, +1)$. Clearly, $\tilde{\lambda}\psi = \psi\tilde{\lambda}$, that is, $\tilde{\lambda} = \psi^{-1}\tilde{\lambda}\psi = \psi\tilde{\lambda}\psi$. Thus

$$\tilde{\lambda}(f_1) = \tilde{\lambda}(x_1, +1) = \psi\tilde{\lambda}\psi(x_1, +1) = \psi\tilde{\lambda}(L(x)_1, +1) \ .$$

In case that $\tilde{\lambda}(f_0) = (L(x)_1, +1)$ we obtain

$$\tilde{\lambda}(f_1) = \psi\tilde{\lambda}(L(x)_1, +1) = \psi(x_0, +1) = (L(x)_0, +1) \ ,$$

that is, $\pi\tilde{\lambda}(f_0) = \pi\tilde{\lambda}(f_1)$. Analogously, in the case $\tilde{\lambda}(f_0) = (L(x)_1, -1)$ we get

$$\tilde{\lambda}(f_1) = \psi\tilde{\lambda}(L(x)_1, +1) = \psi(x_0, -1) = (L(x)_0, -1) \ ,$$

so $\pi\tilde{\lambda}(f_0) = \pi\tilde{\lambda}(f_1)$.

To complete the proof we need to show that the statement holds for any pair of flags g_0 and g_1 of $\tilde{K}$. In accordance with the first part of this proof, we can suppose that

$$\tilde{\lambda}(f_0) = (L(x)_1, i) \qquad \text{and} \qquad \tilde{\lambda}(f_1) = (L(x)_0, i) \ .$$

for some $i \in \{+1, -1\}$. For an arbitrary flag $g_0 = (y_0, +1)$, $y \in D(K)$, we further suppose that

$$\tilde{\lambda}(g_0) = (L(y)_1, j), \qquad \text{for some } j \in \{+1, -1\} .$$

We claim that $\tilde{\lambda}(g_1) = (L(y)_0, j)$; to prove it we show that the other possible case $\tilde{\lambda}(g_1) = (L(y)_0, -j)$ is in contradiction with the assumptions stated above.

Observe that

$$(L(x)_1, i) = \tilde{\lambda}(f_0) = \tilde{\lambda}\psi_g(g_0) = \psi_g\tilde{\lambda}(g_0) = \psi_g(L(y)_1, j) ,$$

that is, $\psi_g(L(y)_1, j) = (L(x)_1, i)$. Supposing that $\tilde{\lambda}(g_1) = (L(y)_0, -j)$ we also obtain

$$(L(x)_0, i) = \tilde{\lambda}(f_1) = \tilde{\lambda}\psi_g(g_1) = \psi_g\tilde{\lambda}(g_1) = \psi_g(L(y)_0, -j) .$$

In considerations that follow we separately examine the two possible values for i. If $i = +1$, we denote $h_0 = (L(y)_0, -j)$. Hence

$$h_0 = (L(y)_0, -j) \xrightarrow{\psi_g} (L(x)_0, +1) \xrightarrow{\psi} (x_0, +1) = f_0 ,$$

and consequently $\psi\psi_g(h_0) = f_0 = \psi_h(h_0)$. But this implies $\psi\psi_g = \psi_h$, and with the computation

$$\begin{aligned} h_1 = (L(y)_1, -j) &= \psi_h^{-1}\theta\psi_h(h_0) = (\psi\psi_g)^{-1}\theta\psi\psi_g(h_0) \\ &= (\psi\psi_g)^{-1}\theta(f_0) = \psi_g^{-1}\psi^{-1}(f_1) = \psi_g^{-1}(L(x)_1, +1) = (L(y)_1, j) \neq h_1 , \end{aligned}$$

we get a contradiction.

In case that $i = -1$, we denote $h_0 = (L(y)_0, j)$. Then analogously

$$h_0 = (L(y)_0, j) \xrightarrow{\psi_g} (L(x)_0, +1) \xrightarrow{\psi} (x_0, +1) = f_0 ,$$

and consequently $\psi\psi_g = \psi_h$. However, the computation

$$\begin{aligned} h_1 = (L(y)_1, j) &= \psi_h^{-1}\theta\psi_h(h_0) = (\psi\psi_g)^{-1}\theta\psi\psi_g(h_0) \\ &= \psi_g^{-1}(L(x)_1, +1) = (L(y)_1, -j) \neq h_1 , \end{aligned}$$

again leads to a contradiction. Thus, $\tilde{\lambda}(g_1) = (L(y)_0, j)$, whence $\pi\tilde{\lambda}(g_0) = \pi\tilde{\lambda}(g_1)$ for every $g \in F(K)$. □

The following result will be also very useful for later discussions.

Proposition 7.5. *Let $\tilde{M}$ be a regular splittable map of the graph $\tilde{K}$ and $\pi\colon \tilde{K} \to K$ be the natural projection. Let $\alpha \in \mathrm{Aut}\,(\tilde{M})$ be any colour-preserving map automorphism. Then*

$$\pi\alpha(h_1) = \pi\alpha(h_0) ,$$

for any $h \in F(K)$.

Proof. Take any flag $h_0 \in F(\tilde{K})$. Since α is a colour-preserving automorphism of the regular map $\tilde{M}$, there exists a unique $g_0 \in F(\tilde{K})$ such that $\alpha(h_0) = g_0$. Recall that we can assign the second coordinates to the flags of $\tilde{K}$ so that $k_1 = \xi_k(k_0)$, for every $k \in F(K)$. Doing that we obtain

$$\psi_g \alpha(h_0) = \psi_g(g_0) = f_0 = \psi_h(h_0) \ .$$

Since $\tilde{M}$ is regular, the automorphisms coincide, that is, $\psi_g \alpha = \psi_h$. Hence,

$$\begin{aligned} \xi_g(g_0) &= \psi_g^{-1}\theta\psi_g(g_0) = (\alpha\alpha^{-1})\psi_g^{-1}\theta\psi_g(\alpha\alpha^{-1})(g_0) \\ &= \alpha(\psi_g\alpha)^{-1}\theta(\psi_g\alpha)\alpha^{-1}(g_0) = \alpha\psi_h^{-1}\theta\psi_h\alpha^{-1}(g_0) = \alpha\xi_h\alpha^{-1}(g_0) \ . \end{aligned}$$

Therefore $\alpha\xi_h = \xi_g\alpha$, and we obtain

$$\pi\alpha(h_1) = \pi\alpha\xi_h(h_0) = \pi\xi_g\alpha(h_0) = \pi\alpha(h_0) \ .$$

Since h was taken to be arbitrary, the statement is proved. □

Proposition 7.6. *Let $\tilde{M} = (\tilde{F}; \tilde{\lambda}, \tilde{\varrho}, \tilde{\tau})$ be a regular splittable map of $\tilde{K}$ and let $\beta \in \mathrm{Aut}\,(\tilde{M})$ be any colour-reversing map automorphism. Then, for any $h \in F(K)$,*

$$\pi\beta(h_1) = \pi\tilde{\tau}^k\beta(h_0) \ ,$$

where $k = 0$ if the twin edges stabilizer G_x in $\mathrm{Aut}\,(\tilde{M})$ *is isomorphic to $\mathbb{Z}_2^3$, and $k = 1$ if G_x is isomorphic to D_8.*

Proof. Let $h_0 \in F(\tilde{K})$ be any flag. Since β is a colour-reversing automorphism, there is a unique $g_1 \in F(\tilde{K})$ such that $\beta(h_0) = g_1$. In consequence,

$$\psi_g\beta(h_0) = \psi_g(g_1) = f_1 = \theta(f_0) = \theta\psi_h(h_0) \ ,$$

thus $\psi_g\beta = \theta\psi_h$. From the computation

$$\begin{aligned} \xi_g(g_1) &= \psi_g^{-1}\theta\psi_g(g_1) = (\beta\beta^{-1})\psi_g^{-1}\theta\psi_g(\beta\beta^{-1})(g_1) = \beta(\psi_g\beta)^{-1}\theta(\psi_g\beta)\beta^{-1}(g_1) \\ &= \beta(\theta\psi_h)^{-1}\theta(\theta\psi_h)\beta^{-1}(g_1) = \beta\psi_h^{-1}\theta\psi_h\beta^{-1}(g_1) = \beta\xi_h\beta^{-1}(g_1) \end{aligned}$$

we conclude that $\beta\xi_h = \xi_g\beta$, therefore

$$\pi\beta(h_1) = \pi\beta\xi_h(h_0) = \pi\xi_g\beta(h_0) = \pi\tilde{\tau}^k\beta(h_0) \ ,$$

with $k \in \{0, 1\}$ depending on the twin edges stabilizer G_x in $\mathrm{Aut}\,(\tilde{M})$ as stated. □

Now we are able to reverse Theorem 7.1 provided that the regular map $\tilde{M}$ is splittable and the graph K is *stable*, i. e. $\mathrm{Aut}\,(\tilde{K}) \cong \mathrm{Aut}\,(K) \times \mathbb{Z}_2$.

Theorem 7.7. *Let K be a connected stable graph and $\tilde{M}$ be a regular splittable map of $\tilde{K}$. Then there exists a regular map N of K and an exponent $t \in$* $\mathrm{Ex}\,(N)$ *such that $\tilde{M} \cong N_t$ and one of the following alternatives occurs:*

(a) *$t = e$ is an involutory integer exponent of N,*
(b) *$t = ep$ is a mixed exponent of N where $e^2 \equiv -1$ (modulo the valency of K).*

Proof. Denote $\tilde{M} = (\tilde{F}; \tilde{\lambda}, \tilde{\varrho}, \tilde{\tau})$, where $\tilde{F} = F(\tilde{K})$. In order to prove our theorem we 'split' the map $\tilde{M}$ of $\tilde{K}$ into a couple of regular maps M_0 and M_1 of K.

Let $f = (x, +1)$ be a fixed flag of K, $x \in D(K)$. Since $\tilde{M}$ is regular, there exists a unique automorphism $\psi \in \mathrm{Aut}\,(\tilde{M})$ mapping the flag $f_0 = (x_0, +1)$ to the flag $L(f)_0 = (L(x)_0, +1)$. The splittability of $\tilde{M}$ implies that ψ^2 is the identity mapping. Recall that $\psi_g \in \mathrm{Aut}\,(\tilde{M})$ denotes the automorphism such that $\psi_g(g_i) = f_i$ for an arbitrary flag $g_i \in \tilde{F}$.

For $i \in \mathbb{Z}_2$ we define permutations of the set $F = F(K)$ by setting

$$\begin{aligned}\lambda_i(g) &= \pi\psi_g^{-1}\psi\psi_g(g_i)\ ,\\ \varrho_i(g) &= \pi\psi_g^{-1}\tilde{\varrho}\psi_g(g_i)\ ,\\ \tau_i(g) &= \pi\psi_g^{-1}\tilde{\tau}\psi_g(g_i)\end{aligned}$$

for any $g \in F$. Observe that

$$\psi_g\tilde{\varrho} = \tilde{\varrho}\psi_g \qquad \text{and} \qquad \psi_g\tilde{\tau} = \tilde{\tau}\psi_g\ ,$$

whence

$$\varrho_i\pi(g_i) = \varrho_i(g) = \pi\tilde{\varrho}(g_i) \qquad \text{and} \qquad \tau_i\pi(g_i) = \tau_i(g) = \pi\tilde{\tau}(g_i)\ ,$$

that is, ϱ_i and τ_i follow on the set F the behaviour of $\tilde{\varrho}$ and $\tilde{\tau}$ on $\tilde{F}$. Since clearly $\lambda_i(g) = \pi\lambda(g_i)$, from Lemma 7.3 we also obtain

$$\lambda_i(g) = \pi\tilde{\lambda}(g_{i+1}) \qquad \text{or} \qquad \lambda_i(g) = \pi\tilde{\lambda}\tilde{\tau}(g_{i+1})\ .$$

i. e., there exists $j \in \mathbb{Z}_2$ such that

$$\lambda_i\pi(g_i) = \lambda_i(g) = \pi\tilde{\lambda}\tilde{\tau}^j(g_{i+1})\ .$$

Consequently,

$$\begin{aligned}\lambda_i^2(g) &= \lambda_i\pi\lambda(g_i) = \pi\lambda^2(g_i) = \pi(g_i) = g\ ,\\ \varrho_i^2(g) &= \varrho_i\pi\tilde{\varrho}(g_i) = \pi\tilde{\varrho}^2(g_i) = \pi(g_i) = g\ ,\\ \tau_i^2(g) &= \tau_i\pi\tilde{\tau}(g_i) = \pi\tilde{\tau}^2(g_i) = \pi(g_i) = g\end{aligned}$$

and

$$\tau_i\lambda_i(g) = \tau_i\pi\tilde{\lambda}\tilde{\tau}^j(g_{i+1}) = \pi\tilde{\tau}\tilde{\lambda}\tilde{\tau}^j(g_{i+1}) = \pi\tilde{\lambda}\tilde{\tau}^j\tilde{\tau}(g_{i+1}) = \lambda_i\pi\tilde{\tau}(g_i) = \lambda_i\tau_i(g)\ .$$

Hence

$$\lambda_i^2 = \varrho_i^2 = \tau_i^2 = (\lambda_i\tau_i)^2 = 1\ .$$

The graph K is connected, so the action of $\langle\lambda, \varrho, \tau\rangle$ is transitive, and we are able to create 3-involutory maps $M_i = (F; \lambda_i, \varrho_i, \tau_i)$ of K, for $i = 0, 1$. We shall prove that M_0 and M_1 are regular maps and there exists an exponent $e \in \mathrm{Ex}\,(K)$ such that $M_1 = (M_0)^e$.

First we show that the maps M_i for both $i \in \{0,1\}$ are regular. To do this we use the projections of colour-preserving automorphisms of $\tilde{M}$. Denote $\Sigma \subseteq \mathrm{Aut}\,(\tilde{M})$ the subgroup consisting of all colour-preserving automorphisms of $\tilde{M}$. The map $\tilde{M}$ is regular, so the index of Σ in $\mathrm{Aut}\,(\tilde{M})$ equals 2. It means that $|\Sigma| = |\,\mathrm{Aut}\,(\tilde{M})|/2$. Since K is stable and $\tilde{M}$ is splittable, Proposition 7.5 implies that

$$\pi\alpha(g_0) = \pi\alpha(g_1)$$

for every colour-preserving automorphism $\alpha \in \Sigma$ and every $g \in F$. Thus we can define the mapping $\alpha'\colon F \to F$ such that

$$\alpha'(g) = \pi\alpha(g_0) = \pi\alpha(g_1)\ .$$

We claim that α' is a map automorphism for both M_0 and M_1. To see this, notice that

$$\alpha'\pi(g_i) = \pi\alpha(g_i)\ .$$

Then we obtain

$$\begin{aligned}
\alpha'\lambda_i(g) &= \alpha'\pi\tilde{\lambda}\tilde{\tau}^j(g_{i+1}) = \pi\alpha\tilde{\lambda}\tilde{\tau}^j(g_{i+1}) = \pi\tilde{\lambda}\tilde{\tau}^j\alpha(g_{i+1}) = \lambda_i\pi\alpha(g_{i+1}) = \lambda_i\alpha'(g)\ ,\\
\alpha'\varrho_i(g) &= \alpha'\pi\tilde{\varrho}(g_i) = \pi\alpha\tilde{\varrho}(g_i) = \pi\tilde{\varrho}\alpha(g_i) = \varrho_i\pi\alpha(g_i) = \varrho_i\alpha'(g)\ ,\\
\alpha'\tau_i(g) &= \alpha'\pi\tilde{\tau}(g_i) = \pi\alpha\tilde{\tau}(g_i) = \pi\tilde{\tau}\alpha(g_i) = \tau_i\pi\alpha(g_i) = \tau_i\alpha'(g)\ ,
\end{aligned}$$

which implies that $\alpha' \in \mathrm{Aut}\,(M_i)$, for $i \in \{0,1\}$. Clearly, the assignment $\alpha \mapsto \alpha'$ is a monomorphism mapping $\Sigma \to \mathrm{Aut}\,(M_i)$. From these observations it follows that

$$|\,\mathrm{Aut}\,(M_i)| \geq |\Sigma| = |\,\mathrm{Aut}\,(\tilde{M})|/2 = |F|\ ,$$

whence both the maps M_i are regular. Moreover, $\mathrm{Aut}\,(M_0) = \mathrm{Aut}\,(M_1) \cong \Sigma$. Since M_i are regular maps with the same underlying graph K and their automorphism groups coincide, By Theorem 5.2 M_i must be congruent. But $\tilde{M}$ is splittable, and Lemma 7.4 implies that

$$\pi\tilde{\lambda}(g_0) = \pi\tilde{\lambda}(g_1)\ ,$$

for any $g \in F$. Consequently, for some $j \in \{0,1\}$,

$$\begin{aligned}
\lambda_1(g) &= \pi\tilde{\lambda}\tilde{\tau}^j(g_0) = \pi\tilde{\tau}^j\tilde{\lambda}(g_0) = \tau_1^j\pi\tilde{\lambda}(g_0) = \tau_0^j\pi\tilde{\lambda}(g_1)\\
&= \pi\tilde{\tau}^j\tilde{\lambda}(g_1) = \pi\tilde{\lambda}\tilde{\tau}^j(g_1) = \lambda_0(g)\ .
\end{aligned}$$

So $\lambda_1 = \lambda_0$. Obviously, $\tau_1 = \tau_0$, and Corollary 5.3 also assures the existence of an exponent e of the graph K such that

$$\varrho_1 = (\varrho_0\tau_0)^{e-1}\varrho_0\ .$$

So we get $M_1 = (F;\, \lambda_0, (\varrho_0\tau_0)^{e-1}\varrho_0, \tau_0) = (M_0)^e$.

Now consider any colour-reversing automorphism $\beta \in \mathrm{Aut}\,(\tilde{M})$. Proposition 7.6 implies that

$$\pi\beta(g_1) = \pi\tilde{\tau}^k\beta(g_0)\ ,$$

for any $g \in F(K)$, where $k \in \{0,1\}$ depends on the stabilizer G_x of twin edges in $\mathrm{Aut}\,(\tilde{M})$. We discuss these two cases for β separately.

(a) Assume first that the stabilizer G_x of any twin edges is isomorphic to $\mathbb{Z}_2^3$. Then $\pi\beta(g_1) = \pi\beta(g_0)$ and we can define a mapping $\beta' \colon F \to F$ such that

$$\beta'(g) = \pi\beta(g_0) = \pi\beta(g_1)$$

Since also

$$\beta'\pi = \pi\beta \ ,$$

for every $g \in F$ and every $i, j \in \mathbb{Z}_2$ we get

$$\begin{aligned} \beta'\lambda_i(g) &= \beta'\pi\tilde{\lambda}\tilde{\tau}^j(g_{i+1}) = \pi\beta\tilde{\lambda}\tilde{\tau}^j(g_{i+1}) = \pi\tilde{\lambda}\tilde{\tau}^j\beta(g_{i+1}) \\ &= \lambda_{i+1}\pi\beta(g_{i+1}) = \lambda_{i+1}\beta'(g) \ , \\ \beta'\varrho_i(g) &= \beta'\pi\tilde{\varrho}(g_i) = \pi\beta\tilde{\varrho}(g_i) = \pi\tilde{\varrho}\beta(g_i) = \varrho_{i+1}\pi\beta(g_i) = \varrho_{i+1}\beta'(g) \ , \\ \beta'\tau_i(g) &= \beta'\pi\tilde{\tau}(g_i) = \pi\beta\tilde{\tau}(g_i) = \pi\tilde{\tau}\beta(g_i) = \tau_{i+1}\pi\beta(g_i) = \tau_{i+1}\beta'(g) \ . \end{aligned}$$

Hence, β' provides both an isomorphism $M_0 \to M_1$ and an isomorphism $M_1 \to M_0$.
Finally, we show that e is an involutory exponent of M_0. In fact, from previous computations we get for $i = 0$

$$\begin{aligned} \beta'\lambda_0 &= \lambda_1\beta' = \lambda_0\beta' \ , \\ \beta'\varrho_0 &= \varrho_1\beta' = (\varrho_0\tau_0)^{e-1}\varrho_0\beta' \ , \\ \beta'\tau_0 &= \tau_1\beta' = \tau_0\beta' \ , \end{aligned}$$

which confirms e to be an integer exponent of M_0. Furthermore, the previous equalities imply that

$$\begin{aligned} \varrho_0 &= (\beta')^{-1}\beta'\varrho_0 = (\beta')^{-1}\varrho_1\beta' = (\beta')^{-1}(\varrho_0\tau_0)^{e-1}\varrho_0\beta' \\ &= (\beta')^{-1}\beta'(\varrho_1\tau_1)^{e-1}\varrho_1 = (\varrho_1\tau_1)^{e-1}\varrho_1 \ . \end{aligned}$$

Combining

$$\varrho_1 = (\varrho_0\tau_0)^{e-1}\varrho_0 \qquad \text{and} \qquad \varrho_0 = (\varrho_1\tau_1)^{e-1}\varrho_1$$

we obtain

$$(\varrho_0\tau_0)^{e^2} = ((\varrho_0\tau_0)^{e-1}\varrho_0\tau_0)^e = (\varrho_1\tau_1)^e = (\varrho_1\tau_1)^{e-1}\varrho_1\tau_1 = \varrho_0\tau_0 \ .$$

Consequently, $e^2 \equiv 1$ (modulo valency of K) and e is the desired involutory integer exponent of M_0. Since $\tilde{M}$ clearly coincides with N_e where either $N = M_0$ or $N = P(M_0)$, the theorem holds for this case.

(b) Further suppose that the stabilizer G_x of every twin edges in $\mathrm{Aut}\,(\tilde{M})$ is isomorphic to D_8. Then $\pi\beta(g_1) = \pi\tilde{\tau}\beta(g_0)$, and we define a pair of mappings $\beta_i \colon F \to F$ such that

$$\beta_i(g) = \pi\beta(g_{i+1}) \ ,$$

for both $i \in \mathbb{Z}_2$. Observe that

$$\beta_0(g) = \pi\beta(g_1) = \pi\tilde{\tau}\beta(g_0) = \tau_1\pi\beta(g_0) = \tau_1\beta_1(g) \ ,$$

from which we also get

$$\beta_1(g) = \tau_1\beta_0(g) = \tau_0\beta_0(g) \ .$$

Notice that in this case

$$\beta_i\pi(g_{i+1}) = \beta_i(g) = \pi\beta(g_{i+1}) \ ,$$

but

$$\beta_i\pi(g_i) = \beta_i(g) = \pi\beta(g_{i+1}) = \pi\tilde{\tau}\beta(g_i) \ .$$

Then we compute, for every $g \in F$ and every $i \in \mathbb{Z}_2$,

$$\begin{aligned}
\beta_i\lambda_{i+1}(g) &= \beta_i\pi\tilde{\lambda}\tilde{\tau}^j(g_i) = \pi\beta\tilde{\lambda}\tilde{\tau}^j(g_i) = \pi\tilde{\lambda}\tilde{\tau}^j\beta(g_i) \\
&= \lambda_i\pi\beta(g_i) = \lambda_i\beta_{i+1}(g) = \lambda_i\tau_i\beta_i(g) \ , \\
\beta_i\varrho_{i+1}(g) &= \beta_i\pi\tilde{\varrho}(g_{i+1}) = \pi\beta\tilde{\varrho}(g_{i+1}) = \pi\tilde{\varrho}\beta(g_{i+1}) = \varrho_i\pi\beta(g_{i+1}) = \varrho_i\beta_i(g) \ , \\
\beta_i\tau_{i+1}(g) &= \beta_i\pi\tilde{\tau}(g_{i+1}) = \pi\beta\tilde{\tau}(g_{i+1}) = \pi\tilde{\tau}\beta(g_{i+1}) = \tau_i\pi\beta(g_{i+1}) = \tau_i\beta_i(g) \ .
\end{aligned}$$

Hence, each β_i provides an isomorphism $M_{i+1} \to P(M_i)$. Notice that $P(M_1) = P((M_0)^e) = (M_0)^{ep}$, thus the map isomorphism $\beta_1\colon M_0 \to (M_0)^{ep}$ is also an exomorphism of M_0 associated with the mixed exponent $ep \in \mathrm{Ex}\,(M_0)$. We show that the integer part e meets the condition $e^2 \equiv -1$ modulo the valency of K.

Since

$$\begin{aligned}
\beta_0\varrho_0(g) &= \beta_0\pi\tilde{\varrho}(g_0) = \pi\tilde{\tau}\beta\tilde{\varrho}(g_0) = \pi\tilde{\tau}\tilde{\varrho}\beta(g_0) = \tau_1\pi\tilde{\varrho}\beta(g_0) \\
&= \tau_1\varrho_1\pi\beta(g_0) = \tau_1\varrho_1\beta_1(g) \ ,
\end{aligned}$$

we obtain

$$\begin{aligned}
\varrho_0 &= (\beta_0)^{-1}\beta_0\varrho_0 = (\beta_0)^{-1}\tau_1\varrho_1\beta_1 = (\beta_0)^{-1}\tau_0(\varrho_0\tau_0)^{e-1}\varrho_0\beta_1 \\
&= (\beta_0)^{-1}\tau_0(\varrho_0\tau_0)^{e-1}\varrho_0\tau_0\beta_0 = (\beta_0)^{-1}\beta_0\tau_1(\varrho_1\tau_1)^{e-1}\varrho_1\tau_1 = (\tau_1\varrho_1)^e\tau_1 \ .
\end{aligned}$$

Consequently,

$$\varrho_1 = (\varrho_0\tau_0)^{e-1}\varrho_0 = ((\tau_1\varrho_1)^e\tau_1\tau_1)^{e-1}(\tau_1\varrho_1)^e\tau_1 = (\tau_1\varrho_1)^{e^2}\tau_1 = \tau_1(\varrho_1\tau_1)^{e^2} \ ,$$

which implies

$$(\varrho_1\tau_1)^{e^2} = \tau_1\varrho_1 = (\varrho_1\tau_1)^{-1} \ .$$

However, this equality holds if and only if $e^2 \equiv -1$ modulo $\mathrm{val}\,(K)$. It follows that ep is the mixed exponent of M_0 meeting the required conditions. In this case, map $\tilde{M}$ coincides with N_{ep} where $N = M_0$ (for $j = 0$) or $N = P(M_0)$ (for $j = 1$). By this, the proof is complete. □

In order to complete the classification of regular maps of $\tilde{K}$ we need to discuss the conditions under which two regular maps of $\tilde{K}$ are isomorphic.

Let $\tilde{M} = M_r$ and $\tilde{N} = N_s$ be isomorphic regular maps with the same underlying graph $\tilde{K}$ and $\varphi\colon \tilde{M} \to \tilde{N}$ be a map isomorphism. Then φ permutes the set of flags $F(\tilde{K}) = F(K) \times \mathbb{Z}_2$, and induces an automorphism of $\tilde{K}$. In case $\tilde{K}$ is a connected bipartite graph, from Lemma 6.1 we know that the induced automorphism of $\tilde{K}$ is either colour-preserving or colour-reversing. Thus it may happen that φ is a colour-reversing permutation of $F(\tilde{K})$.

However, $\tilde{N}$ is a regular map. It means that $\mathrm{Aut}\,(\tilde{N})$ contains colour-reversing automorphisms. Let θ be an arbitrary colour-reversing automorphism of $\tilde{N}$. Then $\theta\varphi\colon \tilde{M} \to \tilde{N}$ is obviously a colour-preserving map isomorphism. That is the reason why we can restrict to colour-preserving isomorphisms mapping $\tilde{M} \to \tilde{N}$. This fact will simplify our considerations.

The following technical result will be also useful for the completion of our classification.

Proposition 7.8. *Let $\tilde{M}$ and $\tilde{N}$ be isomorphic regular splittable maps of $\tilde{K}$, a canonical double covering of a stable graph K. Let $\varphi\colon \tilde{M} \to \tilde{N}$ be a colour-preserving map isomorphism. Then there exists $i \in \{0,1\}$ such that, for all flags $f \in F(K)$,*

$$\pi\varphi(f_0) = \pi\tilde{\tau}_N^i\varphi(f_1) \ ,$$

where $\pi\colon F(\tilde{K}) \to F(K)$ is the index-erasing projection, and $\tilde{\tau}_N$ is the transversal involution of the map $\tilde{N}$.

Proof. Denote $\tilde{M} = (F(\tilde{K}); \tilde{\lambda}_M, \tilde{\varrho}_M, \tilde{\tau}_M)$ and $\tilde{N} = (F(\tilde{K}); \tilde{\lambda}_N, \tilde{\varrho}_N, \tilde{\tau}_N)$. Let $f_0 = (x_0, +1)$ be a fixed flag of $\tilde{K}$, and $\varphi(f_0) = g_0$, for some flag $g_0 = (y_0, j)$, where $x, y \in D(K)$ and $j \in \{+1, -1\}$. The stability of K implies that the automorphism of $\tilde{K}$ induced by the isomorphism φ maps the arc x_1 to the arc y_1. Hence, for $f_1 = (x_1, +1)$ and $g_1 = (y_1, j)$, we have

$$\varphi(f_1) = g_1 \quad \text{or} \quad \varphi(f_1) = \tilde{\tau}_N(g_1) \ .$$

We need to prove that for an arbitrary colour-preserving map isomorphism φ just one of these equalities holds, for any flags $f, g \in F(K)$.

Suppose that $\varphi(f_1) = g_1$. Furthermore, suppose that there exist some flags $h, k \in F(K)$ for which $\varphi(h_0) = k_0$ while $\varphi(h_1) = \tilde{\tau}_N(k_1)$. We shall show that these assumptions lead to a contradiction.

The map $\tilde{N}$ is regular, therefore there exists a unique colour-preserving automorphism $\alpha \in \mathrm{Aut}\,(\tilde{N})$ such that $\alpha(g_0) = k_0$. But $\tilde{N}$ is splittable, whence Proposition 7.5 implies that $\alpha(g_1) = k_1$.

Let η be the permutation of $F(\tilde{K})$ such that

$$\eta = \varphi^{-1}\alpha\varphi \ .$$

The permutation η obviously provides a colour-preserving automorphism of the map $\tilde{M}$. Therefore, by Proposition 7.5,

$$\pi\eta(l_0) = \pi\eta(l_1) \ ,$$

for any flag $l \in F(K)$.

However, for the fixed flag f_0 we obtain

$$\eta(f_0) = \varphi^{-1}\alpha\varphi(f_0) = \varphi^{-1}\alpha(g_0) = \varphi^{-1}(k_0) = h_0 \ ,$$

and

$$\eta(f_1) = \varphi^{-1}\alpha\varphi(f_1) = \varphi^{-1}\alpha(g_1) = \varphi^{-1}(k_1) = \tilde{\tau}_M(h_1) \ .$$

Consequently,

$$\pi\eta(f_0) \neq \pi\eta(f_1) \ ,$$

which contradicts Proposition 7.5 and proves our statement. □

The following theorem provides the third step in the classification of regular maps of $\tilde{K}$ in terms of regular maps of K by determining the conditions under which two regular maps of $\tilde{K}$ are isomorphic. We state and prove it for integer exponents of the base maps.

Theorem 7.9. *Let M and N be regular 3-involutory maps of a stable n-valent graph K, and let r and s be unoriented integer exponents of M and N, respectively. Then M_r is isomorphic to N_s if and only if one of the following statements is true:*
(a) M is isomorphic to N and $r \equiv s$ (mod n),
(b) M is isomorphic to $P(N)$ and $r \equiv -s$ (mod n).

Proof. Denote $M = (F; \lambda_M, \varrho_M, \tau_M)$, $N = (F; \lambda_N, \varrho_N, \tau_N)$, where $F = F(K)$. From the definitions of the derived maps $M_r = (F \times \mathbb{Z}_2; \tilde{\lambda}_M, \tilde{\varrho}_M, \tilde{\tau}_M)$ and $N_s = (F \times \mathbb{Z}_2; \tilde{\lambda}_N, \tilde{\varrho}_N, \tilde{\tau}_N)$ of the graph $\tilde{K}$ we obtain

$$\begin{aligned} \tilde{\lambda}_M(f_i) &= \lambda_M(f)_{i+1} \ , \\ \tilde{\tau}_M(f_i) &= \tau_M(f)_i \ , \\ \tilde{\varrho}_M(f_i) &= (\varrho_M\tau_M)^{r^i-1}\varrho_M(f)_i \ , \end{aligned}$$

and

$$\begin{aligned} \tilde{\lambda}_N(f_i) &= \lambda_N(f)_{i+1} \ , \\ \tilde{\tau}_N(f_i) &= \tau_N(f)_i \ , \\ \tilde{\varrho}_N(f_i) &= (\varrho_N\tau_N)^{s^i-1}\varrho_N(f)_i \ , \end{aligned}$$

for any $f \in F$ and $i \in \mathbb{Z}_2$.

First we shall verify the sufficiency conditions.

(a) Suppose that M is isomorphic to N and $r \equiv s$ (mod n). Let $\varphi\colon M \to N$ be a map isomorphism, i. e.

$$\varphi\lambda_M = \lambda_N\varphi, \qquad \varphi\tau_M = \tau_N\varphi, \qquad \varphi\varrho_M = \varrho_N\varphi \ .$$

Then the mapping $\tilde{\varphi}\colon F \times \mathbb{Z}_2 \to F \times \mathbb{Z}_2$ defined by setting

$$\tilde{\varphi}(f_i) = \varphi(f)_i$$

provides an isomorphism $M_r \to N_s$, since the following equalities hold:

$$\begin{aligned}
\tilde{\varphi}\tilde{\lambda}_M(f_i) &= \tilde{\varphi}(\lambda_M(f)_{i+1}) = \varphi\lambda_M(f)_{i+1} = \lambda_N\varphi(f)_{i+1} = \tilde{\lambda}_N(\varphi(f)_i) = \tilde{\lambda}_N\tilde{\varphi}(f_i) \ , \\
\tilde{\varphi}\tilde{\varrho}_M(f_i) &= \tilde{\varphi}((\varrho_M\tau_M)^{r^i-1}\varrho_M(f)_i) = \varphi(\varrho_M\tau_M)^{r^i-1}\varrho_M(f)_i \\
&= (\varrho_N\tau_N)^{r^i-1}\varrho_N\varphi(f)_i = (\varrho_N\tau_N)^{s^i-1}\varrho_N\varphi(f)_i = \tilde{\varrho}_N(\varphi(f)_i) = \tilde{\varrho}_N\tilde{\varphi}(f_i) \ , \\
\tilde{\varphi}\tilde{\tau}_M(f_i) &= \tilde{\varphi}(\tau_M(f)_i) = \varphi\tau_M(f)_i = \tau_N\varphi(f)_i = \tilde{\tau}_N(\varphi(f)_i) = \tilde{\tau}_N\tilde{\varphi}(f_i) \ ,
\end{aligned}$$

for any $f \in F$ and $i \in \mathbb{Z}_2$. Thus, M_r is isomorphic to N_s.

(b) Let now M be isomorphic to $P(N)$ and $r \equiv -s \pmod{n}$. From previous part of this proof we obtain

$$M_r \cong P(N)_{-s} \ .$$

However, from Proposition 6.4 we know that

$$P(N)_{-s} \cong N_s \ .$$

Hence,

$$M_r \cong N_s \ ,$$

and both the sufficiency conditions are proved.

Conversely, suppose that M_r is isomorphic to N_s and $\varphi\colon M_r \to N_s$ is a map isomorphism. Hence

$$\varphi\tilde{\lambda}_M = \tilde{\lambda}_N\varphi, \qquad \varphi\tilde{\varrho}_M = \tilde{\varrho}_N\varphi, \qquad \text{and} \quad \varphi\tilde{\tau}_M = \tilde{\tau}_N\varphi \ .$$

However, we can take φ to be colour-preserving. Then by Proposition 7.8 we have

$$\pi\varphi(f_0) = \pi\varphi(f_1) \qquad \text{or} \qquad \pi\varphi(f_0) = \pi\tilde{\tau}_N\varphi(f_1) \ ,$$

for any $f \in F$, where $\pi\colon f_i \mapsto f$ is the index-erasing projection, $i \in \mathbb{Z}_2$. In order to prove the necessity of stated conditions, we shall discuss these two cases separately. In each case we shall construct a mapping $\varphi'\colon F \to F$ that provides the desired map isomorphism $M \to N$ and $M \to P(N)$, respectively.

(a) Assume that $\pi\varphi(f_0) = \pi\varphi(f_1)$, and define a mapping φ' by setting

$$\varphi'(f) = \pi\varphi(f_0) = \pi\varphi(f_1) \ ,$$

for any $f \in F$.

We shall show that φ' is a map isomorphism $M \to N$. First observe that

$$\begin{aligned}
&\pi\tilde{\lambda}_N(f_i) = \lambda_N(f) = \lambda_N\pi(f_i) \ , \\
&\pi\tilde{\varrho}_N(f_i) = (\varrho_N\tau_N)^{s^i-1}\varrho_N(f) = (\varrho_N\tau_N)^{s^i-1}\varrho_N\pi(f_i) \ , \\
&\pi\tilde{\tau}_N(f_i) = \tau_N(f) = \tau_N\pi(f_i) \ ,
\end{aligned}$$

for arbitrary $f \in F$ and $i = 0, 1$. Hence we obtain

$$\varphi'\lambda_M(f) = \pi\varphi(\lambda_M(f)_0) = \pi\varphi\tilde{\lambda}_M(f_1) = \pi\tilde{\lambda}_N\varphi(f_1) = \lambda_N\pi\varphi(f_1) = \lambda_N\varphi'(f) \ ,$$
$$\varphi'\varrho_M(f) = \pi\varphi(\varrho_M(f)_0) = \pi\varphi\tilde{\varrho}_M(f_0) = \pi\tilde{\varrho}_N\varphi(f_0) = \varrho_N\pi\varphi(f_0) = \varrho_N\varphi'(f) \ ,$$
$$\varphi'\tau_M(f) = \pi\varphi(\tau_M(f)_0) = \pi\varphi\tilde{\tau}_M(f_0) = \pi\tilde{\tau}_N\varphi(f_0) = \tau_N\pi\varphi(f_0) = \tau_N\varphi'(f) \ .$$

Consequently,

$$\varphi'\lambda_M = \lambda_N\varphi', \qquad \varphi'\varrho_M = \varrho_N\varphi', \qquad \text{and} \quad \varphi'\tau_M = \tau_N\varphi' \ ,$$

and M is isomorphic to N.

Furthermore, φ' also provides a map isomorphism $M^r \to N^s$. Recall that

$$M^r = (F;\ \lambda_M, (\varrho_M\tau_M)^{r-1}\varrho_M, \tau_M)$$

and

$$N^s = (F;\ \lambda_N, (\varrho_N\tau_N)^{s-1}\varrho_N, \tau_N) \ .$$

We already know that

$$\varphi'\lambda_M = \lambda_N\varphi' \quad \textit{and} \quad \varphi'\tau_M = \tau_N\varphi' \ ,$$

whence it suffices to show that

$$\varphi'(\varrho_M\tau_M)^{r-1}\varrho_M = (\varrho_N\tau_N)^{s-1}\varrho_N\varphi' \ .$$

Indeed,

$$\varphi'(\varrho_M\tau_M)^{r-1}\varrho_M(f) = \pi\varphi((\varrho_M\tau_M)^{r-1}\varrho_M(f)_1) = \pi\varphi\tilde{\varrho}_M(f_1) = \pi\tilde{\varrho}_N\varphi(f_1)$$
$$= (\varrho_N\tau_N)^{s-1}\varrho_N\pi\varphi(f_1) = (\varrho_N\tau_N)^{s-1}\varrho_N\varphi'(f) \ .$$

Therefore, combining the previous results we get

$$\varphi'(\varrho_M\tau_M)^{r-1}\varrho_M = (\varrho_N\tau_N)^{s-1}\varrho_N\varphi' = \varphi'(\varrho_M\tau_M)^{s-1}\varrho_M \ ,$$

and multiplying this equality on the left by $(\varphi')^{-1}$ and on the right by τ_M we obtain

$$(\varrho_M\tau_M)^r = (\varrho_M\tau_M)^s \ .$$

Thus, $r \equiv s \pmod{n}$, where $n = \mathrm{val}\,(K)$.

(b) Finally, assume that $\pi\varphi(f_0) = \pi\tilde{\tau}_N\varphi(f_1)$, and φ' is a mapping defined by setting

$$\varphi'(f) = \pi\varphi(f_0) \ ,$$

for any $f \in F$. Note that in this case

$$\varphi'(f) = \pi\tilde{\tau}_N\varphi(f_1) = \tau_N\pi\varphi(f_1) \ .$$

We shall prove that φ' is a map isomorphism $M \to P(N)$. In fact,

$$\begin{aligned}
\varphi'\lambda_M(f) &= \pi\varphi(\lambda_M(f)_0) = \pi\varphi\tilde{\lambda}_M(f_1) = \pi\tilde{\lambda}_N\varphi(f_1) = \lambda_N\pi\varphi(f_1) \\
&= \lambda_N\tau_N(\tau_N\pi\varphi(f_1)) = \lambda_N\tau_N\varphi'(f) \ , \\
\varphi'\varrho_M(f) &= \pi\varphi(\varrho_M(f)_0) = \pi\varphi\tilde{\varrho}_M(f_0) = \pi\tilde{\varrho}_N\varphi(f_0) = \varrho_N\pi\varphi(f_0) = \varrho_N\varphi'(f) \ , \\
\varphi'\tau_M(f) &= \pi\varphi(\tau_M(f)_0) = \pi\varphi\tilde{\tau}_M(f_0) = \pi\tilde{\tau}_N\varphi(f_0) = \tau_N\pi\varphi(f_0) = \tau_N\varphi'(f) \ .
\end{aligned}$$

Therefore,

$$\varphi'\lambda_M = \lambda_N\tau_N\varphi', \qquad \varphi'\varrho_M = \varrho_N\varphi', \qquad \text{and} \quad \varphi'\tau_M = \tau_N\varphi' \ ,$$

and M is isomorphic to $P(N)$.

Moreover, φ' isomorphically maps $M^r \to P(N)^{-s}$. Observe that

$$P(N)^{-s} = (F;\, \lambda_N\tau_N, (\varrho_N\tau_N)^{-s-1}\varrho_N, \tau_N) = (F;\, \lambda_N\tau_N, (\tau_N\varrho_N)^s\tau_N, \tau_N) \ ,$$

since

$$\begin{aligned}
(\varrho_N\tau_N)^{-s-1}\varrho_N &= ((\varrho_N\tau_N)^{-1})^{s+1}\varrho_N = (\tau_N^{-1}\varrho_N^{-1})^{s+1}\varrho_N \\
&= (\tau_N\varrho_N)^{s+1}\varrho_N = (\tau_N\varrho_N)^s\tau_N\varrho_N^2 = (\tau_N\varrho_N)^s\tau_N \ .
\end{aligned}$$

Then it suffices to show that

$$\varphi'(\varrho_M\tau_M)^{r-1}\varrho_M = (\tau_N\varrho_N)^s\tau_N\varphi' \ .$$

But from previous considerations it really follows that

$$\begin{aligned}
\varphi'(\varrho_M\tau_M)^{r-1}\varrho_M(f) &= \tau_N\pi\varphi((\varrho_M\tau_M)^{r-1}\varrho_M(f)_1) = \tau_N\pi\varphi\tilde{\varrho}_M(f_1) \\
&= \tau_N\pi\tilde{\varrho}_N\varphi(f_1) = \tau_N(\varrho_N\tau_N)^{s-1}\varrho_N\pi\varphi(f_1) \\
&= (\tau_N\varrho_N)^s\tau_N(\tau_N\pi\varphi(f_1)) = (\tau_N\varrho_N)^s\tau_N\varphi'(f) \ .
\end{aligned}$$

Combining previous results we obtain

$$\varphi'(\varrho_M\tau_M)^{r-1}\varrho_M = (\tau_N\varrho_N)^s\tau_N\varphi' = \varphi'(\tau_M\varrho_M)^s\tau_M = \varphi'(\varrho_M\tau_M)^{-s-1}\varrho_M \ .$$

Hence,

$$\varphi'(\varrho_M\tau_M)^{r-1}\varrho_M = \varphi'(\varrho_M\tau_M)^{-s-1}\varrho_M \ .$$

When multiplying this equality on the left by $(\varphi')^{-1}$ and on the right by τ_M we have

$$(\varrho_M\tau_M)^r = (\varrho_M\tau_M)^{-s} \ .$$

Thus, $r \equiv -s \pmod{n}$, where $n = \mathrm{val}\,(K)$, and the whole theorem is proved. □

Note that in order to complete the classification of regular maps of canonical double coverings it will be necessary to prove a statement analogous to Theorem 7.9 for mixed exponents as well.

Summing up previous results we obtain the following one.

Corollary 7.10. *Let K be a connected stable n-valent graph, $n \geq 3$. Then there exist at least*

$$\sum |\operatorname{Ex}_{e^2}(M)|/2$$

non-isomorphic regular splittable maps of $\tilde{K}$ where M ranges over all non-isomorphic regular maps of K.

Proof. First observe that the subgroup $\operatorname{Ex}_{e^2}(M)$ consisting of all involutory integer exponents of an arbitrary embedding M of the graph K is of even order whenever $n = \operatorname{val}(K) \geq 3$. In fact, with any involutory integer exponent e, the group $\operatorname{Ex}_{e^2}(M)$ also contains the involutory integer exponent $-e$. Obviously we can take e and $-e$ such that $1 \leq e, -e \leq n-1$. Suppose that $e \equiv -e \pmod{n}$. Then $e = n - e$, that is, $e = n/2$. However, $e^2 \equiv 1 \pmod{n}$ by assumption, so $e^2 = kn+1$ for some nonnegative integer k. Thus we obtain $(n/2)^2 = kn + 1$ which leads to a quadratic equation $n^2 - 4kn - 4 = 0$ with real solutions $n = 2k \pm 2\sqrt{k^2+1}$. But only one of the solutions is a positive integer; for $k = 0$ we get $n = 2$ which is a trivial case out of our interest. Thus, for $n \geq 3$, the exponents e and $-e$ of M are not congruent modulo n, so $|\operatorname{Ex}_{e^2}(M)|$ is even.

Let M be a regular map of K that is not self-Petrie. Then from Theorem 7.9 and the previous part of this proof it follow that M_e and M_{-e} are non-isomorphic regular splittable embeddings of $\tilde{K}$ for any $e \in \operatorname{Ex}_{e^2}(M)$. However, Proposition 6.4 implies that the regular splittable maps $P(M)_e$ and $P(M)_{-e}$ of $\tilde{K}$ are isomorphic to maps M_{-e} and M_e, respectively. Therefore, the number of non-isomorphic regular splittable maps of $\tilde{K}$ derived from each non-isomorphic pair of Petrie duals M and $P(M)$ of K equals $(|\operatorname{Ex}_{e^2}(M)| + |\operatorname{Ex}_{e^2}(P(M))|)/2$.

Finally consider a self-Petrie regular map M of K. By Proposition 6.4 and Corollary 6.7, M_e is isomorphic to M_{-e} for any couple e and $-e$ of involutory integer exponents of M. Hence the number of non-isomorphic regular splittable maps of $\tilde{K}$ derived from each self-Petrie map M of K equals $|\operatorname{Ex}_{e^2}(M)|/2$. □

In the following section we show that the unoriented exponent group of a base map M may also contain mixed exponents which induce derived regular maps non-isomorphic with the regular maps induced by integer exponents of M.

8. Regular embeddings of coctail-party graphs

Now we apply obtained results to the classification of regular splittable embeddings of coctail-party graphs, that is, the tensor products $K_n \otimes K_2$ that are canonical double covers of complete graphs K_n. It is proved in [J, W2] that complete graphs K_n admit an unoriented regular embedding if and only if $n \in \{2, 3, 4, 6\}$. However, K_2 is a bipartite graph, so we do not take the case $n = 2$ into account.

(a) The complete graph K_3 has two regular embeddings, namely the 2-face embedding M on the sphere and the single face embedding N in the projective plane. These two maps are Petrie duals of each other. Since the valency of K_3 is two, its unoriented exponent group is trivial. The canonical double covering graph of K_3 is the cycle C_6 which has two regular embeddings. One of them is the 2-face embedding on the sphere isomorphic to M_1, the other is the 1-face embedding in the projective plane isomorphic to N_1.

(b) The complete graph K_4 has also two regular embeddings. We denote by M the tetrahedron on the sphere and by N the 4-gonal embedding in the projective plane. M and N are Petrie duals. The unoriented exponent group of K_4 consists of two involutory integer exponents, ± 1. The canonical double covering graph of K_4 is the cube Q_3 which has two regular embeddings. The 4-gonal spherical map of Q_3 is isomorphic to M_{-1} and N_1 whereas the less known 6-gonal regular embedding of Q_3 in torus is isomorphic to with M_1 and N_{-1}. The two regular maps of Q_3 are Petrie duals each to the other.

(c) The pair of regular maps of the complete graph K_6 is formed by the triangular embedding M in the projective plane and its Petrie dual N in the nonorientable surface of genus 5. The unoriented exponent group of K_6 consists of two involutory integer exponents ± 1 and two mixed exponents $\pm 2p$. (The fact that $2p$ is an exponent of M was verified using the mathematical software application [GAP].) So the coctail-party graph $K_6 \otimes K_2$ has the following splittable regular maps:

(1) a 6-gonal embedding E_1 isomorphic to M_1 and N_{-1},
(2) a 10-gonal embedding E_2 isomorphic to M_{-1} and N_1,
(3) a 6-gonal embedding E_3 isomorphic to M_{2p} and N_{-2p},
(4) a 4-gonal embedding E_4 isomorphic to M_{-2p} and N_{2p}.

By Corollary 6.5, $P(M_1)$ is isomorphic to M_{-1}, so the length of Petrie polygons in M_1 is equal to the length of the face boundaries in M_{-1}, that is, it equals 10. However, $P(M_{2p})$ is isomorphic to M_{-2p}, hence the length of Petrie polygons in M_{2p} is the same as the length of the face boundaries in M_{-2p} which equals 4. It implies that Petrie polygons in E_1 and E_3 are of different length, so the embeddings E_1 and E_3 are not isomorphic. Therefore, all regular splittable maps of $K_6 \otimes K_2$ are pairwise non-isomorphic.

9. Concluding remarks

In this thesis we have studied the unoriented exponent which can be understood as a generalization of the previously introduced exponent for orientable maps. The unoriented exponent $t = ep^j$ is composed of two parts. The integer part e is derived from an exponent of the embedded graph analogously as in the orientable case whereas the Petrie part p^j reflects the significant role the Petrie operation plays in study of unoriented maps equipped with a high degree of symmetry. This concept seems to be useful for the construction and classification of regular embeddings on nonorientable surfaces. Following the pioneer works on oriented exponents by Nedela and Škoviera we proved analogies of several their results for unoriented maps.

Exponents play an important role at the graph-theoretical approach to the classification of regular maps by underlying graphs. As shown in Section 5, this classification process can be successfully accomplished in the nonorientable case as well. In fact, if we have two regular maps $M = (F; \lambda, \varrho, \tau)$ and N with identical monodromy groups and with the same underlying graph K of valency n, then there exists an exponent e of the graph K and $j \in \mathbb{Z}_2$ such that $N = (F; \lambda\tau^j, (\varrho\tau)^{e-1}\varrho, \tau)$. It implies that the isomorphism classes of maps, whose monodromy groups coincide, correspond to the cosets of the exponent group $\mathrm{Ex}\,(M)$ in $\mathbb{Z}_n^* \times \mathbb{Z}_2$, and their number is $|\ \mathbb{Z}_n^* \times \mathbb{Z}_2 : \mathrm{Ex}\,(M)|$.

We applied the unoriented exponent theory to the classification of regular maps of canonical double covering graphs. From a regular map M of a connected non-bipartite graph K it is possible to derive a regular map M_t of its canonical double cover $\tilde{K}$, for any involutory integer exponent $t = e$ and any mixed exponent $t = ep$ such that $e^2 \equiv -1 \pmod{\operatorname{val}(K)}$. Provided that the base graph K is stable and a regular map of $\tilde{K}$ is splittable, we are able to prove the converse, that is, every splittable regular map of $\tilde{K}$ can be obtained from a regular map of K with this construction. The stability of K is required in the orientable case, too, while the additional condition of splittability seems to be inevitable for unoriented maps. Although we do not know a regular map of a canonical double covering graph which is not splittable, we expect that such maps may exist.

Problem 9.1. *Find a regular map of a canonical double covering graph which is not splittable.*

To complete the classification of regular maps of the canonical double covers it is necessary to find conditions under which these regular maps are pairwise isomorphic. This problem was partially solved in Theorem 7.9 for regular splittable maps that can be obtained from regular maps of the base graph with use of involutory integer exponents.

Problem 9.2. *Let M and N be regular maps of a stable graph K. Let r and s be unoriented exponents of M and N, respectively, and at least one of them be mixed. Find conditions under which M_r is isomorphic to N_s.*

As for the classification of regular embeddings of cocktail-party graphs (Section 8), one also needs to deal with the case of non-splittable maps. If there is such an embedding of $\tilde{K}_n \cong K_n \otimes K_2$, then (since K_n is stable) its automorphism group $\tilde{G}$ projects to a subgroup G of $\operatorname{Aut}(K_n) \cong S_n$ satisfying the following properties:

(1) G is a 2-transitive permutation group of degree n,
(2) $|G| = 2n(n-1)$,
(3) the set-wise stabilizer of two points is isomorphic to Z_4.

However, an analysis of 2-transitive permutation groups shows that there are no such groups. It follows that every regular embedding of $\tilde{K}_n$ is splittable.

References

[BNS] A. Breda d'Azevedo, R. Nedela and J. Širáň, *Classification of regular maps of prime negative Euler characteristic*, Transactions of Amer. Math. Soc. (to appear).

[CM] H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups*, 4th edn, Springer, Berlin, 1984.

[GAP] The GAP-Group, *GAP - Groups, Algorithms, and Programming, Version 4.4*, (http://-www.gap-system.org) (2004).

[GNSS] A. Gardiner, R. Nedela, J. Širáň, and M. Škoviera, *Characterization of graphs which underlie regular maps on closed surfaces*, J. London Math. Soc. **59** (1999), 100–108.

[H] L. Heffter, *Über metacyklische Gruppen und Nachbarconfigurationen*, Math. Ann. **50** (1898), 261–268.

[J1] L. D. James, *Imbeddings of the complete graphs*, Ars Combin. **16-B** (1983), 57–72.

[J2] G. A. Jones, *Maps on surfaces and Galois groups*, Math. Slovaca (1) **47** (1997), 1–33.

[JJ] L. D. James and G. A. Jones, *Regular orientable imbeddings of complete graphs*, J. Combin. Theory Ser. B **39** (1985), 353–367.

[JS] G. A. Jones and D. Singerman, *Theory of maps on orientable surfaces*, Proc. London Math. Soc. (3) **37** (1978), 273–307.

[JT] G. A. Jones and J. S. Thornton, *Operations on maps and outer automorphisms*, J. Combin. Theory Ser. B **35** (1983), 93–103.

[N1] R. Nedela, *Group actions on graphs, maps and surfaces*, DrSc. Thesis (2005).

[N2] R. Nedela, *Regular maps - combinatorial objects relating different fields of mathematics*, J. Korean Math. Soc. **38** (2001), No. 5, 1069–1105.

[NS1] R. Nedela and M. Škoviera, *Exponents of orientable maps*, Proc. London Math. Soc. (3) **75** (1997), 1–31.

[NS2] R. Nedela and M. Škoviera, *Regular embeddings of canonical double coverings of graphs*, J. Combin. Theory Ser. B **67** (1996), 249–277.

[NS3] R. Nedela and M. Škoviera, *Regular maps from voltage assignments and exponent groups*, European J. Combin. **18** (1997), 807–823.

[NSZ] R. Nedela, M. Škoviera and A. Zlatoš, *Bipartite maps, Petrie duality and exponent groups. Collection of papers published in honour of M. Pezzana, edited by C. Gagliardi*, Atti Sem. Mat. Fis. Univ. Modena, Suppl. al **49** (2001), 109–133.

[R] J. J. Rotman, *An introduction to the theory of groups*, 4th edn, Springer, Berlin, 1995.

[V1] A. Vince, *Combinatorial maps*, J. Combin. Theory Ser. B **34** (1983), 1–21.

[V2] A. Vince, *Regular combinatorial maps*, J. Combin. Theory Ser. B **35** (1983), 256–277.

[V3] A. Vince, *Flag transitive maps*, Congr. Numer. **45** (1984), 235–250.

[W1] H. Wielandt, *Finite permutation groups*, Academic Press, New York, 1964.

[W2] S. E. Wilson, *Cantankerous maps and rotary embeddings of K_n*, J. Combin. Theory Ser. B **47** (1989), 262–273.