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TWO PROBLEMS IN GRAPH LAYOUTS

by

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Chapter 1

Introduction

The advent in VLSI technology in recent years has enabled the construction of very complex interconnection networks. Since most experts say that there will be a trend to add more computational units into the circuits rather than increasing computational power of one unit, the interconnection networks are of strong interest in computer science. By using certain number of computational units in circuit certain time and capacity of the circuit is consumed by the communication between the units rather than computational tasks. As the number of units is increasing, the task to manage the communication is getting more complex. One of the most important part of this problem is to find a "suitable" topological structure for the interconnection network. This task is of strong interest of computer scientists and engineers.

The fundamental tool for analyzing the properties of interconnection networks is graph theory. This work is focused on some design problems of interconnection networks. Especially, we study the one- and three-dimensional layouts of their underlying graphs.

The one-dimensional (linear) layouts studied here are rather formulated as graph labeling problems. We focus on the antibandwidth problem. This problem can be formulated as a linear layout problem or a graph labelling problem. The task is to find such a linear layout of a graph in which the length of the shortest edge is maximal over all linear layouts of the graph. The study of the problem is motivated by computer science, enemy facility location problem and coding theory. An interesting application can be found also in tournament scheduling problems.

This problem is almost unexplored - there are some results in the literature but exact results are rare.

The key notion in this thesis is that of linear layout of a graph (see Figure 1.1). It plays an important role also in the second part dedicated to the three-dimensional layouts of graphs. The constructions we use there are based on a simple linear layout of a graph.

The "standard" way the interconnection networks are realized is by a two-dimensional layout. The chip consists of one layer of transistors built in a plane and occupying some area. In recent years the research of the three-dimensional layouts has started. The reasons for this are simple. The three-dimensional layout is more "compact" compared to the two-dimensional one and needs shorter connections links. The shorter connection links mean the lower building costs of the chip. Although at the time of writing of this work, there were no real life applications known to the author, they are expected,

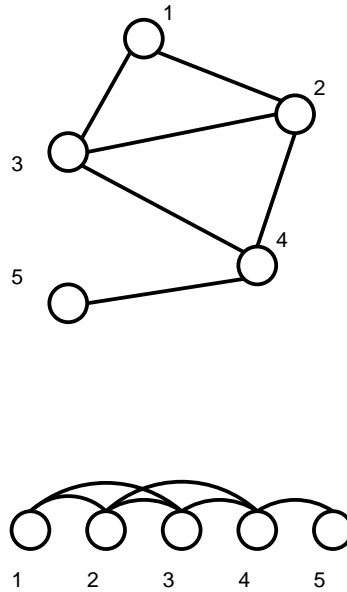


Figure 1.1: A graph and one of its possible linear layouts

according to words of some IT experts, in the near future. In article "IBM bakes new 3D circuit design" from November 11, 2002 IBM says, that that they have devised a new 3D circuit design that uses two or more layers of transistors. This technology, according to IBM, is still fairly green but has some important advantages, like mixing different kinds of circuits in one chip, increasing the number of transistors (higher performance of a chip) and reducing the length of metal interconnections (cheaper and faster production) [1].

This work is organised into two main parts: linear and three dimensional layouts of graphs. In both sections we study the corresponding properties of some well-known classes of interconnection networks.

In the following sections we provide a brief introduction into both problems together with motivation and overview of the previous results.

1.1 Antibandwidth problem

The antibandwidth problem is a special case of the antidilation problem (called separation in literature [41]). Antidilation is a problem of injective embedding of a graph G in a graph H such that the length of the shortest edge of G measured in H is maximised over all possible embeddings of G in H . Antidilation is a dual problem to a well studied dilation problem [36]. The special case of dilation is bandwidth problem where one is looking for such a linear layout of a graph where the longest edge is minimized. The antibandwidth problem is a dual modification of the bandwidth problem. Because of this duality we choose the name "antibandwidth" for the problem.

The antibandwidth problem was originally introduced by Leung, Vornberger and Witthof in [38] under the name *separation number*. Meanwhile, the term *separation number* has been used for another linear layout problem [16]. There was also another name used for this problem. Lin and Yuan called it *dual bandwidth* [39]. Leung, Vornberger and Witthof discuss three modifications

of the well-known bandwidth problem: cyclic bandwidth and the linear and cyclic separation problem (called antibandwidth and cyclic antibandwidth in this work). They show by reductions that directed version of antibandwidth problem is connected to some multiprocessor scheduling problems. Note that directed case of the antibandwidth problem is less difficult comparing the undirected case. NP-completeness of both antibandwidth and cyclic antibandwidth problems is also proved here.

1.1.1 Motivation

The original motivation comes from the area of radio frequency assignment problem [26]. The problem is to assign n different frequencies to n transmitters in such a way that physically neighbouring transmitters have as different frequencies as possible. The transmitters are represented by the vertices of a graph G and their neighbourhood is represented by the adjacency in a graph G . The problem also belongs to the family of obnoxious facility location problems. The "enemy graph" is representing some kind of "enemy" facilities and the task is to arrange them on the line (cycle, mesh,...) such that the minimal distance between any of two enemies is maximised. The "enemy" relation is represented by adjacency in the underlying graph. In this work, we research the linear and cyclic version of the problem, i.e. we consider the embeddings of a graph $G = (V, E)$ into the line or cycle of the length $|V|$. However, the embeddings into general graphs are interesting too. Especially the case where the host graph is a hypercube Q_n and the guest graph is a complete graph K_p . Then antidilation of K_p embedded in Q_n ($p \leq 2^n$) is equal to the minimal Hamming distance of a binary code with p words of length n . The minimal Hamming distance of a code characterizes code's robustness, especially the error correcting property (see for example [40]).

The edge bandwidth problem is studied in [33]. We defined the analogue version of the antibandwidth problem. The task is to label edges by consecutive integers $1, 2, \dots, |E|$ such that the minimal difference of labels on incident edges is maximized. There is a nice real-life application of edge antibandwidth problem in tournament schedulings. Consider a tournament with n players. There are m pairs of players. One such pair is a game. We want to schedule these games in such an order where the break between two games involving the same player is maximised (so the player can rest as long as possible without delaying the schedule). A tournament can be represented by a n -vertex m -edge graph G . Then our tournament scheduling problem is equivalent to finding the edge antibandwidth of the graph G . Motivated by this application we started to research also this modification of the original antibandwidth problem.

1.1.2 Goals and results

In this thesis we provide results concerning the basic versions of antibandwidth problem: linear, cyclic and edge antibandwidth. Our goal is to research and determine the invariant for most common interconnection networks and show its relation to other graph invariants. This goal is given by the trends in this area of research. In this part we solved several open problems. For interconnection networks like meshes and hypercubes there were previously

known the upper and lower bounds obtained by Miller and Pritikin [41]. They proved the following results for $m \times n$ meshes:

$$\left\lfloor \frac{n(m-1)}{2} \right\rfloor \leq \text{ab}(P_m \times P_n) \leq \left\lfloor \frac{mn}{2} \right\rfloor.$$

And for hypercubes on 2^n vertices:

$$2^{n-1} - \frac{2^n}{\sqrt{2\pi n}}(1 + o(1)) \leq \text{ab}(Q_n) \leq 2^{n-1}.$$

We proved the exact values. Namely for meshes we have:

$$\text{ab}(P_m \times P_n) = \left\lfloor \frac{n(m-1)}{2} \right\rfloor.$$

And for hypercubes:

$$\text{ab}(Q_n) = 2^{n-1} - \frac{2^n}{\sqrt{2\pi n}}(1 + o(1)).$$

Moreover, we solve and determine the antibandwidth value for some other classes of interconnection networks like tori, complete k -ary trees and three-dimensional meshes.

We extend the existing results by a research of the cyclic antibandwidth problem which is motivated by an existence of similar research of cyclic bandwidth problem. We determine the cyclic antibandwidth problem for meshes, toroidal meshes and hypercubes. Moreover, we found a simple but nice real life application for the edge antibandwidth problem. Note that there exists an analogue research for the edge bandwidth problem which is the other motivation for studying this version of the problem. Motivated by our simple real life application we start to research this modification of the problem for complete, complete bipartite and complete k -partite graphs and provide the exact results for complete and complete bipartite graphs.

The antibandwidth problem still offers a lot of unexplored open problems. We summarize some of them at the end of thesis.

1.2 Three-dimensional layouts of graphs

In this part of the thesis we are concerned with three-dimensional layouts of graphs. Our work is motivated mainly by computer science, especially the three-dimensional VLSI layouts. Our results can be used in the area of three-dimensional graph drawings as well. One of our main results here is a general lower bound for volume-optimal $3D$ layout of a graph. This bound depends on a parameter called *cutwidth* which can be defined using the terminology of linear layouts discussed in the previous section. Our goal is to minimize the volume occupied by the graph in space. The graph is drawn orthogonally in a three-dimensional mesh realized on integer coordinates with unit spacing.

1.2.1 Motivation

The research on three-dimensional circuit layouts has started in seminal works [46, 50]. As we showed in the introduction, only recently the technology advanced such that $3D$ circuits can be build physically. Meanwhile, theoretical research based on models continued.

Rosenberg in his work [50] pointed out the importance of presenting interconnection networks in 3D and proved some basic results concerning two models of layouts: all-volume model (basically our general model) and one-active-layer model. Preparata in [46] used these models and proved results for cube-connected cycles.

Models considered in both works were natural generalization of 2D models from [54].

Since it was formally proved that 3D layouts may essentially reduce material [37], measured as volume, several basic results appeared. This problem can be considered also as a special orthogonal three-dimensional drawing of graphs, see for example [19].

The main limitation of all early results in this area was the degree of a node. Only the vertices with degree at most 6 were considered. This limitation was given by the model used for representing the circuit where the node was just a point in a 3D mesh.

There are only few results allowing the degree of vertex to be arbitrarily high [7, 10, 62]. Models considered there represent vertex as a rectangle [10], square [62] or axis-aligned box [7]. There are two trends concerning the routing of the edges in 3D layout models. Generally, we can say that overlapping of the edges is forbidden. There are models that allow crossing of the edges in one point [10] and also the models which allows crossing (or better said touching) of the edges in their endpoints only [7] (motivated more by "nice" 3D graph drawing).

1.2.2 Goals and results

In our work, we consider the one-active-layer and general model. In both cases we use the representation of a vertex which allows the degree of a vertex to be arbitrarily large.

Calamoneri and Massini addressed two open problems concerning the study of optimal volume of the hypercube under two-active-layer and general model [10]. In our work we solve both of them. Calamoneri and Massini obtained the following results concerning the two-active-layers model:

$$\text{VOL}_{2-AL}(Q_{\log N}) = \Omega(N^{\frac{3}{2}} \log^{\frac{1}{2}} N).$$

$$\text{VOL}_{2-AL}(Q_{\log N}) = O(N^{\frac{3}{2}} \log N).$$

We prove the optimal volume under one-active-layer layout and use it to build up the volume optimal two-active-layer layout:

$$\text{VOL}_{1-AL}(Q_{\log N}) = \Theta(N^{\frac{3}{2}} \log N).$$

Using two one-active-layer layouts we obtain the following volume for two-active-layers model:

$$\text{VOL}_{2-AL}(Q_{\log N}) = \Theta(N^{\frac{3}{2}} \log N).$$

For the volume of the layout under the general model we prove the following optimal value:

$$\text{VOL}(Q_{\log N}) = \Theta(N^{\frac{3}{2}}).$$

The volume-optimal construction for both models employs the so called colinear layout of a graph. It is a layout which belongs to a large family of linear layouts of graphs. This layout is a key to our optimal solution and it can be applied for the cartesian products only. This fact leads to a problem which is still open according to our knowledge. It is a problem of finding a general construction providing a volume-optimal layout of a general graph.

Motivated by a similar research by Fernandez and Efe [21] in the area of two-dimensional layouts we extend our results from hypercube to any cartesian product graph with isomorphic factors. We provide here an asymptotically exact results and constructions for several other classes of product graphs (cube-connected-cycles, star graph, linear array (path), and others).

1.3 Structure of the thesis

In the next chapter we define the terms and notions we use in this thesis. We define just the notions which are new or less common. The studied interconnection networks are defined here as well.

Chapter 3 is dedicated to existing results in both antibandwidth and three-dimensional layout problems. We provide here a short overview of both problems.

Chapters 4,5 and 6 discuss our results concerning three modifications of the antibandwidth problem: linear antibandwidth, cyclic antibandwidth and edge antibandwidth, respectively. Chapters 4 and 5 provides exact or asymptotically exact results for networks like meshes, toroidal meshes, three-dimensional meshes, complete k-ary trees and hypercubes. The edge antibandwidth is discussed for complete, complete bipartite and complete k-partite graphs in Chapter 6.

Chapter 7 discuss our results in the volume optimal three-dimensional layouts problem. We provide here a general lower bounds and an optimal constructions for hypercube in both models. Then we generalize obtained results for any cartesian product graph constructed from isomorphic factors and provide asymptotically exact values for products of some known interconnection networks.

Chapter 8 concludes obtained results and provides some interesting open problems for both parts of the thesis the antibandwidth and volume optimal three-dimensional layouts of graphs.

Chapter 2

Notions and definitions

In this chapter we define the terms we use in this thesis together with some classes of interconnection networks we study here. We follow the standard notation for common terms so we define just the new problems and less common notions.

This chapter consists of three sections. The first one is dedicated to the definition of the antibandwidth problem and its related notions. In the second section we define the problems and notions related to the three-dimensional layouts of graphs. In the last section we define some of the known interconnection networks which are discussed later in this work.

2.1 Antibandwidth problem

The antibandwidth problem is a special case of the so called antidilation problem. First, we define the embedding of a graph G in a graph H .

Definition 2.1.1. *Let $G = (V_1, E_1), H = (V_2, E_2)$ be graphs such that $|V_1| \leq |V_2|$. Then the embedding of a graph G in a graph H is a injective mapping of vertices of G into vertices of H .*

Definition 2.1.2. *The antidilation problem is to embed a graph $G = (V_1, E_1)$ into a graph $H = (V_2, E_2)$, $|V_1| \leq |V_2|$, in such a way that the minimal distance of adjacent vertices of G measured in H is maximised over all possible embeddings of G into H .*

We call G "the guest graph" and H "the host graph".

We denote the antidilation of a graph G in a graph H by

$$\text{adil}(G, H).$$

We provide the relation between antidilation and its dual variant dilation later in this work. The dilation is defined in a dual way to the antidilation problem.

Definition 2.1.3. *The dilation problem is to embed a graph $G = (V_1, E_1)$ into a graph $H = (V_2, E_2)$, $|V_1| \leq |V_2|$, in such a way that the maximal distance of adjacent vertices of G measured in H is minimised over all possible embeddings of G into H .*

We denote the dilation of a graph G in a graph H by

$$\text{dil}(G, H).$$

In the case where the host graph is a path the antidilation problem reduces to its special case - the antibandwidth problem. The antibandwidth problem is to embed a graph $G = (V, E)$ into a path of length $|V|$ in such a way that the shortest edge is maximised over all possible embeddings of G . See the example in the Figure 2.1. We call this the "linear antibandwidth problem" or simply the "antibandwidth problem".

Usually, it is useful to look at the antibandwidth problem as at a graph labelling problem. More formally:

Definition 2.1.4. For a nonempty graph $G = (V, E)$, let f be a one-to-one labelling

$$f : V \rightarrow \{0, 1, 2, 3, \dots, |V| - 1\}.$$

Define the antibandwidth of G according to f as

$$\text{ab}(G, f) = \min_{uv \in E} |f(u) - f(v)|.$$

The antibandwidth of G is defined as

$$\text{ab}(G) = \max_f \text{ab}(G, f).$$

The antibandwidth problem is, in fact, a dual problem to the well known bandwidth problem [16] where one asks for such a linear layout of the graph, where the length of the longest edge is minimised.

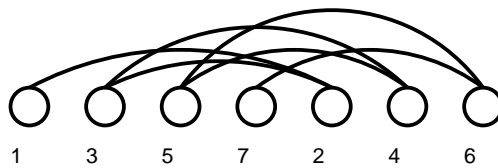
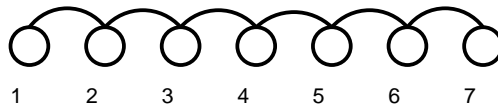


Figure 2.1: Optimal layout of P_7

We discuss here the cyclic version of the antibandwidth problem too. This is a modification of the problem where a graph G is embedded into a cycle. Formally:

Definition 2.1.5. For a nonempty graph $G = (V, E)$, let f be a one-to-one labelling:

$$f : V \rightarrow \{0, 1, 2, 3, \dots, |V| - 1\}.$$

Define the cyclic antibandwidth of a connected graph G according to f as

$$\text{cab}(G, f) = \min_{uv \in E} \{|f(u) - f(v)|, |V| - |f(u) - f(v)|\}.$$

The antibandwidth of G is defined as

$$\text{cab}(G) = \max_f \text{cab}(G, f).$$

In the cyclic antibandwidth problem the vertices are mapped bijectively into $C_{|V|}$ such that the minimal distance, measured in the cycle, of adjacent vertices is maximised.

Because of the duality between the bandwidth and antibandwidth problems we try to follow the research trends in the bandwidth problem. One of these trends is a research of so called *edge bandwidth* problem. This problem is basically similar to standard bandwidth problem. The difference is that in edge bandwidth problem one is labelling the edges of a graph not the vertices. The goal is similar - to minimize the maximal difference of labels of incident edges. In fact, this problem can be simply turned into standard bandwidth problem by performing a line graph operation on originally researched graph. After this conversion one can solve the standard bandwidth problem on a converted graph. The same holds for *edge antibandwidth problem* which we define as follows.

Definition 2.1.6. *The edge antibandwidth problem is to label edges of an m -edge graph bijectively by $0, 1, 2, 3, \dots, m - 1$ such that the minimal difference of labels on incident edges is maximised.*

Let us remind that the line graph operation on a graph G turns the edges of G into vertices of $L(G)$. Any two vertices of $L(G)$ are connected by an edge iff the corresponding edges in G were adjacent. See the Figure 2.2 for example.

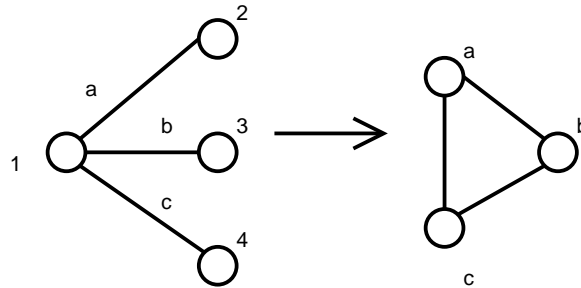


Figure 2.2: Line graph operation on star on 4 vertices

In the text we denote the edge antibandwidth of a graph G by the following notation:

$$\text{ab}(L(G)).$$

To complete the necessary definitions we need to define the following notions.

Definition 2.1.7. *Let G be a graph. Let $\partial(A)$ denote the vertex boundary of a set $A \subseteq V$, i.e., the set of all vertices from $V - A$ having a neighbour in A .*

Definition 2.1.8. Let $G = (V_1, V_2, E)$ be a bipartite graph. Let $\partial_b(A)$ denote the vertex boundary of a set $A \subseteq V_1$, i.e. the set of all vertices from V_2 having a neighbour in A . We call it the bipartite vertex boundary.

Let us define the *simplicial order* in a graph G as follows [8].

Definition 2.1.9. Let $x = (x_1, x_2, x_3, \dots, x_n)$ and $y = (y_1, y_2, y_3, \dots, y_n)$. Then $x < y$ if either $\sum x_i < \sum y_i$, or $\sum x_i = \sum y_i$, and for some j we have $x_j > y_j$ and $x_i = y_i$ for all $i < j$.

In the proof of antibandwidth of complete k -ary tree we use the following two notions: vertex bisector and edge bisector.

Definition 2.1.10. We say that a set of vertices U in a graph $G = (V, E)$ is a vertex r -bisector if removing U the remaining vertices are partitioned into disjoint sets V_1, V_2 , s.t. $|V_1|, |V_2| \leq r$ and every path between V_1 and V_2 contains a vertex from U .

And in the similar way we define the edge bisector.

Definition 2.1.11. We say that a set of edges F in a graph $G = (V, E)$ is an edge $\lceil n/2 \rceil$ -bisector if removing F the vertices are partitioned into disjoint sets V_1, V_2 , s.t. $|V_1|, |V_2| \leq \lceil n/2 \rceil$ and every edge between V_1 and V_2 belongs to F .

2.2 Three-dimensional layouts of graphs

All three-dimensional layouts discussed here are realized in the three-dimensional integer mesh with edges of unit length. The exact rules for three-dimensional layouts are given by models. For our work the following two models are important.

Definition 2.2.1. The three-dimensional one-active-layer layout of a graph G is a mapping of G into the three-dimensional mesh such that the following conditions are satisfied:

- A vertex of degree d is represented by a square of integer coordinates of side d lying in the basic plane given by $z = 0$. The sides of the square are parallel to the x and y axes.
- Two vertices (squares) do not touch.
- Edges are represented as edge disjoint paths in the 3D mesh graph in the halfspace above the basic plane. A path touches only two squares which represent the endvertices of the corresponding edge.

Definition 2.2.2. The three-dimensional general layout of a graph G is a mapping of G into the three-dimensional integer mesh such the following conditions are satisfied:

- A vertex of degree d is represented by a cube of integer coordinates of the side of length d . The sides of the cube are parallel to the axes.
- Two vertices (cubes) do not touch.

- *Edges are represented as edge disjoint paths in the mesh graph. A path touches only two cubes which represent the endvertices of the corresponding edge.*

These two models cover main trends in research and with small differences (vertex representation mainly) they are used in most of the cited papers.

With use of one-active-layer and general model we can define our goal.

Definition 2.2.3. *The volume-optimal one-active-layer (general) layout of a graph G is a layout minimizing its volume in the one-active-layer (general) model.*

The volume of a graph G in one-active-layer model is denoted by

$$\text{VOL}_{1\text{-AL}}(G).$$

The volume of a graph G in general model is denoted by

$$\text{VOL}(G).$$

Remark. The models we described, differs from the so called multilayer model [13], in which vertices are represented as boxes lying on the bottom layer and edges are routed in layers parallel to the basic plane.

Most of our results in this part of the thesis are connected with cutwidth parameter of a graph.

Definition 2.2.4. *Let $\phi : V \rightarrow 1, 2, 3, \dots, |V|$ be a 1-1 labelling of vertices of a graph $G = (V, E)$. Define*

$$\text{cw}(G, \phi) = \max_i \{|\{uv \in E : \phi(u) \leq i < \phi(v)\}|\}.$$

The cutwidth of the graph G is defined as

$$\text{cw}(G) = \min_{\phi} \{\text{cw}(G, \phi)\}.$$

The cutwidth is strongly related to linear layouts and, roughly saying, represents the largest edge cut in a graph which is embedded in a line.

The *bisection width* is defined in a similar way to cutwidth. The only difference is that only the cut in the middle of the layout is minimised. More formally

Definition 2.2.5. *Let $\phi : V \rightarrow 1, 2, 3, \dots, |V|$ be a 1-1 labelling of vertices of a graph $G = (V, E)$. Define*

$$\text{bw}(G, \phi) = \max\{|\{uv \in E : \phi(u) \leq |V|/2 < \phi(v)\}|\}.$$

The bisection width of the graph G is defined as

$$\text{bw}(G) = \min_{\phi} \{\text{bw}(G, \phi)\}.$$

The basic building block for three-dimensional layouts we discuss here is a colinear layout. It is a slight modification of the linear layout of a graph. It is defined as follows.

Definition 2.2.6. *In the colinear layout the vertex is represented as a square with side length equal to its degree. The vertices are arranged in the line in a ordering. The edges are routed in edge-disjoint manner in the two-dimensional integer mesh situated in the halfplane above the vertices.*

The height of colinear layout of a graph G is defined as $\Delta(G) + t$, where $\Delta(G)$ is maximal vertex degree in G and t is the number of horizontal tracks used for routing of the edges.

Note that from the definition of cutwidth follows that the optimal colinear layout of G has number of tracks equal to cutwidth of G .

In the section dedicated to volumes of cartesian product graphs the following parameter plays an important role.

Definition 2.2.7. [31] *Let the routing ρ be defined as follows. For every two distinct vertices u, v of G there exist 2 paths, from u to v and from v to u . The number of paths of ρ is then $n(n - 1)$. The edge forwarding index of (G, ρ) denoted by $\pi(G, \rho)$ is the maximum number of paths, specified by routing ρ going through any edge of G . More precisely :*

$$\pi(G, \rho) = \max\{\pi_e(G, \rho) : e \in E(G)\},$$

and the edge-forwarding index of G is defined as

$$\pi(G) = \min\{\pi(G, \rho) : \forall \rho\}.$$

2.3 Interconnection networks

The main task of an interconnection network is to provide the communication scheme between the computational units in a circuit. The properties, and therefore the abilities, of the interconnection networks to satisfy the communication needs of the circuit depend on its topological structure. It has been shown practically that graph theory is a very powerful tool for designing and analyzing of the topological structure of interconnection networks.

One of the most popular network topologies is hypercube. On the other hand, there are also other topologies that are of strong interest of researchers because of their interesting properties.

In the following sections we show the definitions of some of these networks, especially those which are mentioned later in this work. We refer the interested reader to the nice monograph on interconnection networks [60] for more details on particular topologies and techniques used for their design.

2.3.1 Hypercube networks

The hypercube became the first choice for the topological structure of parallel processing and computing systems. This structure has been intensively studied in graph theory [30] and several commercial machines are based on hypercube architecture.

Definition 2.3.1. *The vertex set of the hypercube Q_n consists of all binary sequences of length n on the set $\{0, 1\}$, i.e.*

$$V = \{x_1x_2\dots x_n : x_i \in \{0, 1\}, i = 1, 2, \dots, n\}$$

Two vertices $x = x_1x_2\dots x_n$ and $y = y_1y_2\dots y_n$ are linked by an edge if and only if x and y differ exactly in one coordinate, i.e. $\sum_{i=1}^n |x_i - y_i| = 1$.

Let us note that there is also other equivalent definition of the hypercube based on cartesian product method [60].

2.3.2 Mesh and tori networks

Definition 2.3.2. *The structure of a mesh network is defined as a cartesian product of two undirected paths of lengths m, n : $P_m \times P_n$.*

We often refer to such a mesh as a $m \times n$ mesh.

Natural generalization of this definition leads to a definition of *multidimensional meshes*.

Definition 2.3.3. *The n -dimensional mesh network is defined as $P_{m_1} \times P_{m_2} \times \dots \times P_{m_n}$ where P_{m_i} is undirected path of length m_i .*

The torus network is constructed in the same way as the mesh network with use of cycles C_i instead of paths P_i , for $i = 1, 2, \dots, n$.

2.3.3 De Bruijn networks

This network has certain advantages over the hypercube, i.e. it can connect more processors for the same values of degree and diameter, it has unique shortest path between any ordered vertices, what gives an easy routing algorithm, and others.

Definition 2.3.4. *The vertex set V of de Bruijn graph $B(d, n)$ is*

$$V = \{x_1x_2\dots x_n : x_i \in \{0, 1, 2, \dots, d-1\}, i = 1, 2, \dots, n\}$$

and edge set E consists of all edges from one vertex $x_1x_2\dots x_n$ to d other vertices $x_2\dots x_{n-1}\alpha$, where $\alpha \in \{0, 1, \dots, d-1\}$.

For other equivalent definitions (using iterated line graphs or arithmetic method) see [60].

2.3.4 Cube-connected cycles

Definition 2.3.5. *The n -dimensional cube-connected cycles, denoted by $CCC(n)$ is constructed from n -dimensional hypercube Q_n by replacing each vertex of Q_n with undirected cycle of length n . The i th dimensional edge incident to a vertex of Q_n is then connected to the i th vertex of the corresponding cycle of $CCC(n)$.*

2.3.5 Star graph

Definition 2.3.6. *An n -dimensional star graph is a symmetric graph that has $N = n!$ nodes of degree $n-1$. Each node in an n -star is assigned a label, which is a distinct permutation of the set of n symbols $\{1, 2, 3, \dots, n\}$. Two nodes are connected with a dimension- i link, $2 \leq i \leq n$, iff the label of one node can be obtained from the other by interchanging the first symbol and the i th symbol.*

An n -star contains n disjoint $(n-1)$ -stars as subgraphs, each pair of which are connected by $(n-2)!$ links [61].

2.3.6 Complete k -ary trees

Definition 2.3.7. *The complete k -ary tree, denoted by $T_{k,h}$ is a rooted tree in which the root and every internal node has k children and every leaf is at distance h from the root.*

2.3.7 Complete transposition graph

Definition 2.3.8. *An m -dimensional complete transposition graph is a symmetric graph that has $n = m!$ nodes of degree $m - 1$. Each node of this graph is assigned a label, which is a distinct permutation of the set of m symbols $\{1, 2, \dots, m\}$. Two nodes are connected by an edge if, and only if, their labels differ in exactly two positions.*

2.3.8 Butterfly graph

Definition 2.3.9. *The m -dimensional butterfly graph, denoted by $BF(m)$, has vertex set $V = \{(x; i) : x \in V(Q_m), 0 \leq i \leq m\}$. Two vertices $(x; i)$ and $(y; j)$ are connected by an edge if and only if $j = i + 1$ and either $x = y$ or x differs from y in precisely the j th bit.*

Chapter 3

Previous results

In this chapter we summarize the previously known results. Both antibandwidth and three-dimensional layout problems are discussed here. We provide the overview of both problems with accent on the results which have direct connection to the results obtained in this thesis.

3.1 Known results on the antibandwidth problem

Because of the name confusions the searching for the results in literature is not simple. Since the antibandwidth problem is a dual one to the well-studied bandwidth problem there was some effort to find a connection between these two problems. So far, nothing strong has been proven. We provide a relation later in this work.

Motivation for studying this problem comes mainly from the area of theoretical computer science. This gives the main trend in the literature discussing the antibandwidth problem - to research the invariant itself and to determine its value for known interconnection networks. In our work, we follow this framework and our research is focused on this two points mainly. Before that, we start with some known general results characterizing the invariant itself. We refer the interested reader to a survey by Miller and Pritikin [42].

3.1.1 Upper bounds

We start with the upper bound showing a relation between the antibandwidth of a graph G and the size of its largest clique $\omega(G)$ [24]. Since it holds that $\omega(G) \leq \chi(G)$ the similar claim can be obtained as we will see later.

Theorem 3.1.1. *Let $G = (V, E)$ be a graph on $n = |V|$ vertices. Let $\omega(G)$ denote the size of its largest clique. Then*

$$\text{ab}(G) \leq \left\lfloor \frac{n-1}{\omega(G)-1} \right\rfloor.$$

There is an interesting connection between the linear antibandwidth and the chromatic index of a graph G given by the following theorem. This result can be found in [39].

Theorem 3.1.2. *Let $G = (V, E)$ be a graph with chromatic number $\chi(G)$. Then*

$$\text{ab}(G) < \frac{n}{\chi(G)-1}.$$

The following claim [24] provides the relation between the antibandwidth of a graph and its minimal degree. This upper bound is sharp and it is reached by a path P_n as we will see later.

Theorem 3.1.3. *Let $G = (V, E)$ be a connected graph on n vertices having minimal degree $\delta(G)$. Then*

$$\text{ab}(G) \leq \lfloor \frac{n}{2} \rfloor - \lfloor \frac{\delta(G)}{2} \rfloor.$$

The following relation involves antibandwidth of a graph and its largest independent set [24].

Theorem 3.1.4. *Let $G = (V, E)$ be a graph with independence number $\alpha(G)$. Then*

$$\text{ab}(G) \leq \alpha(G).$$

There can be found a few more upper bounds in the survey [42] but these upper bounds are sharp for extremal graphs only (complete graphs or paths).

3.1.2 Lower bounds

The lower bound for antibandwidth problem is obtained by a construction. In the following paragraphs we show some known constructions for paths, cycles, trees, meshes and hypercubes.

First we show a simple lower bound for bipartite graphs.

Theorem 3.1.5. [39] *For a bipartite graph G ,*

$$\text{ab}(G) \geq \left\lceil \frac{D(G)}{2} \right\rceil,$$

where $D(G)$ denotes the diameter of G .

Theorem 3.1.6. [41] *For F any forest,*

$$\text{ab}(F) \geq \text{MIN}(F),$$

where $\text{MIN}(F)$ denotes the minority part of the graph bipartition.

The antibandwidth value is interesting also for disconnected graphs. Note that this is not true for the bandwidth problem where the bandwidth value for disconnected graphs is simply the maximal one of the values of all of its components. In [24] there are results proved for disconnected graphs consisting of isolated copies of paths, cycles and complete graphs. For the sake of completeness we provide these results in the following theorems. All of them are due to [24].

Theorem 3.1.7. *Let G be the union of paths P_i with $i = q_j > 0 (j = 1, \dots, m)$ and $\sum_i^m q_i = n$. Then*

$$\text{ab}(G) = \left\lfloor \frac{n}{2} \right\rfloor.$$

Theorem 3.1.8. *Let C_n be a cycle on n vertices. Let kC_n consists of k isolated copies of C_n . Then for n even*

$$\text{ab}(kC_n) = \frac{kn}{2} - 1$$

And for n odd

$$\text{ab}(kC_n) = \frac{k(n-1)}{2}.$$

Theorem 3.1.9. *Let kK_n consist of k isolated copies of K_n . Then*

$$\text{ab}(kK_n) = k$$

Now we continue with exact results for connected graphs. These are known for simple graphs like paths, cycles and complete binary trees.

Paths. The following result comes from [39].

Theorem 3.1.10. *Let P_n be a path of length n . Then*

$$\text{ab}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$$

Cycles. The similar result can be obtained for the antibandwidth of cycle.

Theorem 3.1.11. *Let C_n be a cycle of length n . Then*

$$\text{ab}(C_n) = \left\lceil \frac{n}{2} \right\rceil - 1$$

Trees. The complete binary tree is one of the fundamental structures in computer science. The antibandwidth invariant of complete binary trees has been determined by Miller and Pritikin in [41] and later in [59].

Theorem 3.1.12. *Denote by $T_{2,h}$ a complete binary tree of height h . Then*

$$\text{ab}(T_{2,h}) = 2^h - 1$$

There are some other special trees discussed in [59]. Namely, the exact results are provided for complete binary trees, double stars and (m, n) -caterpillars.

Meshes. The bounds for antibandwidth of meshes and hypercubes has been determined by Miller and Pritikin in [41]. We do not show the proofs for meshes and hypercubes here. We will prove exact results for both graphs in chapter dedicated to our results on antibandwidth problem.

Theorem 3.1.13. *[41] Let $P_m \times P_n$, $m \geq n$ be a $m \times n$ mesh. Then*

$$\left\lfloor \frac{n(m-1)}{2} \right\rfloor \leq \text{ab}(P_m \times P_n) \leq \left\lfloor \frac{mn}{2} \right\rfloor.$$

The upper bound comes from Theorem 3.1.3. In this thesis we extend this results and prove that lower bound is the optimal one. In addition, we show an optimal results for three-dimensional meshes.

Theorem 3.1.14. *[41] Let Q_n be a hypercube of an order n . Then*

$$2^{n-1} - \frac{2^n}{\sqrt{2\pi n}}(1 + o(1)) \leq \text{ab}(Q_n) \leq 2^{n-1}.$$

The upper bound comes from Theorem 3.1.3 again. We will prove later that the claimed lower bound is the optimal one.

3.1.3 The complexity of the antibandwidth problem

It is a good way to prove that a problem is hard before researching it deeply. In other case there might be a general polynomial time algorithm which can solve it efficiently. This is not the case of antibandwidth problem as it was shown in [38] and later in [39].

Theorem 3.1.15. *The problem of deciding $ab(G) \geq 2$ is NP-complete.*

Theorem 3.1.16. *The problem of deciding $cab(G) \geq 2$ is NP-complete.*

As both theorems say, the antibandwidth and cyclic antibandwidth problem are computationally hard. According to known results, the problem is polynomially solvable for some special classes of graphs. These special classes of graphs consist of the complements of arborescent comparability, interval and threshold graphs [17, 32].

In this work, we provide some new labelling algorithms which solve the antibandwidth problem for other graph classes. Those labelling algorithms which give the exact results, solve the appropriate problems efficiently.

3.2 Known results on the three-dimensional layouts

Since the question of optimal three-dimensional layout is not simple there are many research directions in this area. This is because we can consider a layout to be optimal when it occupies not only the minimal possible volume. The other point of view is the layout optimality as a function of the total wiring length or number of crossings of the edges.

In our work we discuss just the volume-optimality of the layout. In this section we describe some known results which have more or less direct connection to our work.

The networks of main interest in this chapter are the product networks, i.e. the networks which can be built by a cartesian product operation. These networks have some nice properties. We can use some of the results for lower dimension layouts. The most natural is to start with one-dimensional (i.e. linear) layout and extend it to a higher dimensions.

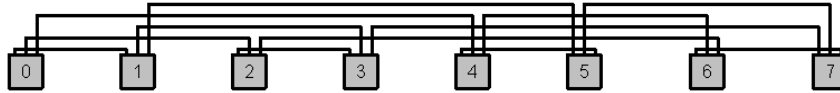
Since we use the hypercube as a model network in the part of the thesis considering the volume-optimal three-dimensional layouts, let us start with the hypercube.

In a colinear layout of the hypercube we ask for such a linear ordering of vertices where the number of the wiring tracks above the line of vertices is as small as possible. In fact, this question is equivalent to finding the cutwidth of the hypercube. It turns out that the normal layout (vertices in natural order, see Figure 3.1) minimizes total wire length and intercolumn wire density [27, 28, 43, 4, 45]. We use it in our construction because it provides the minimal number of wiring tracks. The minimal number of tracks needed in this layout is $\lfloor \frac{2N}{3} \rfloor$, what is, in fact, the cutwidth of the hypercube.

It is useful (at least for comparison) to show the result for optimal two-dimensional layout of hypercube. An efficient layouts for several hypercubic networks has been proposed in [63]:

Theorem 3.2.1. *An N -node hypercube can be laid out in*

$$\frac{4}{9}N^2 + O(N^2).$$

Figure 3.1: Optimal colinear layout of Q_3

area.

Theorem 3.2.2. *An N -node cube-connected-cycles network can be laid out in*

$$\frac{4N^2}{9\log_2^2 N} + O\left(\frac{N^2}{\log^2 N}\right)$$

area.

Not only the hypercubic networks are discussed in the literature. Also VLSI layouts of other interconnection networks are in strong interest of researchers. An efficient colinear and two-dimensional VLSI layouts of complete and star graphs are proved in [61].

Theorem 3.2.3. *An N -node complete graph can be laid out in*

$$\frac{N^4}{16} + O(N^4)$$

area.

The result for VLSI layout of the star graph (Definition 2.3.6) shows that it occupies smaller area than the similar size hypercube [61]:

Theorem 3.2.4. *An n -star can be laid out in $\frac{N^2}{16} + O(N^2)$ area, where $N = n!$.*

As we mentioned in the introduction the first papers discussing the three-dimensional layouts of graphs deal with the bounded vertex degree. The vertex is represented by a point in three-dimensional mesh and therefore its degree can not be higher than 6. However, some of these early results provide solutions which can be applied to general graphs. These papers [5, 15, 18, 19] are studying the optimal three-dimensional graph drawing and deal with a bounded vertex degree. On the other hand, the research of three-dimensional circuit layouts with bounded degree is represented for example by the paper of Leighton and Rosenberg [29]. The main results of this paper are the facts that significant savings in the material can be obtained by three-dimensional layout with only one active layer comparing the two-dimensional one. The second interesting result is that such a layouts can be obtained algorithmically. The algorithms which can convert an existing two-dimensional layout of a network to a more efficient three-dimensional one are provided there.

Now we recall some results discussing the three-dimensional layouts of hypercubes. The following bounds are due to Calamoneri and Massini [10].

Theorem 3.2.5. *A lower bound for a 3D mesh layout of Q_n is*

$$\Omega(N^{\frac{3}{2}} \log^{\frac{1}{2}} N)$$

if the layout has the following properties

- all nodes are represented as rectangular slices with area proportional to their degree,
- all nodes lie on two planes, corresponding to opposite sides of the bounding box,
- nodes are partitioned into two equally sized sets.

An upper bound:

Theorem 3.2.6. *There exists a three-dimensional layout of Q_n having volume*

$$\Theta(N^{\frac{3}{2}} \log N).$$

Calamoneri and Massini provide two open problems in [10]. The first one is asking for an optimal two-active layer layout of hypercube, i.e. it is a problem of eliminating the difference between their upper and lower bound from previous theorems. The second problem asks for study of a layout of the hypercube in three dimensions without any restrictions on the vertex position in the volume. Later in this work we provide solutions for both open problems.

One of the suitable replacements for the hypercube is de Bruijn network. Its three-dimensional layout is studied in [62]. Their construction uses 2D layouts of de Bruijn networks aligned in more layers.

Theorem 3.2.7. *The upper bound for the volume of de Bruijn network (given by the construction) is*

$$O\left(\frac{N^{\frac{3}{2}}}{\log^{\frac{3}{2}} N}\right),$$

with wire length

$$O\left(\sqrt{\frac{N}{\log N}}\right).$$

In this work we deal with the volumes of three-dimensional layouts of product graphs too. There exists a parallel research with results for area of two-dimensional layouts of some product graphs by Fernandez and Efe [21]. We provide results for the same product networks as Fernandez and Efe. Our results obtained in our thesis employ a completely different technique than that of Fernandez and Efe. However, we use a lemma by Fernandez and Efe.

The following lemma [21] provides the lower bound for the bisection width of product graph G^r . Since the bisection width is a lower bound for cutwidth, this lemma is useful for the approximation of the cutwidth of product graph G^r .

Lemma 3.2.1. *If the edge forwarding index of factor graph G with n vertices is $\pi(G)$ then the bisection width of product graph G^r satisfies*

$$bw(G^r) \geq \frac{n^{2r} - 1}{2\pi(G)n^{r-1}}.$$

Chapter 4

Linear antibandwidth problem

This chapter is dedicated to the linear antibandwidth of some known interconnection networks and consists of our results published in [58, 12, 48, 53]. Our results provided here are exact or at least asymptotically exact. Provided results include values of the antibandwidth invariant for meshes, three-dimensional meshes, tori, complete k-ary trees and hypercubes. Results for meshes and hypercubes are improvements over an existing bounds. Almost all results discussed here are obtained via our new method suitable for bipartite graphs.

Before we start with research of interconnection networks, we introduce some general relations between antibandwidth and other graph invariants.

4.1 General observations

First we provide a general relation between antidilation and its dual problem dilation. When the special case of this relation is used, i.e. when the host graph G_2 is a path, we obtain a relation between antibandwidth and bandwidth problem.

Theorem 4.1.1. *Let G_1 be a guest graph and G_2 be a host graph with \overline{G} standing for the complement of graph G . Then*

$$\text{adil}(G_1, G_2) \leq \text{dil}(\overline{G_1}, G_2) + 1.$$

Proof. If $\text{adil}(G_1, G_2) = 1$, the claim trivially holds. Let $\text{adil}(G_1, G_2) = k \geq 2$. Then $G_2^{k-1} \subseteq \overline{G_1}$. Let $\text{dil}(\overline{G_1}, G_2) = l \geq 1$. Then $\overline{G_1} \subseteq G_2^l$. Hence $G_2^{k-1} \subseteq G_2^l$, which implies $k - 1 \leq l$. \square

Corollary 4.1.1. *Let G be a graph with \overline{G} standing for its complement. Then*

$$\text{ab}(G) \leq \text{b}(\overline{G}) + 1.$$

In the following sections we start research the invariant value for particular interconnection networks.

4.2 Antibandwidth of meshes

The previously known result for meshes provided the upper and lower bounds for antibandwidth value. In this section we prove the exact value with use of previously known lower bound construction.

Theorem 4.2.1. For any $m \geq n \geq 2$

$$\text{ab}(P_m \times P_n) = \left\lceil \frac{(m-1)n}{2} \right\rceil.$$

Proof. The lower bound construction is showed in [41]. An example of the numbering of $P_8 \times P_7$ is in Table 4.1.

38	13	45	20	51	25	55	28
7	39	14	46	21	52	26	56
33	8	40	15	47	22	53	27
3	34	9	41	16	48	23	54
30	4	35	10	42	17	49	24
1	31	5	36	11	43	18	50
29	2	32	6	37	12	44	19

Table 4.1: Optimal numbering of the vertices of $P_8 \times P_7$ mesh.

Upper Bound. Define an interval $I = [L, R]$, where

$$L = \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)^2 + 1 \text{ and } R = \left\lceil \frac{mn}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor - 1.$$

A recent result of Bezrukov and Piotrowski [6] shows that p -vertex subsets of the mesh of minimal bipartite vertex boundary are obtained as the first p vertices in the simplicial order restricted to one partition only depicted in Table 4.2 for $m = 11$ and $n = 7$.

10		17		24		31		36		39
	11		18		25		32		37	
5		12		19		26		33		38
	6		13		20		27		34	
2		7		14		21		28		35
	3		8		15		22		29	
1		4		9		16		23		30

Table 4.2: Optimal numbering of the vertices of $P_{11} \times P_7$ mesh with respect to the simplicial ordering of vertices.

This especially implies the following. Let A be a subset of one partition set of the mesh. If $|A| \in I$ then

$$|\partial_b(A)| \geq |A| + \left\lfloor \frac{n}{2} \right\rfloor. \tag{4.1}$$

The interval I is shown in bold.

Denote $k = \text{ab}(P_m \times P_n)$. We know that $\lceil (m-1)n/2 \rceil \leq k \leq mn/2$. Consider a linear layout of the mesh. For a set S of consecutive k vertices of the layout denote $A_1 = S \cap V_1$, and $A_2 = S \cap V_2$.

1. Assume that there exist S such that the corresponding A_1 satisfies $|A_1| \in I$. Observe that

$$|\partial_b(A_1)| + k - |A_1| \leq |V_2|.$$

Thus by (4.1)

$$k \leq \left\lfloor \frac{mn}{2} \right\rfloor - (|\partial_b(A_1)| - |A_1|) \leq \left\lfloor \frac{mn}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{(m-1)n}{2} \right\rfloor.$$

2. Assume that for all sets S , $|A_1| \notin I$. We show that this leads to a contradiction. Distinguish 3 cases.

- (a) There exist S, S' such that $|A_1| < L$ and $|A'_1| > R$. Then one can easily compute that $|A_1| < |A_2|$ and $|A'_1| > |A'_2|$. This forces the existence of S'' such that $||A''_1| - |A''_2|| \leq 1$ which gives

$$\frac{k-1}{2} \leq |A''_1| \leq \frac{k+1}{2}.$$

Comparing this interval with I we conclude that $|A''_1| \in I$, a contradiction.

- (b) Assume that for all S , $|A_1| < L$. Then $|A_2| = k - |A_1| > k - L$. Take two disjoint S and S' with $|A_2| > k - L$ and $|A'_2| > k - L$. We get $|A_2| + |A'_2| > 2k - 2L \geq mn/2$, a contradiction.
- (c) Assume that for all S , $|A_1| > R$. This case is symmetric to the previous case.

□

4.3 Antibandwidth of tori

Consider a 2-dimensional toroidal mesh $C_n \times C_n$, defined by means of the cartesian product of two cycles.

Theorem 4.3.1. *For even n*

$$\text{ab}(C_n \times C_n) = \frac{n(n-2)}{2}.$$

Proof.

Lower bound. The optimal numbering is depicted in Table 4.3, for $n = 8$. The numbering in the first row from bottom is generally described by a vector

$$R_1 = (0, n + \frac{n^2}{2}, 2n, 3n + \frac{n^2}{2}, 4(n-1), \dots, (\frac{n}{2} - 1)n + \frac{n^2}{2}, (n-1)\frac{n}{2}, \\ (n-3)\frac{n}{2} + \frac{n^2}{2}, (n-5)\frac{n}{2}, (n-7)\frac{n}{2} + \frac{n^2}{2}, \dots, 3\frac{n}{2}, \frac{n}{2} + \frac{n^2}{2}).$$

Define

$$A_1 = (\frac{n^2}{2}, -\frac{n^2}{2}, \frac{n^2}{2}, -\frac{n^2}{2}, \dots, \frac{n^2}{2}, -\frac{n^2}{2}),$$

35	11	51	27	63	23	47	7
3	43	19	59	31	55	15	39
34	10	50	26	62	22	46	6
2	42	18	58	30	54	14	38
33	9	49	25	61	21	45	5
1	41	17	57	29	53	13	37
32	8	48	24	60	20	44	4
0	40	16	56	28	52	12	36

Table 4.3: Optimal numbering of the vertices of $C_8 \times C_8$ torus.

$$A_2 = \left(-\frac{n^2}{2} + 1, \frac{n^2}{2} + 1, -\frac{n^2}{2} + 1, \frac{n^2}{2} + 1, \dots, -\frac{n^2}{2} + 1, -\frac{n^2}{2} + 1\right).$$

The numberings in rows $R_i, i > 1$, are described recursively by

$$R_2 = R_1 + A_1, R_3 = R_2 + A_2, R_4 = R_3 + A_1, \dots$$

The numbering is bijective. In fact, in all rows, the even (odd) labels form increasing arithmetic progressions of length $n/2$ and distance 1. The beginning elements of all progression are multiples of $n/2$, so the progressions are distinct. The distances of neighbours in the first row, including the wrap-around edge, are at least $\frac{n(n-2)}{2}$. By definition, the same distances are preserved in all rows. The distances in columns, except for the wrap-around edges, are at least $\frac{n^2}{2} - 1$. The row R_n satisfies

$$R_n = R_1 + A,$$

where

$$A = \left(-1 + \frac{n}{2} + \frac{n^2}{2}, -1 + \frac{n}{2} - \frac{n^2}{2}, \dots, -1 + \frac{n}{2} + \frac{n^2}{2}, -1 + \frac{n}{2} - \frac{n^2}{2}\right).$$

The components of A taken $(\text{mod } n^2)$ are the distances of labels of rows R_1 and R_n .

Remark. Another optimal numbering is obtained from the numbering of the even torus in [34, 49]. See for illustration Table 4.4. First we number the vertices of the first partition of the torus by restricting the original order to the first partition using numbers $1, 2, \dots, n^2/2$ and then repeating this process for the second partition. One can formally prove that the minimal difference is $n^2/2 - n$.

Upper bound. The even torus is a bipartite graph with partite sets V_1 and V_2 , $|V_1| = |V_2| = n^2/2$. We use the same ideas as for the mesh. Denote by $I = [L, R]$, an interval with $L = n^2/4 - n + 2$ and $R = n^2/4 - 1$. It is known that an m -vertex set with the minimal vertex boundary in any even torus is given by the first m vertices in the numbering given in [34, 49]. Bezrukov and Piotrowski [6] proved that an m -vertex set in any even torus with the minimal bipartite vertex boundary is given by taking the first m vertices in the above numbering restricted to one partite set only. An example of such numbering is shown in Table 4.4, for $n = 10$. A simple analysis shows that if A is a subgraph of one of the partite sets and $n^2/4 - n + 2 \leq |A| \leq n^2/4 - 1$, then $|\partial_b(A)| - |A| \geq n$.

99	46	91	34	75	30	88	44	98	50
45	90	33	74	16	70	29	87	43	97
89	32	73	15	59	12	69	28	86	42
31	72	14	58	4	56	11	68	27	85
71	13	57	3	51	2	55	10	67	26
21	63	7	53	1	52	5	60	17	76
81	22	64	8	54	6	61	18	77	35
39	82	23	65	9	62	19	78	36	92
95	40	83	24	66	20	79	37	93	47
49	96	41	84	25	80	38	94	48	100

Table 4.4: Optimal numbering of the vertices of $C_{10} \times C_{10}$ torus.

Consider a linear layout. Let $k = \text{ad}(C_n \times C_n)$. We know that

$$n^2/2 - n \leq k \leq n^2/2 - 2.$$

Assume indirectly that $k \geq n^2/2 - n + 1$. For any set S of k consecutive vertices on the line let $A_i = V_i \cap S$ for $i = 1, 2$. Let S be a set of consecutive k vertices on the line.

1. Assume that there exist S such that the corresponding A_1 satisfies $|A_1| \in I$. Observe that

$$|\partial_b(A_1)| + k - |A_1| \leq |V_2|.$$

Hence similarly as for the mesh

$$k \leq \frac{n^2}{2} - (|\partial_b(A_1)| - |A_1|) \leq \frac{n^2}{2} - n,$$

a contradiction.

2. Assume that for all S , it holds $|A_1| \notin I$. Distinguish 3 cases.

- (a) There exist S, S' such that $|A_1| < L$ and $|A'_1| > R$. Then one can easily compute that $|A_1| \leq |A_2|$ and $|A'_1| \geq |A'_2|$. This forces the existence of S'' such that $||A''_1| - |A''_2|| \leq 1$ which gives

$$\frac{k-1}{2} \leq |A''_1| \leq \frac{k+1}{2}.$$

Comparing this interval with I we conclude that $|A''_1| \in I$, a contradiction.

- (b) Assume that for all S , $|A_1| < L$. Then $|A_2| = k - |A_1| \geq n^2/4$. Take two disjoint S and S' with $|A_2| \geq n^2/4$. and $|A'_2| \geq n^2/4$. As $|V_2| = n^2/2$, we have $|A_2| = |A'_2| = n^2/4$ and consequently if $v \notin S \cup S'$ then $v \in V_1$. Consider now 2 special choices of the sets S and S' .

- $S \cup S'$ is the set of the first (last) $2k$ vertices on the line.
- S (resp. S') is the set of the first (last) k vertices on the line.

This implies the following structure of vertices on the line: The vertices are divided into consecutive sets Z_1, Z_2, Z_3, Z_4, Z_5 from the left, where $Z_1, Z_3, Z_5 \subseteq V_1$, $|Z_1| = |Z_3| = |Z_5| = n^2 - 2k$, and $|Z_2| = |Z_4| = 3k - n^2 < n^2/2$. Consider a Hamiltonian cycle in the torus. Let $u, v \in V_2$ be two vertices in the distance 2 on the cycle. These two vertices have a common neighbour $w \in V_1$. If u, v are from different sets, say $u \in Z_2, v \in Z_4$. Then for any position of w we get a distance smaller than k , a contradiction. This forces that if say, $u \in Z_2$ and $v \in V_2$ then its neighbour on the cycle in the distance 2 must lie in Z_2 too. Continuing this process we conclude that $V_2 \subseteq Z_2$, a contradiction again.

(c) Assume that for all S , $|A_1| > R$. This case is symmetric to the previous case.

□

Theorem 4.3.2. For odd n

$$\text{ab}(C_n \times C_n) = \frac{(n-2)(n+1)}{2}.$$

Proof. Lower bound. Consider a labelling $f : V(C_n \times C_n) \rightarrow \{0, 1, 2, \dots, n^2-1\}$ defined by

$$f(i, j) = \frac{(n^2 + n + 2)i + (n^2 - n)j}{2} \pmod{n^2}.$$

First we show that f is a one-to-one mapping. Assume indirectly that for some $(i, j) \neq (i', j')$: $f(i, j) - f(i', j') = 0$. This implies

$$\frac{(n^2 + n + 2)(i - i') + (n^2 - n)(j - j')}{2} \equiv 0 \pmod{n^2} \equiv 0 \pmod{n}. \quad (4.2)$$

As $(n^2 + n + 2)/2 \equiv 1 \pmod{n}$ and $(n^2 - n)/2 \equiv 0 \pmod{n}$ we have $i - i' \equiv 0 \pmod{n}$, hence $i = i'$. Substituting this into (4.2) we get

$$\frac{n(n-1)}{2}(j - j') \equiv 0 \pmod{n^2}.$$

Then

$$\begin{aligned} \frac{n-1}{2}(j - j') &\equiv 0 \pmod{n}, \\ (n-1)(j - j') &\equiv 0 \pmod{2n} \equiv 0 \pmod{n}, \\ j' - j &\equiv 0 \pmod{n}, \end{aligned}$$

which implies $j' = j$, a contradiction.

Now we compute the absolute difference of two adjacent labels in the toroidal mesh. By the definition of f

$$\begin{aligned} |f((i+1) \bmod n, j) - f(i, j)| &= \\ &= \frac{n^2 + n + 2}{2}((i+1) \bmod n - i) \pmod{n^2}. \end{aligned} \quad (4.3)$$

If $(i+1) \bmod n - i = 1$ then the right hand side of (4.3) equals $(n^2 + n + 2)/2$. If $(i+1) \bmod n - i = -1$ then the right hand side of (4.3) equals

$$n^2 - (n^2 + n + 2)/2 = (n-2)(n+1)/2.$$

27	48	20	41	13	34	6
47	19	40	12	33	5	26
18	39	11	32	4	25	46
38	10	31	3	24	45	17
9	30	2	23	44	16	37
29	1	22	43	15	36	8
0	21	42	14	35	7	28

Table 4.5: Optimal numbering of the vertices of $C_7 \times C_7$ torus.

The second case, i.e., the labels are column-adjacent, is similar. Table 4.5 shows the optimal numbering of torus the $C_7 \times C_7$.

Upper bound. Consider any labelling of the vertices of the toroidal mesh by $\{0, 1, \dots, n^2 - 1\}$. We say that a label is small if its value is at most $(n^2 - 1)/2$, otherwise it is large. Consider n row cycles of the mesh. In any row cycle there is at least one pair xy of adjacent labels such that $x > y$ and either $y > (n^2 - 1)/2$ or $x \leq (n^2 - 1)/2$, i.e. any row contains a pair of adjacent labels which are both either small or large. Distinguish 2 cases.

Assume that there are at least $(n + 1)/2$ pairs of large row adjacent labels. Among them find a pair x_0y_0 such that y_0 is the largest number. Then

$$y_0 \geq \frac{n^2 - 1}{2} + \frac{n + 1}{2} \text{ and } x_0 - y_0 \leq n^2 - 1 - \frac{n^2 + n}{2} = \frac{(n - 2)(n + 1)}{2}.$$

This gives

$$\min_{uv \in E} |f(u) - f(v)| \leq \frac{(n - 2)(n + 1)}{2},$$

for any f , which implies the result.

Assume that there are at least $(n + 1)/2$ pairs of small row adjacent labels. Repeating the above argument we get a pair x_0y_0 such that

$$x_0 - y_0 \leq \frac{n^2 - n}{2},$$

with equality for $x_0 = (n^2 - 1)/2$ and $y_0 = (n - 1)/2$. If either $x_0 \neq (n^2 - 1)/2$ or $y_0 \neq (n - 1)/2$ then the claim follows. So assume that $x_0 = (n^2 - 1)/2$ and $y_0 = (n - 1)/2$. Now we repeat the previous arguments for column cycles and in the worst case we find a pair $x'_0y'_0$ with $x'_0 = (n^2 - 1)/2$ and $y'_0 = (n - 1)/2$, a contradiction. \square

4.4 Antibandwidth of hypercubes

In this section we prove new upper bound for the antibandwidth of hypercubes. As a result we obtain the value which is equal to the known lower bound up to the a constant factor.

The vertices of a hypercube Q_n can be naturally partitioned into sets $X_i, i = 0, 1, 2, \dots, n$ according to their distance from the vertex $0\dots 00$. Note that there are edges between X_i and $X_{i+1}, i = 0, 1, 2, \dots, n - 1$ only and $|X_i| = \binom{n}{i}$.

Before proving the upper bound we recall some facts about vertex isoperimetric properties of hypercubes. It is known [26] that the set of cardinality

m of the minimal vertex boundary in the hypercube Q_n is obtained by taking the first m vertices in the following order: X_0, X_1, \dots, X_n , while the order in the set X_i is lexicographic. Moreover m can be uniquely expressed in the form

$$m = \binom{n}{n} + \binom{n}{n-1} + \dots + \binom{n}{j+1} + \binom{x}{j},$$

for some real x . Then for any set A , $|A| = m$, following Frankl [22]

$$|\partial(A)| \geq \binom{n}{j} + \binom{x}{j-1}.$$

It is known [3, 35, 55] that the set of cardinality m of minimal vertex boundary of a set $A \subseteq V_1$ in the hypercube Q_n is obtained by taking the first m vertices in the following order: X_0, X_2, X_4, \dots , while the order in the set X_{2i} is lexicographic. In fact, it is nothing else than the simplicial order restricted to one partition only. Moreover m can be uniquely expressed in the form

$$m = \binom{n}{n} + \binom{n}{n-2} + \dots + \binom{n}{j+2} + \binom{x}{j},$$

for some real x and j of the same parity as n . One can easily prove an analogue of the Frankl's result for the bipartite vertex boundaries in hypercubes.

Lemma 4.4.1. *For any set $A \subseteq V_1$ of the hypercube Q_n , with $|A| = m$,*

$$|\partial_b(A)| - |A| \geq \binom{n-1}{j} + \binom{x}{j-1} - \binom{x}{j}. \quad (4.4)$$

Now we are prepared to prove the result.

Theorem 4.4.1. *For the n -dimensional hypercube Q_n*

$$\text{ab}(Q_n) = 2^{n-1} - \frac{2^n}{\sqrt{2\pi n}} (1 + o(1)).$$

Proof. Lower bound. For the lower bound construction see [41]. Upper bound. For the upper bound we prove that

$$\text{ab}(Q_n) \leq \begin{cases} 2^{n-1} - \binom{n-1}{\frac{n}{2}}, & \text{for } n \equiv 0 \pmod{4}, \\ 2^{n-1} - \frac{1}{2} \binom{n-1}{\frac{n-1}{2}}, & \text{for } n \equiv 1 \pmod{4}, \\ 2^{n-1} - \binom{n-1}{\frac{n-2}{2}}, & \text{for } n \equiv 2 \pmod{4}, \\ 2^{n-1} - \binom{n-1}{\frac{n-1}{2}}, & \text{for } n \equiv 3 \pmod{4}. \end{cases}$$

Let n be divisible by 4. Other cases differ in details only. We use a similar method as for meshes and even tori. Denote $k = \text{ad}(Q_n)$. Let $k \leq 2^{n-1} - \binom{n-1}{\frac{n}{2}}$. Then we are done and note that

$$\binom{n-1}{\frac{n}{2}} = \frac{2^n}{\sqrt{2\pi n}} (1 + o(1)).$$

Assume indirectly that $k \geq 2^{n-1} - \binom{n-1}{\frac{n}{2}} + 1$. Note that $k \leq 2^{n-1} - 1$. Let $I = [L, R]$, where

$$L = 2^{n-2} - \binom{n-1}{\frac{n}{2}} + 1 \text{ and } R = 2^{n-2} - 1.$$

Consider a linear layout. For a set S of consecutive k vertices of the linear layout denote $A_1 = S \cap V_1$, and $A_2 = S \cap V_2$.

1. Assume that there exist S such that the corresponding A_1 satisfies $|A_1| \in I$. Then

$$\begin{aligned} |A_1| &= 2^{n-2} - \binom{n-1}{\frac{n}{2}} + 1 \\ &= \binom{n}{n} + \binom{n}{n-2} + \dots + \binom{n}{\frac{n}{2}+2} + \frac{1}{2} \binom{n}{\frac{n}{2}} - \binom{n-1}{\frac{n}{2}} + 1 \\ &= \binom{n}{n} + \binom{n}{n-2} + \dots + \binom{n}{\frac{n}{2}+2} + 1, \end{aligned}$$

and consequently $|A_1|$ can be expressed as

$$|A_1| = \binom{n}{n} + \binom{n}{n-2} + \dots + \binom{n}{\frac{n}{2}+2} + \binom{x}{\frac{n}{2}},$$

for some real $n/2 \leq x \leq n-1$. The last inequality comes from the fact that $|A_1| \leq \frac{k}{2} < 2^{n-2}$. From Lemma 4.4.1 we have

$$|\partial_b(A_1)| - |A_1| \geq \binom{n-1}{\frac{n}{2}} + \binom{x}{\frac{n}{2}-1} - \binom{x}{\frac{n}{2}} \geq \binom{n-1}{\frac{n}{2}}.$$

Observe that $|\partial_b(A_1)| + k - |A_1| \leq |V_2|$, which implies

$$k \leq 2^{n-1} - (|\partial_b(A_1)| - |A_1|) \leq 2^{n-1} - \binom{n-1}{\frac{n}{2}},$$

a contradiction.

2. Assume that for all S , it holds $|A_1| \notin I$. Distinguish 3 cases.

- (a) There exist S, S' such that $|A_1| < L$ and $|A'_1| > R$. Then one can easily compute that $|A_1| \leq |A_2|$ and $|A'_1| \geq |A'_2|$. This forces the existence of S'' such that $||A''_1| - |A''_2|| \leq 1$ which gives

$$\frac{k-1}{2} \leq |A''_1| \leq \frac{k+1}{2}.$$

Comparing this interval with I we conclude that $|A''_1| \in I$, a contradiction.

- (b) Assume that for all S , $|A_1| < L$. Then $|A_2| = k - |A_1| \geq 2^{n-2} + 1$. Take two such disjoint S and S' we get $|A_2| + |A'_2| > 2^{n-1}$, a contradiction.

- (c) Assume that for all S , $|A_1| > R$. This case is symmetric to the previous one.

□

4.5 Antibandwidth of three-dimensional mesh

We provide the antibandwidth value for a three-dimensional mesh which is exact up to the constant factor. Previously, only lower and upper bounds for two-dimensional meshes were discussed in literature.

Denote $M_3 = P_n \times P_n \times P_n$, for $n \geq 3$. The vertices of P_n are $\{0, 1, 2, \dots, n-1\}$ and edges $\{(i, i+1) | i = 0, 1, 2, \dots, n-2\}$. Let $M_3 = (V_1, V_2, E)$, where V_1 and V_2 are the partition sets and $(0, 0, 0) \in V_1$.

The diameter of M_3 is $3(n-1)$. For $r = 0, 1, 2, \dots, 3(n-1)$, let $B(r)$ denote the set of vertices of M_3 in the distance r from $(0, 0, 0)$. It is easy to see that for $r = 0, 1, 2, \dots, 3(n-1)$

$$|B(r)| = |B(3(n-1) - r)|, \quad (4.5)$$

and

$$|B(r)| = \binom{r+2}{2} - 3 \binom{r-n+2}{2}, \quad (4.6)$$

for $r \leq 2(n-1)$. Further it holds

$$\sum_{r=0}^{3(n-1)} (-1)^r |B(r)| = n \pmod{2}. \quad (4.7)$$

4.5.1 Upper bound

In the following paragraphs we prove the upper bound for antibandwidth of three-dimensional mesh. The proof is simplified to one case because the proof of other cases runs very similarly.

Theorem 4.5.1. *For $n \geq 3$*

$$\text{ab}(M_3) \leq \frac{n^3}{2} - \frac{3n^2}{8} + O(1).$$

Proof. Assume $n \equiv 1 \pmod{4}$. Other cases are similar.

Denote $t = 3(n-1)/4$. In this case (4.5) and (4.7) imply

$$|B(0)| + |B(2)| + |B(4)| + \dots + |B(2t-2)| + \frac{1}{2}|B(2t)| = \frac{n^3}{4}. \quad (4.8)$$

Further, (4.6) gives

$$|B(2t)| = |B(\frac{3(n-1)}{2})| = \frac{3n^2+1}{4}.$$

Consider an optimal linear layout of M_3 . Denote $k = \text{ab}(M_3)$. If

$$k \leq \frac{n^3 - |B(2t)|}{2} + 2,$$

then we are done. Suppose indirectly that

$$k \geq \frac{n^3 - |B(2t)|}{2} + 3,$$

We know that $k < n^3/2$. Let S be a set of consecutive k vertices on the line. Denote $A_i = V_i \cap S$, for $i = 1, 2$. Denote by $J = [L, R]$, an interval with

$$L = \frac{n^3}{4} - \frac{1}{2}|B(2t)| + 1, \quad R = \frac{n^3}{4}.$$

Distinguish two cases.

Case 1. Assume that there exist S such that the corresponding A_1 satisfies $|A_1| \in J$. Observe that

$$|\partial_b(A_1)| + k - |A_1| \leq |V_2|.$$

Hence

$$k \leq \frac{n^3 - 1}{2} - (|\partial_b(A_1)| - |A_1|).$$

In what follows we will show that $|\partial_b(A_1)| - |A_1| \geq |B(2t)|/2 - 2$, which immediately gives a contradiction. The equation (4.8) implies that

$$|A_1| = |B(0)| + |B(2)| + |B(4)| + \dots + |B(2t - 2)| + \alpha|B(2t)|,$$

for some constant $0 < \alpha \leq 1/2$. Let I_1 be the set of the first $|A_1|$ vertices from V_1 in the simplicial order. Bezrukov and Piotrowski [6] proved that

$$|\partial_b(A_1)| \geq |\partial_b(I_1)|. \quad (4.9)$$

Let $A \in V_1 \cup V_2$ be a set of cardinality

$$|A| = |B(0)| + |B(1)| + |B(2)| + \dots + |B(2t - 1)| + \alpha|B(2t)|.$$

Let I be the set of the first $|A|$ vertices from $V_1 \cup V_2$ in the simplicial order. Bollobás and Leader [8] proved

$$|\partial_b(A)| \geq |\partial_b(I)|. \quad (4.10)$$

From the definitions of I_1 and I we have

$$|\partial_b(I_1)| = |B(1)| + |B(3)| + |B(5)| + \dots + |B(2t - 1)| + |\partial(I)| - (1 - \alpha)|B(2t)| \quad (4.11)$$

Moreover Bollobás and Leader [8, 9] showed that

$$|\partial(I)| \geq (1 - \alpha)|B(2t)| + \alpha|B(2t + 1)|. \quad (4.12)$$

Combining (4.9),(4.10),(4.11) and (4.12) we obtain

$$\begin{aligned} |\partial_b(A_1)| - |A_1| &\geq |\partial_b(I_1)| - (|B(0)| + |B(2)| + \dots + |B(2t - 2)| + \alpha|B(2t)|) \\ &\geq \sum_{i=0}^{2t-1} (-1)^{i+1} |B(i)| - \alpha(|B(2t)| - |B(2t + 1)|) \\ &= -\frac{1}{2} + \frac{1}{2}|B(2t)| - \alpha(|B(2t)| - |B(2t + 1)|) \\ &\geq \frac{1}{2}|B(2t)| - \frac{3}{2}. \end{aligned}$$

Case 2. Assume that for all sets S , $|A_1| \notin J$. We show that this leads to a contradiction. Distinguish 3 cases.

1. There exist S, S' such that $|A_1| < L$ and $|A'_1| > R$. Then one can easily compute that $|A_1| < |A_2|$ and $|A'_1| > |A'_2|$. This forces the existence of S'' such that $||A''_1| - |A''_2|| \leq 1$ which gives

$$\frac{k-1}{2} \leq |A''_1| \leq \frac{k+1}{2}.$$

Comparing this interval with J we conclude that $|A''_1| \in J$, a contradiction.

2. Assume that for all S , $|A_1| < L$. Then $|A_2| = k - |A_1| > k - L$. Take two disjoint S and S' with $|A_2| > k - L$ and $|A'_2| > k - L$. We get $|A_2| + |A'_2| > 2k - 2L \geq \frac{n^3}{2} + 4$ a contradiction.
3. Assume that for all S , $|A_1| > R$. This case is symmetric to the previous case.

□

4.5.2 Lower bound

In this section we describe the optimal (up to the third order term) labelling of three-dimensional mesh. First, we describe the algorithm for labelling the vertices and then we prove that the minimal difference of neighbouring labels in this labelling is matching the upper bound up to the third order term.

The labelling of vertices of M_3 proceeds in two phases. The first phase is the labelling of all $v \in B(r)$ for $r = 0, 2, 4, \dots$, r is even. The second phase continues with $r = 1, 3, 5, \dots$, r is odd. The labelling of one $B(r)$ proceeds as follows. Start at the top of the cut $B(r)$, i.e. in the vertex with maximal z and y -axis. We call this level of a cut a "base level". See the Figure 4.1.

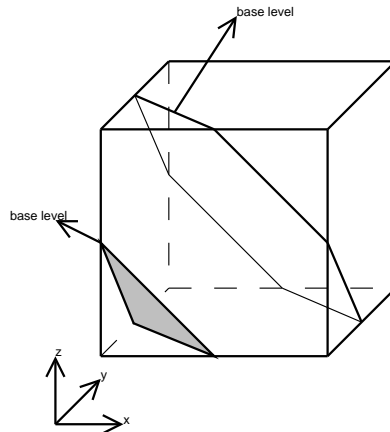


Figure 4.1: Cuts in a 3D mesh

This vertex obtains the first label. Then continue from the left to right and from the top to bottom with the rest of vertices of $B(r)$. See the example of the labelling in the Table 4.6.

In the next paragraphs we analyze the labelling algorithm. First, we start with some simple observations. Consider a position of a vertex v_1 in a cut denoted by a triplet r, h, w where r is a distance of the cut from the vertex $(0, 0, 0)$, h is a height of vertex v_1 counted from the top of the cut and w is the order in the h -th row of the cut.

Observation 4.5.1. *Let v_1 and v_2 be two neighbouring vertices in M_3 and let r be even. Then the neighbour of a vertex $v_1 \in B(r)$, with maximal label is a vertex v_2 at position $r - 1, h, w - 1$.*

Denote by L_i^r the i th level of a cut $B(r)$ counted from its base level, i.e. the level consisting of vertices with the greatest z -axis. The base level of $B(r)$ is denoted by L_0^r .

42	17	56	29
5	43	18	57
34	6	44	19
1	35	7	45

1st level

13	53	27	63
39	14	54	28
3	40	15	55
33	4	41	16

2nd level

49	24	61	32
10	50	25	62
37	11	51	26
2	38	12	52

3rd level

20	58	30	64
46	21	59	31
8	47	22	60
36	9	48	23

4th level

Table 4.6: Labelling of particular levels of $P_4 \times P_4 \times P_4$ mesh

Observation 4.5.2. *Let $R = 3n/2$ then*

1. $|L_i^R| - |L_i^{R-1}| = 1$ for all $i \leq n/2 - 2$
2. $|L_i^R| - |L_i^{R-1}| = -1$ for all $i : n/2 - 2 < i < n$

Theorem 4.5.2. $\text{ab}(M_3) \geq \left\lceil \frac{(4n-3)(n^2-1)}{8} \right\rceil$

Proof. We show an exact proof for the case when n is divisible by four. The other cases are similar.

Let r be even and let v_1 and v_2 be two neighbouring vertices, i.e. $v_1 \in B(r)$ and $v_2 \in B(r-1)$. Moreover, let v_2 be the smallest neighbour of v_1 .

If we denote the label assigned to a vertex v by $f(v)$ we get

$$f(v_1) = |B(0)| + |B(2)| + \dots + |B(r-2)| + C(v_1) \quad (4.13)$$

$$f(v_2) = \frac{n^3}{2} + |B(1)| + |B(3)| + \dots + |B(r-3)| + C(v_2) \quad (4.14)$$

where $C(v_1)$ and $C(v_2)$ stand for orders of v_1 and v_2 in cuts $B(r)$ and $B(r-1)$ respectively.

$$\begin{aligned} f(v_2) - f(v_1) &= \frac{n^3}{2} + |B(1)| - |B(2)| + \\ &\quad |B(3)| - |B(4)| + \dots + |B(r-3)| - |B(r-2)| \\ &\quad - |B(0)| + C(v_2) - C(v_1) \end{aligned} \quad (4.15)$$

Using the formula for $|B(k)|$ and after some algebraic manipulations we get

$$|B(k)| - |B(k-1)| = 2k - 3n + 4 \quad (4.16)$$

for $k \geq n-1$ and

$$|B(k)| - |B(k-1)| = -k - 2 \quad (4.17)$$

for $k < n-1$.

Substituting (4.16) and (4.17) into (4.15), using $|B(0)| = 1$ we get

$$f(v_2) - f(v_1) = \frac{2n^3 - 3n^2 - 6rn + 2r^2 + 4}{4} + C(v_2) - C(v_1) \quad (4.18)$$

The Equation (4.18) consists of two terms: the fraction and the difference $C(v_2) - C(v_1)$. Our aim is to minimize the value of (4.18). Considering the fraction from (4.18) as a function of r we get that its value is minimal for $r = 3n/2$.

After substitution we have

$$f(v_2) - f(v_1) \geq \frac{4n^3 - 3n^2}{8} + C(v_2) - C(v_1) \quad (4.19)$$

Now, we discuss the difference $C(v_2) - C(v_1)$. An easy observation shows that the minimal difference between two cuts is realised in the middle of the mesh. Moreover, this observation is confirmed by the way the inequality (4.19) is obtained. Therefore, we concentrate our attention to the cuts in the middle of the mesh, i.e. around the value $r = 3n/2$. For larger meshes these cuts are of hexagonal shape. See the Figure 4.2. The difference between two

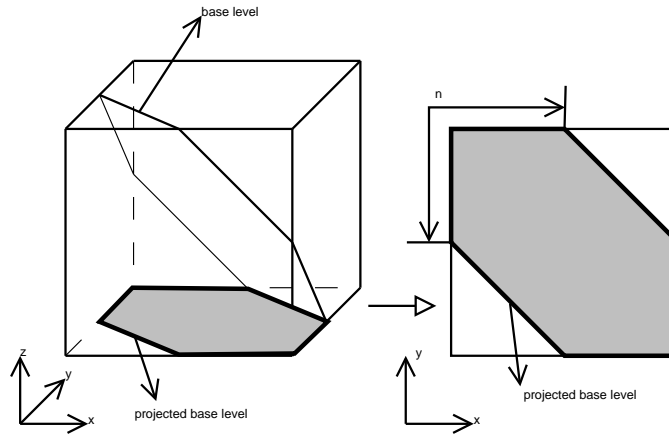


Figure 4.2: Projection of a cut in a mesh

corresponding levels in these cuts is at most 1. One cut has at most n levels. See the Figure 4.2 for illustration. So far, we can say that

$$C(v_2) - C(v_1) = O(n) \quad (4.20)$$

Let $R = 3n/2$. We need to compute the difference (4.20) for two neighbouring cuts $B(R)$ and $B(R - 1)$. Consider the projection of both cuts into the basic plane of the mesh. Note that number of vertices of a projected cut is the same as the original one.

Both cuts $B(R)$ and $B(R - 1)$ are labelled starting with their base levels. There is exactly $n/2 - 2$ consecutive levels in $B(R)$ such that each one of them is one vertex longer than appropriate level in $B(R - 1)$. Since we are looking for the greatest value of $C(v_1)$ this will be obtained according to Observation 4.5.2 in level $n/2 - 2$. Then, with use of Observation 4.5.1 we have $C(v_2) - C(v_1) = -n/2 + 2 - 1$. And finally

$$f(v_2) - f(v_1) \geq \frac{4n^3 - 3n^2}{8} - n/2 + 1 = \left\lceil \frac{(4n - 3)(n^2 - 1)}{8} \right\rceil$$

□

Conjecture 4.5.1. *We conjecture that the lower bound given by Theorem 4.5.2 is optimal.*

4.6 Antibandwidth of complete k-ary trees

The value of antibandwidth of complete binary trees is known (see Theorem 3.1.12). We extend known results by proving the value of antibandwidth of complete k-ary trees. Proof is divided into two parts according to parity of k .

4.6.1 Even k Case

In this section we prove the exact value of the antibandwidth of a complete k -ary tree, when k is even.

Theorem 4.6.1. *For even $k \geq 4$,*

$$ab(T(k, n)) = \frac{n + 1 - k}{2}.$$

Proof. Lower bound. We prove the lower bound by providing a labelling. The idea is to assign the middle label to the root, to assign the smallest and largest labels to nodes at the first level, and to proceed by assigning labels to the children of a node labelled l by using labels as far as possible from l . So, the root, at level 1, is labelled by $(n + 1)/2$. The vertices at level 2 have labels consecutively from the left $1, 2, 3, \dots, \frac{k}{2}, n - \frac{k}{2} + 1, n - \frac{k}{2} + 2, \dots, n$. The vertices at level 3 have labels consecutively from the left $n - \frac{k}{2} - \frac{k^2}{2} + 1, n - \frac{k}{2} - \frac{k^2}{2} + 2, \dots, n - \frac{k}{2}, \frac{k}{2} + 1, \frac{k}{2} + 2, \dots, \frac{k}{2} + \frac{k^2}{2}$

and so on. As an example, see Figure 4.3. One can check that the minimum difference of labels is as claimed.

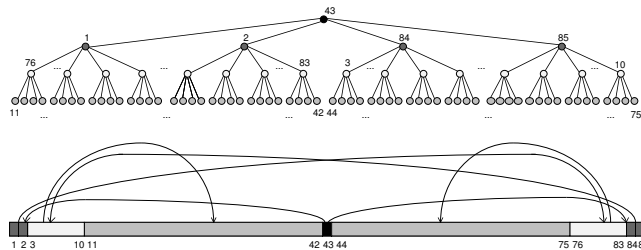


Figure 4.3: An example of labelling of a complete 4-ary tree.

Upper Bound. We proceed by contradiction, so let us assume that

$$ab(T(k, n)) \geq \frac{n + 1 - k}{2} + 1.$$

Let $f : V_T \rightarrow \{1, 2, \dots, n\}$ be a bijective labelling of the vertices of $T(k, n)$. Then, two cases can arise:

1. There exists a vertex v with neighbours u and w , such that

$$f(u) < f(v) < f(w).$$

Hence $d(v) \geq k$. Then, we can define two integer values l and $r = d(v) - l$, both ≥ 1 such that u_1, u_2, \dots, u_l and w_1, w_2, \dots, w_r are neighbours of v and

$$f(u_1) < f(u_2) < \dots < f(u_l) < f(v) < f(w_1) < f(w_2) < \dots < f(w_r)$$

(for a visualization, see Fig. 4.3).

It follows that $f(w_1) - f(u_l) \leq n - 1 - (l - 1) - (r - 1) \leq n + 1 - k$ since $l + r \geq k$.

Hence $\min\{f(v) - f(u_l), f(w_1) - f(v)\} \leq \frac{n+1-k}{2}$, a contradiction.

2. For any v with neighbours $u_1, u_2, \dots, u_{d(v)}$ either $f(u_i) < f(v)$, for all $i = 1, 2, \dots, d(v)$ or $f(u_i) > f(v)$, for all $i = 1, 2, \dots, d(v)$. Let I be the interval $[(n + 1 - k)/2, (n + 1 + k)/2]$ and let us focus on the vertices with degree strictly greater than 1.

- (a) Assume there exists v , with $d(v) > 1$, s.t. $f(v) \in I$. W.l.o.g. assume that $f(v) \leq (n + 1)/2$.

If for all neighbours $u_1, u_2, \dots, u_{d(v)}$ of v it holds

$$f(u_1) < f(u_2) < \dots < f(u_j) < f(v),$$

then

$$f(v) - f(u_j) \leq \frac{n+1}{2} - 1 - (j-1) \leq \frac{n+1-k}{2},$$

a contradiction.

If, on the contrary, for all neighbours $u_1, u_2, \dots, u_{d(v)}$ of v , it holds $f(v) < f(u_1) < f(u_2) < \dots < f(u_j)$ then

$$f(u_1) - f(v) \leq n - \frac{n+1}{2} + \frac{k}{2} - (j-1) \leq \frac{n+1-k}{2},$$

again a contradiction.

- (b) Assume that for all v with $d(v) > 1$, it holds $f(v) \notin I$. Consider the root r . As $f(r) \notin I$, w.l.o.g. assume that $f(r) \leq \frac{n+1-k}{2} - 1$. Then for all vertices w on level 2 we have $f(w) \geq \frac{n+1-k}{2} + 1$. Similarly, for vertices w on level 3 we have $f(w) \leq \frac{n+1-k}{2} - 1$, etc., until we reach the vertices on level h . W.l.o.g. assume that for all vertices w on level h we have $f(w) \geq \frac{n+1-k}{2} + 1$.

As $k^h \geq \frac{n-1+k}{2}$, at least one leaf w satisfies:

$$f(w) \geq \frac{n+1-k}{2} + 1.$$

Clearly, for the parent p of w : $f(w) < f(p)$. Hence

$$f(p) - f(w) \leq n - \frac{n-1+k}{2} = \frac{n+1-k}{2},$$

a contradiction.

□

4.6.2 Odd k Case

In this section we provide upper and lower bounds for the antibandwidth that differ in a lower order term, in the case k odd. Unfortunately, in this case, the symmetric construction exploited in the even case cannot be applied, so we will use a completely different technique.

Theorem 4.6.2. For odd $k \geq 3$ and $h \geq 3$,

$$\text{ab}(T(k, n)) \leq \frac{n}{2} - \max\left\{\frac{k}{2}, \frac{h}{8} - o(h)\right\}.$$

Proof. The upper bound of the form $(n - k)/2$ can be obtained in a similar way as for the k even case. For the second upper bound assume that h is odd. The even h case can be proven similarly. Let S be the smallest set of vertices whose removal divides the vertices of the resulting forest into independent sets X and Y , s.t. $|X|, |Y| \leq n/2$. We claim that

$$\text{ab}(T(k, n)) \leq \frac{n - |S|}{2}.$$

To prove this, consider an optimal layout. Removing the last $n - 2\text{ab}(T(k, n))$ vertices we get 2 independent sets: the first one is the set on positions

$$1, 2, 3, \dots, \text{ab}(T(k, n)),$$

and the second one is the set on the positions

$$\text{ab}(T(k, n)) + 1, \dots, 2\text{ab}(T(k, n)).$$

Note that there are possible edges between the two sets only, otherwise we get an edge of length smaller than $\text{ab}(T(k, n))$.

As $\text{ab}(T(k, n)) \leq n/2$ we have

$$|S| \leq n - 2\text{ab}(T(k, n)),$$

which proves the claim.

In what follows we prove that $|S| \geq h/4 - o(h)$. We need some new notations. Let L_i , for $i = 1, 2, 3, \dots, h + 1$ denote the set of vertices of the i -th level of the tree, while L_1 contains the root. Set $x_i = |L_i \cap X|, y_i = |L_i \cap Y|, s_i = |L_i \cap S|$. Observe that, for $i \geq 2$, as X, Y and S are defined, and in view of the structure of a complete k -ary tree, we have that

$$k(x_{i-1} + y_{i-1} + s_{i-1}) = x_i + y_i + s_i. \quad (4.21)$$

Furthermore, the properties of X, Y and S imply that the children of vertices in $L_{i-1} \cap X$ must be in $L_i \cap (S \cup Y)$, hence $y_i + s_i \geq kx_{i-1}$. By (4.21), this is equivalent to $ky_{i-1} + ks_{i-1} \geq x_i$. Repeating this argument for $L_{i-1} \cap Y$ we derive the following:

$$x_i - ks_{i-1} \leq ky_{i-1} \leq x_i + s_i, \quad (4.22)$$

$$y_i - ks_{i-1} \leq kx_{i-1} \leq y_i + s_i. \quad (4.23)$$

Now we show that S is a vertex $(n/2 + 7|S|/2)$ -bisector. It is easy to see that the sets

$$V_1 = \cup_{\text{even } i} (L_i \cap X) \cup \cup_{\text{odd } i} (L_i \cap Y),$$

$$V_2 = \cup_{\text{odd } i} (L_i \cap X) \cup \cup_{\text{even } i} (L_i \cap Y)$$

are distinct and any path between them contains a vertex from S . Hence S is a vertex r -bisector. Let us estimate r .

$$\begin{aligned}
|V_1| &= \sum_{\text{even } i} x_i + \sum_{\text{odd } i} y_i \\
&\leq \sum_{\text{even } i} x_i + \frac{1}{k} \sum_{\text{even } i} (x_i + s_i) \\
&\leq \frac{k+1}{k} \sum_{\text{even } i} x_i + \frac{1}{k} |S|. \tag{4.24}
\end{aligned}$$

To estimate the last sum we need estimations for every x_i , for even i . From the left hand side of inequality (4.23) we have

$$\begin{aligned}
\sum_{i=2}^{h+1} (y_i - ks_{i-1}) &\leq k \sum_{i=2}^{h+1} x_{i-1} \\
|Y| - y_1 - k|S| &\leq k(|X| - x_{h+1}) \\
n - |X| - |S| - y_1 - k|S| &\leq k(|X| - x_{h+1}) \\
kx_{h+1} &\leq (k+1)|X| - n + (k+1)|S| + 1 \leq \frac{(k+1)n}{2} - n + (k+2)|S| \\
x_{h+1} &\leq \frac{k-1}{2k}n + \frac{k+2}{k}|S|. \tag{4.25}
\end{aligned}$$

Combining right hand sides of inequalities (4.22) and (4.23) we have:

$$x_{i-2} \leq \frac{1}{k}(y_{i-1} + s_{i-1}) \leq \frac{1}{k}\left(\frac{1}{k}(x_i + s_i) + s_{i-1}\right) = \frac{1}{k^2}(x_i + s_i + ks_{i-1}).$$

Iterating this inequality backwards, starting with $i = h+1$ we get for even $i \geq 2$

$$x_i \leq \frac{1}{k^{h-i+1}}(x_{h+1} + \sum_{j=i+1}^{h+1} k^{h+1-j} s_j).$$

Using this estimation we compute

$$\begin{aligned}
\sum_{\text{even } i \geq 2}^{h+1} x_i &\leq \sum_{\text{even } i \geq 2}^{h+1} \frac{x_{h+1}}{k^{h-i+1}} + \sum_{\text{even } i \geq 2}^{h+1} \sum_{j=i+1}^{h+1} \frac{s_j}{k^{i-j}} \\
&< x_{h+1} \sum_{\text{even } t \geq 0}^{h-1} \frac{1}{k^t} + \sum_{j=3}^{h+1} \left(\frac{1}{k} + \frac{1}{k^3} + \dots + \frac{1}{k^{h-2}} \right) s_j \\
&< x_{h+1} \sum_{\text{even } t \geq 0}^{\infty} \frac{1}{k^t} + \sum_{j=3}^{h+1} \frac{k}{k^2-1} s_j \\
&< \frac{k^2}{k^2-1} x_{h+1} + \frac{k}{k^2-1} |S|.
\end{aligned}$$

Substituting (4.26) into (4.24) and using (4.25) we obtain

$$\begin{aligned}
|V_1| &\leq \frac{k}{k-1} x_{h+1} + \frac{2}{k-1} |S| \\
&\leq \frac{k}{k-1} \left(\frac{k-1}{2k} n + \frac{k+2}{k} |S| \right) + \frac{2}{k-1} |S| \\
&\leq \frac{n}{2} + \frac{k+4}{k-1} |S| \\
&\leq \frac{n}{2} + \frac{7}{2} |S|.
\end{aligned}$$

Repeating the same calculations for $|V_2|$ we get the same bound, hence concluding that S is a vertex $(n/2 + 7|S|/2)$ -bisector. Assume $|V_1| \leq |V_2|$. Let $|V_2| = n/2 + p$. By deleting a suitable set of at most $\log_k p + 1$ vertices we can separate p vertices from V_2 and add them to V_1 . To see this, observe that p can be expressed in the form

$$p = \sum_{i=1}^z \alpha_i \frac{k^i - 1}{k - 1},$$

where $0 \leq \alpha_i \leq k$ are integers, and z is the smallest number s.t. $(k^{z+1} - 1)/(k - 1) > p$, i.e., $z \leq \log_k p + 1$. And note that by removing a suitable vertex from V_2 we get k complete subtrees of size $(k^j - 1)/(k - 1)$, where $j \leq z$.

Thus we get a vertex $n/2$ -bisector. Its size is

$$|S| + \log_k p + 1 \leq |S| + \log_k \frac{7}{2}|S| + 1.$$

Further, removing all edges incident to the vertices of the vertex $n/2$ -bisector and distributing the isolated vertices among the current sets V_1 and V_2 in such a way that neither of them contains more than $n/2$ vertices we get an edge $n/2$ -bisector of the size at most

$$(k + 1)(|S| + \log_k \frac{7}{2}|S| + 1).$$

It is known [44] that the size of the smallest edge $\lceil n/2 \rceil$ -bisector of the complete k -ary n -vertex tree of height h is at least

$$\frac{k - 1}{2}(h - \log_k h - 1).$$

Thus we have

$$(k + 1)(|S| + \log_k \frac{7}{2}|S| + 1) \geq \frac{k - 1}{2}(h - \log_k h - 1).$$

Hence

$$|S| \geq \frac{k - 1}{2(k + 1)}(h - \log_k h - 1) - \log_k \frac{7}{2}|S| - 1.$$

As $|S| \leq h$, this yields

$$|S| \geq \frac{k - 1}{2(k + 1)}h - o(h) \geq \frac{h}{4} - o(h).$$

□

In the following paragraphs, for the sake of completeness, we shortly repeat the algorithm by Miller and Pritikin [41]. This algorithm provides reasonably good layout for forests and we use its slight modification in the lower bound construction in the next theorem.

For a bipartite graph B with a specified bipartition M, N with $|M| \leq |N|$, we refer to the *minority* $MIN(B) = |M|$ and *majority* $MAJ(B) = |N|$ of B and refer to M and N as being the minority and majority sides, respectively.

Given any bipartition X, Y of a forest with $|X| \leq |Y|$, there always exists a vertex $y \in Y$ of degree 0 or 1 since the average degree of the majority side vertices is at most $(|X| + |Y| - 1)/|Y|$, which is less than two.

Let a forest F_1 have minority side M_1 and majority side N_1 . For each $i \in [1, MAJ(F_1)]$, recursively define y_i, x_i, M_i, N_i as follows. Let $y_i \in N_i$ be a vertex of degree 0 or 1 in F_i . If y_i has degree 1 in F_i choose x_i as its sole neighbour. If M_i is empty, choose $x_i = y_i$. In any other case, choose x_i to be any element of M_i . Let $F_{i+1} = F_i - x_i - y_i$, $M_{i+1} = M_i - x_i$, $N_{i+1} = N_i - y_i$. The resulting layout is obtained by the following labelling. Assign $f(x_i) = i$ for each $i \in [1, MIN(F_1)]$ and $f(y_i) = MIN(F_1) + i$ for each $i \in [1, MAJ(F_1)]$. This leads to a construction with

$$ab(F) \geq MIN(F).$$

Theorem 4.6.3. For odd $k \geq 3$ and $h \geq 3$

$$ab(T(k, n)) \geq \frac{n}{2} - O(k^2h).$$

Proof. We proceed with the following construction.

1. Number the levels of the tree by $1, 2, \dots, h + 1$. For every level $i : i \geq 2$ number the vertices from left to right by integers $1, 2, \dots, k^{i-1}$. Then delete the vertex with label $\lfloor \frac{k^{i-1}}{2} \rfloor$ together with its adjacent edges. Define the set D consisting of deleted vertices. The remaining parts of the tree define the forest F .
2. Divide the vertices of F into two partitions X and Y s.t.

$$|Y| - 1 \leq |X| \leq |Y|.$$

3. For every $v : v \in Y$ such that v was adjacent to some $d \in D$ define the priority to 2. For all neighbours of every such v define the priority value to 1. The rest of vertices of F obtain priority value 3. The higher priority the lower its value.
4. Use the modified Miller/Pritikin algorithm to get the layout of F with $ab(F) \geq \lfloor \frac{n-h}{2} \rfloor$. The modification of used algorithm simply follows the priorities of vertices defined in the previous step. If it is not possible to label a vertex w with priority 1 directly, i.e. the vertex w do not have neighbour from Y of degree 1 or there is no vertex from Y with degree 0, label one of the leaves from Y of degree 1 and its parent from X and remove them from the forest. This operation creates $k - 1$ isolated vertices from Y which can be used for labelling the vertices with priority 1.
5. Place the vertices from set D in the middle of the layout, between the sets X and Y .

The algorithm places the vertices from the sets X, Y, D in the order X, D, Y . For the final lower bound the distance from the neighbours of D to D is important. Let P_i be the set of vertices of priority i . Since every deleted vertex except the last one has $(k + 1)$ neighbours, approximately half of them belongs to the set Y , i.e. $|P_2| = (k + 1)h/2$. Every vertex from P_2 has k

neighbours from X , i.e. $|P_1| = (k + 1)kh/2$. To label the vertices of P_1 we need $|P_1|$ vertices from Y of degree 0. These can be easily produced from leaves (see step 4 of the algorithm). With a simple analysis we get that the labelling of P_1 needs

$$\frac{hk(k + 1)}{2} \cdot \frac{(k + 1)}{k} = \frac{h(k + 1)^2}{2}$$

leaves. Labelling of P_1 vertices will make all of P_2 vertices from Y isolated and therefore they can be used to label the second half of P_2 vertices from X . In resulting layout there will be $h(k + 1)^2/2$ P_1 vertices, then $h(k + 1)/2$ P_2 vertices. Since P_2 are the neighbours of D , then

$$ab(T(k, n)) \geq n/2 - h(k + 1)^2/2 - h(k + 1)/2 = n/2 - O(hk^2).$$

□

Combining our methods we are able to prove that:

Theorem 4.6.4. For odd $k \geq 3$ and $h = 2$

$$ab(T(k, n)) = \frac{k^2 + 1}{2}.$$

Proof. Lower bound. The lower bound construction is an analogue of the construction of even case from the proof of Theorem 4.6.1. The root obtains label $\lceil \frac{n}{2} \rceil$. The second level is labelled such that the left $\lceil \frac{k}{2} \rceil$ children of the root obtains labels $1, 2, \dots, \lceil \frac{k}{2} \rceil$. The rest of the vertices is labelled in similar way as in construction from the proof of Theorem 4.6.1. For an example see the Figure 4.4.

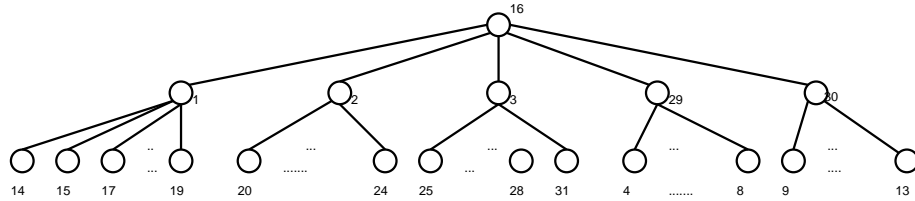


Figure 4.4: An example of labelling of a complete 5-ary tree of height 2.

Upper bound. We proceed by a contradiction. Assume that

$$ab(T(k, n)) = \frac{k^2 + 1}{2} + 1 = \frac{k^2 + 3}{2}.$$

Then we continue as in proof of lower bound in Theorem 4.6.1.

□

Chapter 5

Cyclic antibandwidth problem

In this chapter we show our results for the cyclic modification of the antibandwidth problem. The exact results are provided for paths, cycles, two-dimensional meshes, tori and hypercubes [48].

5.1 Basic observations

Lemma 5.1.1. *Let $G = (V, E)$, $|V| = n$, Then*

$$\text{cab}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Equality holds, e.g., for G consisting of a matching.

Lemma 5.1.2. *For n -vertex paths and cycles,*

$$\text{cab}(P_n) = \text{cab}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Let \overline{G} denote the complement of G . Let G^i , for $i \geq 1$, denote the graph obtained from G by joining all vertices in a distance of at most i .

Lemma 5.1.3. *For n -vertex graph G and $k \geq 2$*

$$\text{ab}(G) \geq k \quad \text{iff } P_n^{k-1} \subseteq \overline{G},$$

$$\text{ab}(G) = 1 \quad \text{iff } \overline{G} \text{ does not contain a Hamiltonian path,}$$

$$\text{cab}(G) \geq k \quad \text{iff } C_n^{k-1} \subseteq \overline{G},$$

$$\text{cab}(G) = 1 \quad \text{iff } \overline{G} \text{ does not contain a Hamiltonian cycle.}$$

Lemma 5.1.4.

$$\frac{1}{2}\text{ab}(G) \leq \text{cab}(G) \leq \text{ab}(G),$$

and both bounds are attainable.

Proof. The right inequality is evident and the equality can be obtained by a path. Let $\text{ab}(G) = k \geq 2$. This implies that \overline{G} contains P_n^{k-1} and $P_n^{2\lfloor \frac{k-1}{2} \rfloor}$. As $C_n \subseteq P_n^2$, we have $C_n^{\lfloor \frac{k-1}{2} \rfloor} \subseteq P_n^{2\lfloor \frac{k-1}{2} \rfloor}$. By Lemma 5.1.3 it holds $C_n^{\lfloor \frac{k-1}{2} \rfloor} \subseteq \overline{G}$ and hence

$$\text{cab}(G) \geq \left\lfloor \frac{k-1}{2} \right\rfloor + 1 = \left\lfloor \frac{\text{ab}(G) + 1}{2} \right\rfloor.$$

The left inequality is sharp this can be proved in the following way. Let G be the complement of P_n^i , for $i \geq 2$. We claim that

$$\text{ab}(G) = i + 1 \quad \text{and} \quad \text{cab}(G) = \lfloor \frac{i}{2} \rfloor + 1.$$

The first equality is trivial. For the second one, clearly, P_n^i contains $C_n^{\lfloor \frac{i}{2} \rfloor}$ which implies $\text{cab}(G) \geq \lfloor \frac{i}{2} \rfloor + 1$. The last inequality is equality for the following reason. The degree of the first vertex in P_n^i is $2i$. Hence the maximum C_n^j contained in P_n^i satisfies $2j \leq i$, i.e., $j = \lfloor \frac{i}{2} \rfloor$. \square

The next Lemma is obvious but useful. All results for meshes, tori and hypercubes are obtained by the Lemma.

Lemma 5.1.5. *Let $G = (E, V)$, $|V| = n$. Let f be a labelling of G . then*

$$\text{cab}(G) \geq \min\{\text{ab}(G, f), n - \max_{uv \in E} |f(u) - f(v)|\}. \quad (5.1)$$

5.2 Cyclic antibandwidth of meshes

Theorem 5.2.1. *Let m be even and n be odd and $m \geq n$. Then*

$$\left\lfloor \frac{n(m-1)}{2} \right\rfloor \leq \text{cab}(P_m \times P_n) \leq \left\lceil \frac{n(m-1)}{2} \right\rceil.$$

Otherwise

$$\text{cab}(P_m \times P_n) = \frac{n(m-1)}{2}.$$

Proof. Upper bound follows from $\text{cab}(P_m \times P_n) \leq \text{ab}(P_m \times P_n) = \left\lceil \frac{(m-1)n}{2} \right\rceil$.

Lower bounds is based on Lemma 5.1.5. Distinguish two cases according to parity of n .

Let n be even. For this case we define a new optimal labelling f with respect to antibandwidth (see Table 5.1).

3	27	9	33	15	39	21
24	6	30	12	36	18	42
2	26	8	32	14	38	20
23	5	29	11	35	17	41
1	25	7	31	13	37	19
22	4	28	10	34	16	40

Table 5.1: Optimal labelling of 7×6 mesh

We describe this labelling formally and then we show that this labelling is optimal with respect to antibandwidth. The first column of the labelling in the Table 5.1 is described by the vector

$$K_1 = \left(\frac{n}{2}, \frac{mn}{2} + \frac{n}{2}, \frac{n}{2} - 1, \dots, \frac{mn}{2} + 2, 1, \frac{mn}{2} + 1 \right). \quad (5.2)$$

Define

$$A_1 = \left(\frac{mn}{2} + \frac{n}{2}, -\frac{mn}{2} + \frac{n}{2}, \frac{mn}{2} + \frac{n}{2}, -\frac{mn}{2} + \frac{n}{2}, \dots, -\frac{mn}{2} + \frac{n}{2} \right),$$

$$A_2 = (n, n, n, \dots, n).$$

The numbering in columns $K_i, i > 1$ is defined as follows.

$$K_2 = K_1 + A_1, K_3 = K_1 + A_2, K_4 = K_2 + A_1,$$

$$K_5 = K_2 + A_2, \dots, K_n = K_{n-3} + A_{n-3}.$$

The smallest difference of labels of adjacent vertices is given by the smallest absolute value in the vector A_1 which is $|\lfloor -\frac{mn}{2} + \frac{n}{2} \rfloor| = \frac{n(m-1)}{2}$. Hence the labelling is optimal with respect to antibandwidth. The maximum difference of labels of adjacent vertices is $\lfloor mn/2 + n/2 \rfloor = \lfloor (m+1)n/2 \rfloor$. Substituting this into (5.1) we get the lower bound.

Let n be odd.

We use the same way of labelling as in previous case (see Table 5.2).

20	5	25	10	30	15	35
2	22	7	27	12	32	17
19	4	24	9	29	14	34
1	21	6	26	11	31	16
18	3	23	8	28	13	33

Table 5.2: Optimal labelling of 7×5 mesh

Formally the first column is described by the vector:

$$K_1 = \left(\left\lfloor \frac{mn}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{mn}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil - 1, \left\lfloor \frac{n}{2} \right\rfloor - 1, \dots, \left\lfloor \frac{mn}{2} \right\rfloor + 1, 1, \left\lfloor \frac{mn}{2} \right\rfloor \right).$$

Define

$$A_1 = \left(-\left\lfloor \frac{mn}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{mn}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil, -\left\lfloor \frac{mn}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor, \dots, -\left\lfloor \frac{mn}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \right),$$

$$A_2 = (n, n, n, \dots, n).$$

The following terms define the labelling for columns $K_i, i > 1$:

$$K_2 = K_1 + A_1, K_3 = K_1 + A_2, K_4 = K_2 + A_1,$$

$$K_5 = K_2 + A_2, \dots, K_n = K_{n-3} + A_{n-3}$$

One can check that the smallest difference of adjacent labels is $\lceil (m-1)n/2 \rceil$, i.e. the labelling is antibandwidth optimal. The maximum difference of adjacent labels is $\lceil (m+1)n/2 \rceil$. Substituting this into (5.1) we get the result. \square

5.3 Cyclic antibandwidth of tori

Theorem 5.3.1. *For even n*

$$\text{cab}(C_n \times C_n) = \text{ab}(C_n \times C_n) = \frac{n(n-2)}{2}.$$

Proof. Upper bound follows from $\text{cab}(C_n \times C_n) \leq \text{ab}(C_n \times C_n) = \frac{n(n-2)}{2}$. We apply Lemma 5.1.5 again. The antibandwidth optimal labelling f is shown in Theorem 4.3.1. The maximum difference between adjacent labels is $n + \frac{n^2}{2}$. After substitution of this value into (5.1.5) we get the claimed result. \square

Theorem 5.3.2. *For odd n*

$$\text{cab}(C_n \times C_n) = \frac{(n-2)(n+1)}{2}$$

Proof. In the same way as in previous case we determine the maximum difference of adjacent labels in the antibandwidth optimal labelling described in Theorem 4.3.2. To see the differences between any two adjacent labels in the torus we examine the following four cases:

1. Neighbours in a column.

$$\begin{aligned} |f(i+1, j) - f(i, j)| &= \frac{(n^2 + n + 2)(i+1) + (n^2 - n)j}{2} \\ &\quad - \frac{(n^2 + n + 2)i + (n^2 - n)j}{2} \pmod{n^2} \\ |f(i+1, j) - f(i, j)| &= \frac{n^2 + n + 2}{2} \end{aligned}$$

2. Neighbours in a row.

$$|f(i, j+1) - f(i, j)| = \frac{n^2 - n}{2}$$

3. First and last vertex in a row.

$$\begin{aligned} |f(i, (j+1) \bmod n) - f(i, j)| &= \tag{5.3} \\ &= \frac{(n^2 - n)((j+1) \bmod n - j)}{2} \pmod{n^2} \end{aligned}$$

If $(j+1) \bmod n - j = 1$ then the right hand side of (5.3) equals $(n^2 - n)/2$.
If $(j+1) \bmod n - j = -1$ then the right hand side of (5.3) equals $(n^2 + n)/2$.

4. First and last vertex in a column.

$$\begin{aligned} |f((i+1) \bmod n, j) - f(i, j)| &= \tag{5.4} \\ &= \frac{n^2 + n + 2}{2} ((i+1) \bmod n - i) \pmod{n^2}. \end{aligned}$$

If $(i+1) \bmod n - i = 1$ then the right hand side of (5.4) equals

$$(n^2 + n + 2)/2.$$

If $(i+1) \bmod n - i = -1$ then the right hand side of (5.4) equals

$$n^2 - (n^2 + n + 2)/2 = (n-2)(n+1)/2.$$

Thus, the maximum difference of adjacent labels is $(n^2 + n + 2)/2$, which by Lemma 5.1.5 implies the claim. \square

5.4 Cyclic antibandwidth of hypercubes

Theorem 5.4.1. *For the n -dimensional hypercube Q_n*

$$\text{cab}(Q_n) = 2^{n-1} - \frac{2^{n-1}}{\sqrt{2\pi n}} (1 + o(1)).$$

Proof. The upper bound comes from Theorem 4.4.1. For the lower bound we apply Lemma 5.1.5. Miller and Pritikin [41] proposed the following labelling f of the hypercube. Place the sets X_i on the line in the order $X_1, X_3, X_5, \dots, X_0, X_4, X_6, \dots$, while the order of vertices in the sets is decreasing according to the corresponding binary numbers. They proved

$$\text{ab}(Q_n, f) = 2^{n-1} - \frac{2^{n-1}}{\sqrt{2\pi n}} (1 + o(1)).$$

Now we estimate the maximum distance of any two adjacent vertices under the labelling f . Realize that

$$\max_{uv \in E(Q_n)} |f(u) - f(v)| = \max_{k \geq 1} \{ \max\{D_1(k) + D_2(k)\}, 2^{n-1} \}, \quad (5.5)$$

where

$$D_1(k) = |X_{2k+1}| + |X_{2k+3}| + \dots + |X_0| + |X_2| + \dots + |X_{2k-2}|.$$

and $D_2(k)$ is the maximum distance between two adjacent vertices u and v , where $u \in X_{2k}$ and $v \in X_{2k-1}$, in the situation where the set X_{2k} is placed immediately after X_{2k-1} . First we determine $D_1(k)$.

$$\begin{aligned} D_1(k) &= \binom{n}{2k+1} + \binom{n}{2k+3} + \dots \\ &+ \binom{n}{0} + \binom{n}{2} + \dots + \binom{n}{2k-2} \\ &= \binom{n}{1} + \binom{n}{3} + \dots + \binom{n}{2k-1} + \binom{n}{2k+1} + \dots + \binom{n}{0} + \binom{n}{2} \\ &+ \dots + \binom{n}{2k-2} - \left(\binom{n}{1} + \binom{n}{3} + \dots + \binom{n}{2k-1} \right) \\ &= 2^{n-1} - \binom{n-1}{2k-1}. \end{aligned} \quad (5.6)$$

Now we determine D_2 . Let the vertex u be represented by a number containing ones on positions p_{2k}, \dots, p_2, p_1 . Note that $i \leq p_i \leq n - 2k + i$. The position p_1 is the least significant. The rank of the vertex u in X_{2k} counted from the rightmost vertex is

$$\binom{p_{2k}-1}{2k} + \binom{p_{2k-1}-1}{2k-1} + \dots + \binom{p_2-1}{2} + \binom{p_1-1}{1} + 1.$$

The farthest neighbour of u in X_{2k-1} is the vertex v which contains ones on positions

$p_{2k}, p_{2k-1}, \dots, p_2$. Its rank in X_{2k-1} counted from the rightmost vertex is

$$\binom{p_{2k}-1}{2k-1} + \binom{p_{2k-1}-1}{2k-2} + \dots + \binom{p_2-1}{1} + 1.$$

Hence

$$\begin{aligned} D_2(k) &= \binom{n}{2k} - \sum_{i=1}^{2k} \binom{p_i - 1}{i} + \sum_{i=2}^{2k} \binom{p_i - 1}{i-1} \\ &= \binom{n}{2k} - p_1 + 1 + \sum_{i=2}^{2k} \binom{p_i - 1}{i-1} - \binom{p_i - 1}{i}. \end{aligned} \quad (5.7)$$

Adding (5.6) and (5.7) and setting $p_1 = 1$

$$D_1(k) + D_2(k) = 2^{n-1} + \binom{n-1}{2k} + \sum_{i=2}^{2k} \binom{p_i - 1}{i-1} - \binom{p_i - 1}{i}.$$

Note that for $j \geq 1$, and integer x , the function

$$\binom{x}{j} - \binom{x}{j+1}$$

is nonnegative and nondecreasing in the interval $[j, 2j]$ which can be easily checked by computing the ratio of two consecutive values for x and $x - 1$. Then

$$\max_{[j, 2j]} \left\{ \binom{x}{j} - \binom{x}{j+1} \right\} = \binom{2j}{j} - \binom{2j}{j+1} = \frac{1}{j+1} \binom{2j}{j}. \quad (5.8)$$

It follows that

$$\binom{p_i - 1}{i-1} - \binom{p_i - 1}{i}$$

is maximized when $p_i - 1 = \min\{2i - 2, n - 2k - 1 + i\}$. Distinguish 2 cases:

- Let $4k \leq n + 1$. Then $2i - 2 \leq n - 2k - 1 + i$ and consequently

$$D_1(k) + D_2(k) = 2^{n-1} + \binom{n-1}{2k} + \sum_{i=1}^{2k-1} \frac{1}{i+1} \binom{2i}{i}.$$

The right hand side is maximized when $k = \lfloor (n+1)/4 \rfloor$. Hence

$$\max_{4k \leq n+1} \{D_1(k) + D_2(k)\} = 2^{n-1} + \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} + \sum_{i=1}^{2\lfloor \frac{n+1}{4} \rfloor - 1} \frac{1}{i+1} \binom{2i}{i}. \quad (5.9)$$

- Let $4k \geq n + 2$. If $i \geq n - 2k + 2$ then $p_i - 1 = n - 2k - 1 + i$, otherwise $p_i = 2(i - 1)$. This gives

$$\begin{aligned} D_2(k) &= \binom{n}{2k} + \binom{n-1}{2k-1} - \binom{n-1}{2k} + \binom{n-2}{2k-2} - \binom{n-2}{2k-1} + \dots \\ &+ \binom{2n-4k+1}{n-2k+1} - \binom{2n-4k+1}{n-2k+2} + \sum_{i=1}^{n-2k} \frac{1}{i+1} \binom{2i}{i} \\ &= \binom{n}{2k-1} + \sum_{i=1}^{n-2k} \frac{1}{i+1} \binom{2i}{i}. \end{aligned}$$

Hence

$$D_1(k) + D_2(k) = 2^{n-1} + \binom{n-1}{2k} + \sum_{i=1}^{n-2k} \frac{1}{i+1} \binom{2i}{i}.$$

The right hand side is maximized when $k \geq \lceil (n+2)/4 \rceil$. Then

$$\max_{4k \geq n+2} \{D_1(k) + D_2(k)\} = 2^{n-1} + \binom{n-1}{2\lfloor \frac{n+2}{4} \rfloor} + \sum_{i=1}^{n-2\lfloor \frac{n+2}{4} \rfloor} \frac{1}{i+1} \binom{2i}{i}. \quad (5.10)$$

Now realize that the right hand side of (5.10) does not exceed the right hand side of (5.9). It is easy to see that the sum of Catalan numbers in (5.9) is of order $\theta(2^n/n^{\frac{3}{2}})$ and by Stirling approximation:

$$\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} = \frac{2^{n-1}}{\sqrt{2\pi n}} (1 + o(1)). \quad (5.11)$$

Then

$$\max_k \{D_1(k) + D_2(k)\} \leq 2^{n-1} + \frac{2^n}{\sqrt{2\pi n}} (1 + o(1)).$$

Finally, substituting the last expression into (5.5) and (5.1) we get the result. \square

Chapter 6

Edge antibandwidth problem

This chapter is dedicated to the results concerning the edge antibandwidth problem. It is a research motivated by a simple tournament scheduling application and by an existence of similar research discussing the edge bandwidth problem. Since our motivation from tournament schedulings comes from real life we focused on classes of graphs representing this type of problems - the complete graphs. These graphs represent the tournaments of type "everyone against everyone". In general, complete k -partite graphs represent the situation where every player from one team has to play a game against every player from every other team and players of one team are not playing against players from the same team. The solution of edge antibandwidth problem comes with an ordering of games where the time between two games involving the same player (i.e. player can rest during this period of time) is maximized without delaying the schedule.

Let us start with a simple corollary of Theorem 3.1.1.

Corollary 6.0.1. *Let G be a graph on n vertices and m edges with maximal degree Δ . Then*

$$\text{ab}(L(G)) \leq \frac{m-1}{\Delta-1}$$

Proof. Consider the vertex with maximal degree Δ . This vertex is incident with Δ edges and after the line graph operation these edges are turned into vertices. Moreover, according to definition of the line graph operation, these vertices are all connected because original edges were incident in G . This forms the largest clique in the line graph $L(G)$. Afterwards, considering the upper bound for antibandwidth of a graph G with largest clique $\omega(G)$ (see Theorem 3.1.1)

$$\text{ab}(G) \leq \frac{n-1}{\omega(G)-1}$$

we obtain the result of this corollary.

In the next sections we determine exact results for complete graphs and complete bipartite graphs.

6.1 Complete graph K_n

Theorem 6.1.1. *For $n \geq 1$*

$$\text{ab}(L(K_n)) = \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Proof. Lower bound. It is useful to imagine the antibandwidth problem as placing the vertices of G into consecutive integer points on a line such that the minimal distance of adjacent vertices is maximized. Label the vertices of K_n by $0, 1, 2, \dots, n-1$. An edge between i and j will be denoted by the set $\{i, j\}$. Then the vertices of $L(K_n)$ are $\{\{i, j\} : i, j = 0, 1, 2, \dots, n-1, i \neq j\}$.

For $i = 0, 1, 2, \dots, \lfloor n/2 \rfloor - 1$, define

$$S_i = \{i, 1+i\}, \{n-1+i, 2+i\}, \{n-2+i, 3+i\}, \dots, \left\{\left\lceil \frac{n}{2} \right\rceil + 1 + i, \left\lceil \frac{n}{2} \right\rceil + i\right\},$$

$$T_i = \{i, 2+i\}\{n-1+i, 3+i\}\{n-2+i, 4+i\}, \dots, \left\{\left\lceil \frac{n}{2} \right\rceil + 2 + i, \left\lceil \frac{n}{2} \right\rceil + i\right\},$$

where all additions are taken mod n . Now place the vertices of $L(K_n)$ into consecutive integer points on a line in the order

$$S_0, T_0, S_1, T_1, \dots, S_{\lfloor \frac{n}{2} \rfloor - 1}, T_{\lfloor \frac{n}{2} \rfloor - 1}.$$

This order gives a labelling of $L(K_n)$ by $1, 2, 3, \dots, \binom{n}{2}$. One can see that the minimal distance of labels of adjacent vertices is $\lceil n/2 \rceil - 1$. Due to the definition of the order, it is sufficient to check the distances of adjacent vertices in S_0, T_0, S_1 only.

Upper bound. Assume that the vertices of $L(K_n)$ are placed on a line into integer points $1, 2, 3, \dots, \binom{n}{2}$. As there are n vertices of degree $n-1$ in K_n , the graph $L(K_n)$ contains n cliques isomorphic to K_{n-1} . Any two cliques have exactly one vertex in common. For the i -th clique, $i = 1, 2, 3, \dots, n$, let $a_i(b_i)$ be the position of the leftmost (rightmost) vertex of the clique on the line. Clearly

$$\text{ab}(L(K_n)) \leq \frac{\min_i(b_i - a_i)}{n-2}. \quad (6.1)$$

We claim that

$$\min_i(b_i - a_i) \leq \frac{n^2 - 2n - 2}{2}.$$

If the claim is true Theorem follows. Let $a_k = \max_i a_i$. Then necessarily $a_k \geq n/2$.

If $a_k \geq n/2 + 1$ then $\min_i(b_i - a_i) \leq (n^2 - 2n - 2)/2$.

If $a_k = n/2$, then there exist $j \neq k$, s.t. $a_j = a_k = n/2$. At least one of the values b_j, b_k is smaller than $\binom{n}{2}$ which proves our claim.

Second, consider the odd n case. Again, let $a_k = \max_i a_i$. Then it follows that $a_k \geq \frac{n+1}{2}$.

If $a_k \geq \frac{n+1}{2} + 1$, then

$$\min_i(b_i - a_i) \leq \frac{n^2 - 2n - 3}{2}.$$

If $a_k = \frac{n+1}{2}$, then there exists $j \neq k$, s.t. $a_j = a_k = \frac{n+1}{2}$. At least one of the values b_j, b_k is smaller than $\binom{n}{2}$ which gives again

$$\min_i(b_i - a_i) \leq \frac{n^2 - 2n - 3}{2}.$$

After the substitution into (6.1) and few algebraic manipulations we get the claim of our theorem. \square

6.2 Complete bipartite graph $K_{n,n}$

Theorem 6.2.1. For $n \geq 1$

$$\text{ab}(L(K_{n,n})) = n - 1.$$

Proof. Lower bound. We show construction based on the following model. Assume we have two teams A and B each of n players and they will play a match (chess for example) such that every player from team A has to play one game against every player from team B . The notation $\{a, b\}$ means a game between $a \in A$ and $b \in B$. The graph model is constructed as follows. The vertices are all couples $\{a, b\}$ and the edges are connecting vertices which contains the same player. Note that $\{a, b\}$ and $\{b, a\}$ are considered as different games. This construction gives the complete bipartite graph $K_{n,n}$. The lower bound ordering:

$$\begin{aligned} & \{1, 1\}, \{2, 2\}, \{3, 3\}, \dots, \{n, n\}, \{1, 2\}, \{2, 3\}, \dots, \\ & \{n - 1, n\}, \dots, \{1, n\}, \{2, 1\}, \dots, \{n, n - 1\}. \end{aligned}$$

The minimal difference between two games involving the same player is $n - 1$.

Upper bound. We proceed by a contradiction. Let us assume that

$\text{ab}(L(K_{n,n})) = n$. Consider the optimal linear ordering of edges (vertex couples) of $K_{n,n}$. Take any n consecutive couples anywhere in the middle of the layout. Denote this set by V' . From our assumption, these couples are independent. Now take a couple X on the position one unit to the right from the end of V' . Since $|V'| = n$ it contains all possible couples without the repeating of some player. Then the couple X contains both players already used in V' . Since couples can not repeat X have two different neighbours in V' . The closest of that neighbours can be at most $n - 1$ units away from X what contradicts our assumption and $\text{ab}(L(K_{n,n})) \leq n - 1$. \square

6.3 Complete k-partite graph $K_{n,n,\dots,n}$

Theorem 6.3.1. For k -partite graph with partition size n ; $n \geq 1$ and even $k > 2$

$$\frac{k}{2}(n - 1) \leq \text{ab}(L(K_{n,n,\dots,n})) \leq \frac{kn}{2}.$$

Proof. Lower bound. Consider similar "game" model as in previous proof. The number of teams is now k . The games are ordered such that the first $k/2$ games are between first players of each team, the second $k/2$ games between the second players of each team and so on. After $nk/2$ games played we start with combination first player from team 1, second player from team 2 and so on. By starting this series of games the players are playing their second game. The distance between two games involving the same player is $\frac{k}{2}(n - 1)$. The lower bound ordering follows.

$$\begin{aligned} & \{1_1, 1_2\}, \{1_3, 1_4\}, \dots, \{1_{k-1}, 1_k\}, \{2_1, 2_2\}, \dots, \\ & \{2_{k-1}, 2_k\}, \dots, \{n_1, n_2\}, \dots, \{n_{k-1}, n_k\}, \{1_1, 2_2\}, \\ & \{1_2, 2_3\}, \dots, \{1_{k-1}, 2_k\}, \dots, \{1_n, 2_1\}. \end{aligned}$$

Upper bound comes from Observation 6.0.1. \square

Conjecture 6.3.1. For k -partite graph with partition size n ; $n \geq 1$ and even $k > 2$

$$\text{ab}(L(K_{n,n,\dots,n})) = \frac{k}{2}(n - 1).$$

Chapter 7

Layout volumes of cartesian products

In this chapter we discuss the layout volumes of cartesian product graphs. We start with the hypercube and then we generalise obtained results for volumes of cartesian products of some other known interconnection networks. Our results obtained for hypercubes solve the open problems addressed at Graph Drawing conference in 2004 by Calamoneri and Massini [10]. They asked to find the optimal volume of the hypercube in the two-active-layer model and general model. We solve both problems here. The first one we solve by proving the optimal volume of the hypercube under the one-active-layer model and using this layout we obtain the optimal two-active-layer layout asked by Calamoneri and Massini. The solution for the second problem is the result concerning the optimal volume of the hypercube under the general model, i.e. the model where there are no restrictions on the vertex position in the layout. The techniques used for both models can be applied to a general cartesian product graph constructed from isomorphic factors. Since there exists a parallel research by Fernandez and Efe [21] in the area of two dimensional layouts we generalise results for hypercube for any cartesian products with isomorphic factors. To achieve this goal we develop a technique different than that of Fernandez and Efe.

7.1 General lower bounds on layout volume

In this section we prove a lower bounds which are of special interest for their general use and easy applicability. The results provided in this section and the following chapter come from [56, 57].

Theorem 7.1.1. *The volume of the 3-dimensional 1-active layer layout of any graph G of cutwidth $cw(G)$ satisfies*

$$\text{VOL}_{1-AL}(G) \geq cw(G) \sqrt{\sum_{v \in V} \text{deg}^2(v)}.$$

Proof. Assume we have an optimal 1-active layer layout of G of the volume $\text{VOL}_{1-AL}(G)$. The layout is put into a box of the width W , length L and

height H , where W, L, H are measured along the x, y, z axes, respectively. Thus

$$\text{VOL}_{1-AL}(G) = WLH. \tag{7.1}$$

As two vertices do not touch we have

$$WL \geq \sum_{v \in V} (\deg(v) + 1)^2. \tag{7.2}$$

Consider now the first layer. For every vertex (square) remove its sides except for the left one. It is easy to see that the edges which ended on those 3 sides can be prolonged and connected to the left side using the original tracks and such that the edges are edge disjoint. See Fig. 7.1

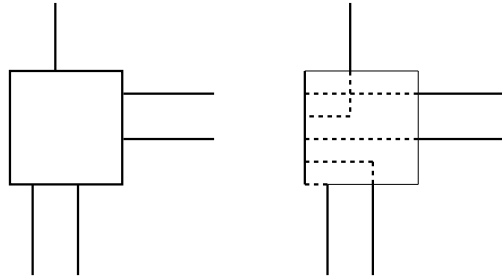


Figure 7.1: Replacing a 5×5 square vertex by a straight-line segment

We obtain a new layout in which the square vertices are replaced by straight-line segments. Now we use a similar idea as in earlier paper [47] (Lemma 2.1), for estimating the area of 2-dimensional layouts. For the sake of completeness we repeat the argument. Any such segment is identified by its coordinates given by the position of the lower end of the segment. Sort the segments according to their coordinates lexicographically. Label the segments by $1, 2, \dots, |V|$, according to the lexicographic order. Let ϕ denote this labelling. Take the first i segments in this labelling. We can find a surface, normal to xy -plane, which separates precisely i segments from the rest, as in Fig. 7.2.

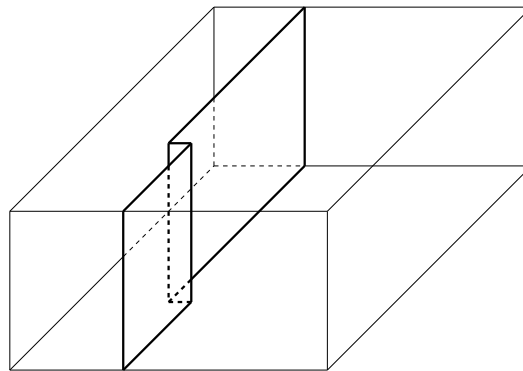


Figure 7.2: A 3D cut of the volume

The area of the surface is $(L + 1)H$. Clearly the number of edges between the two parts of the layout is given by

$$\{|\{uv \in E : \phi(u) \leq i < \phi(v)\}|\} \leq (L + 1)H.$$

Maximizing the left hand side over all i 's and then minimizing it over all ϕ 's we get the cutwidth of G on the left side. By rearranging the terms we have

$$(L + 1)H \geq \text{cw}(G). \tag{7.3}$$

We may repeat the above argument by changing the role of L and W . We get

$$(W + 1)H \geq \text{cw}(G). \tag{7.4}$$

Combining relations (7.1),(7.2),(7.4) and (7.3) we have

$$\begin{aligned} \text{VOL}_{1-AL}(G) &\geq (\text{cw}(G) - H) \sqrt{\sum_{v \in G} (\text{deg}(v) + 1)^2} \\ &= \left(\text{cw}(G) - \frac{\text{VOL}_{1-AL}(G)}{WL} \right) \sqrt{\sum_{v \in G} (\text{deg}(v) + 1)^2} \\ &\geq \left(\text{cw}(G) - \frac{\text{VOL}_{1-AL}(G)}{\sum_{v \in G} (\text{deg}(v) + 1)^2} \right) \sqrt{\sum_{v \in G} (\text{deg}(v) + 1)^2} \\ &\geq \text{cw}(G) \sqrt{\sum_{v \in G} (\text{deg}(v) + 1)^2} - \frac{\text{VOL}_{1-AL}(G)}{\sqrt{\sum_{v \in G} (\text{deg}(v) + 1)^2}}. \end{aligned}$$

By eliminating $\text{VOL}_{1-AL}(G)$ and some algebraic manipulations we get

$$\text{VOL}_{1-AL}(G) \geq \text{cw}(G) \left(\sqrt{\sum_{v \in V} (\text{deg}(v) + 1)^2} - 1 \right) \geq \text{cw}(G) \sqrt{\sum_{v \in V} \text{deg}^2(v)}.$$

□

Remark. The idea of cutting of the 3-dimensional layout into special parts was used in [11] in a model for bounded degree graphs, where they considered the so called *special bisection width* of a graph which is however in general a smaller quantity than the cutwidth.

Theorem 7.1.2. *The optimal volume of the 3-dimensional layout of any graph $G = (V, E)$ satisfies*

$$\text{VOL}(G) \geq (\text{cw}(G) - \sqrt{2\text{cw}(G)})^{\frac{3}{2}}.$$

Proof. Consider an optimal 3-dimensional layout of G . Assume the layout is put into a box of sizes W, L, H such that

$$W \geq L \geq H. \tag{7.5}$$

Take any vertex (a cube of sizes $n \times n \times n$) and choose one of its vertical "edges" (a straightline segment). Delete the cube except for that segment. Observe that the graph edges originally attached to the cube can be prolonged and connected to the segment using edge disjoint paths along the tracks. Repeating this operation for all vertices we get a new layout in which the vertices are replaced by segments of length n and edges are routed in the edge disjoint manner. Similarly as in the proof of Theorem 7.2.1, we can find a surface, see Fig. 7.3, which separates the segments into two parts such that there are at least $\text{cw}(G)$ edges between the segments.

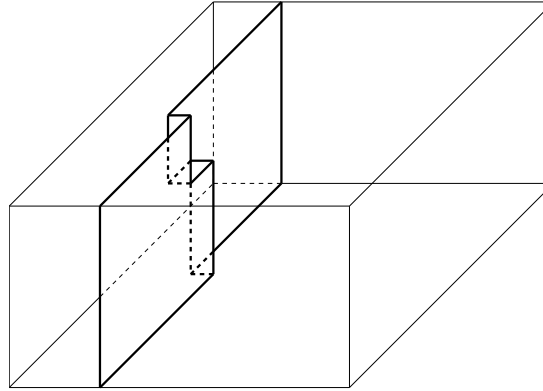


Figure 7.3: A 3D cut of the volume

The edges must cross the surface. The area of the surface is $HL + H + 1$. It follows

$$HL + H + 1 \geq \text{cw}(G).$$

Then

$$HL \geq \text{cw}(G) - 1 - H. \tag{7.6}$$

The relation (7.5) implies

$$LW \geq \text{cw}(G) - 1 - H, \tag{7.7}$$

$$WH \geq \text{cw}(G) - 1 - H. \tag{7.8}$$

Multiplying (7.6)(7.7) and (7.8) and taking the cube root we have

$$\text{VOL}^{\frac{2}{3}}(G) \geq \text{cw}(G) - 1 - H \geq \text{cw}(G) - 1 - \text{VOL}^{\frac{1}{3}}(G).$$

Solving this quadratic inequality for $\text{VOL}^{\frac{1}{3}}(G)$ we get the lower bound. \square

7.2 Layout volumes of hypercube

In this section we discuss the layouts of hypercubes in detail. We start with lower bounds proved in previous section and then we continue with constructions of optimal volumes under the one-active-layer and general models.

7.2.1 Lower bounds

The following two corollaries follows from the general lower bounds for both models from previous chapter (Theorems 7.1.1 and 7.1.2).

Corollary 7.2.1. *The optimal volume of the 3-dimensional 1-active layer layout of the N -vertex hypercube $Q_{\log N}$ satisfies*

$$\text{VOL}_{1-AL}(Q_{\log N}) \geq \frac{2}{3}N^{\frac{3}{2}} \log N + O(\sqrt{N} \log N).$$

Proof. Several papers proved that $\text{cw}(Q_{\log N}) = \lfloor 2N/3 \rfloor$, e.g., [2, 4]. □

Corollary 7.2.2. *The optimal volume of the 3-dimensional layout of the N -vertex hypercube $Q_{\log N}$ satisfies*

$$\text{VOL}(Q_{\log N}) \geq \frac{2\sqrt{6}}{9}N^{\frac{3}{2}} + O(N).$$

7.2.2 Upper bounds

In this section we discuss the upper bounds, i.e. the constructions of layouts of hypercube under the particular models.

One-active-layer layout

Theorem 7.2.1. *The optimal volume of the one-active-layer layout of the $\log N$ -dimensional hypercube satisfies*

$$\text{VOL}_{1-AL}(Q_{\log N}) \leq \frac{2^{\frac{i}{2}+1}}{3}N^{\frac{3}{2}} \log N + O(N^{\frac{3}{2}}),$$

where $i = \log N \pmod{2}$.

Proof. Assume $\log N$ is divisible by 2. The second case is similar. Our basic building block is a linear layout of the m -dimensional hypercube Q_m using $\text{cw}(Q_m) = \lfloor 2^{m+1}/3 \rfloor$ horizontal tracks, while every vertex is represented by a square of size m . See Figure 7.4 for the case $m = 3$. Such a layout

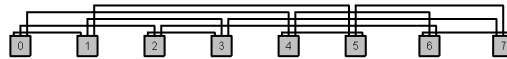


Figure 7.4: Linear layout of Q_3

can be obtained by placing the vertices on the line in the natural order and assigning the edges to tracks properly. This was observed in several papers, e.g., [14, 20, 25]. We utilize a fact that $Q_{\log N}$ can be represented as a cartesian product $Q_{\frac{\log N}{2}} \times Q_{\frac{\log N}{2}}$. Using the linear layout minimizing the cutwidth for

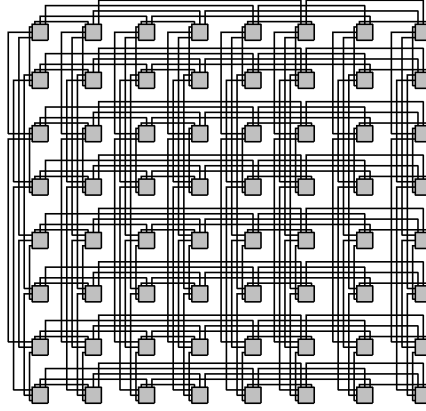


Figure 7.5: 2-dimensional layout of $Q_6 = Q_3 \times Q_3$.

$Q_{\frac{\log N}{2}}$, one can easily design a 2-dimensional layout of $Q_{\log N}$ as shown in Figure 7.5, for $N=64$.

Now rotating every edge in the angle $\pi/2$ around the line defined by its endpoints and compacting the vertices in the natural way we get a one-active-layer layout of the volume

$$\left\lceil \frac{2}{3} N^{\frac{1}{2}} \right\rceil \times N(\log N + 1)^2.$$

The first term stands for the height of the layout and the second one for the occupied area on the 1st layer. We decrease the height of the layout by a factor of $\log N$ in the following way. Consider the layout of any $Q_{\frac{\log N}{2}}$ - the sublayout of the global layout of $Q_{\log N}$. Assume the vertices of $Q_{\frac{\log N}{2}}$ are aligned along the x -axis. The edges occupy $\lfloor 2N^{1/2}/3 \rfloor$ tracks parallel to x . Divide the edges evenly into $\log N$ groups according to the distance of the corresponding tracks from the basic plane. I.e., the first group occupy the first $t = \lfloor \lfloor 2N^{1/2}/3 \rfloor / \log N \rfloor$ tracks when counting from the basic plane, the second group occupy the second t tracks, and so on. Reattach the edges to their corresponding vertices such that:

- i) The endpoints of edges of the same group has the same y coordinate.
- ii) The endpoints of edges of different groups have different y coordinate.
- iii) The endpoints of edges lies in the first or the third “quadrant” of the square vertex. This can be viewed as a “shifting” of edges in y direction on a proper place on its square endvertices. Finally, decrease the height h of every track to $h \bmod t + 1$ and reroute correspondingly the edges. See Figure 7.6.

We repeat this operation for all $Q_{\frac{\log N}{2}}$'s in x and y directions. For the y direction, the endpoints of edges lie in the second and fourth quadrants of square vertices. This avoids overlapping of vertical parts of two edges starting in the same square vertex. The new height of the layout is t , which implies the claimed volume. \square

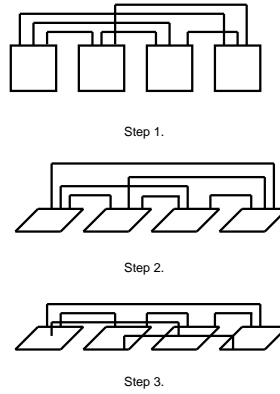


Figure 7.6: Construction of 1-AL layout from optimal colinear layout

General layout

Theorem 7.2.2. *The optimal volume of the general 3-dimensional layout of the log N -dimensional hypercube satisfies*

$$\text{VOL}(Q_{\log N}) \leq \frac{2^{\frac{i}{2}+1}\sqrt{6}}{9}N^{\frac{3}{2}} + O(N^{\frac{4}{3}} \log N),$$

where $i = \log N \pmod{3}$.

Proof. Let $\log N$ be divisible by 3. The other 2 cases are similar. Consider a cube of edge length $n = \log N$ in the 3D mesh. Consider another cube of edge length $s + n$ such that the first cube is positioned "centrally" in the second one and

$$s = \left\lceil \sqrt{\left\lfloor \frac{2N^{\frac{1}{3}}}{3} \right\rfloor} \right\rceil.$$

Place $N^{1/3}$ copies of the second cube (with the first one inside) along the x axis with unit spacing such that they form a box C of size

$$(s + n + 1)N^{1/3} \times (s + n) \times (s + n).$$

Note that in the box C , there are at least

$$(s + n)^2 - n^2 \geq \left\lfloor \frac{2N^{\frac{1}{3}}}{3} \right\rfloor$$

tracks, parallel to the x -axis, which do not cross the cubes of sides n . Now let the $N^{1/3}$ small cubes be vertices of a $Q_{\frac{\log N}{3}}$ placed along the x -axis in the natural order. Thus having cutwidth $\lfloor 2N^{1/3}/3 \rfloor$, if we assume that the edges are drawn as in the linear layout in one plane. It is easy to redraw the edges of $Q_{\frac{\log N}{3}}$ such that they lie in the box C and if two edges shared the same track in the linear layout they will share the same track in C . Moreover the edges are attached to the opposite sides of a vertex only. See Figure 7.7.

Finally, we use again the fact that $Q_{\log N} = Q_{\frac{\log N}{3}} \times Q_{\frac{\log N}{3}} \times Q_{\frac{\log N}{3}}$. Repeating the above construction for all $Q_{\frac{\log N}{3}}$'s in all 3 dimensions we get a layout for

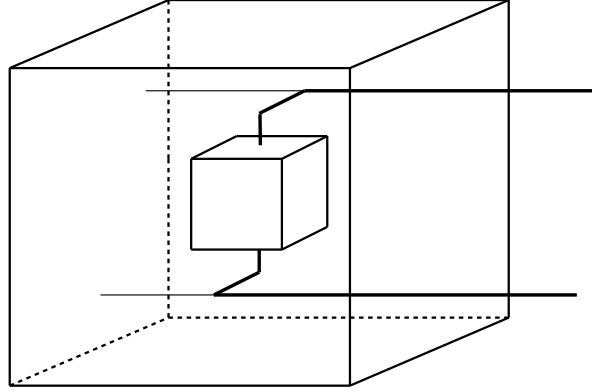


Figure 7.7: Routing edges in the general model

$Q_{\log N}$. One can check that the layout satisfies the assumptions of the model. The total volume is

$$(s + n + 1)^3 N = \frac{2\sqrt{6}}{9} N^{\frac{3}{2}} + O(N^{\frac{4}{3}} \log N).$$

□

2-active-layers layout

Calamoneri and Massini [10] proposed a 2-active layer layout model for bipartite graphs. In this model, all vertices are represented as rectangles and lie on two opposite sides of the bounding box of the layout volume. The edges are distributed evenly between the two layers. The other properties of the model are as in the one-active-layer layout model. Particularly, they studied layouts of $\log N$ -dimensional hypercubes, assuming that each vertex is represented as a $1 \times \lceil \frac{\log N}{2} - 1 \rceil$ rectangle, and proved

$$\text{VOL}_{2-AL}(Q_{\log N}) = \Omega(N^{\frac{3}{2}} \log^{\frac{1}{2}} N),$$

$$\text{VOL}_{2-AL}(Q_{\log N}) = O(N^{\frac{3}{2}} \log N).$$

We follow the same model but we represent every vertex of the hypercube as a square of side $\log N$. We have

Theorem 7.2.3. *The optimal volume of the 3-dimensional 2-active layer layout of the N -vertex hypercube $Q_{\log N}$ satisfies*

$$\text{VOL}_{2-AL}(Q_{\log N}) = \Theta(N^{\frac{3}{2}} \log N).$$

Proof. The lower bound is proved in a similar way as in Theorem 7.1.1. The matching upper bound is obtained by placing two $(\log N - 1)$ -dimensional hypercubes on the two opposite layers using the one-active-layout from Theorem 7.2.1 and adding the edges of the $\log N$ -th dimension as straight-line segments between the layers. □

7.3 Layout volumes of homogenous product graphs

The cartesian product is well-known operation defined on graphs. When applied, the cartesian product combines a set of "factor" graphs into a "product" graph. Several well-known networks are instances of product networks, including the mesh, the hypercube and the torus. In this section, we consider only homogenous products, i.e. the factor graphs are *isomorphic*.

The following facts follow from definition of cartesian product.

$$\Delta(G_1 \times G_2) = \Delta(G_1) + \Delta(G_2),$$

$$n = n_1 \times n_2,$$

where n, n_1, n_2 denote the numbers of vertices of graphs $G_1 \times G_2, G_1, G_2$ respectively.

This section generalises the results for hypercubes. We consider homogenous products of some known interconnection networks. We provide the lower bounds and the construction of the upper bound for the layout volumes in both models.

First we introduce a lemma which offers the relation between the cutwidths of cartesian product graphs and their factors.

Lemma 7.3.1. *Let G^r be the homogenous product graph with factor graph G , and let n be the number of vertices of G . Then*

$$cw(G^r) \leq \frac{n^r - 1}{n - 1} cw(G) = O(n^{r-1} cw(G)).$$

Proof. We proceed by induction on r . The claim is true for $r = 2$ as one can easily see that

$$cw(G_1 \times G_2) \leq cw(G_1) + n_1 cw(G_2), \tag{7.9}$$

where n_1 is the number of vertices in G_1 . Let $r \geq 3$. Assume the claim is true for $r - 1 \geq 2$. Then by (7.9) we have

$$\begin{aligned} cw(G^r) &\leq cw(G^{r-1}) + |V(G^{r-1})| \cdot cw(G), \\ cw(G^{r-1} \times G) &\leq \frac{n^{r-1} - 1}{n - 1} cw(G) + n^{r-1} cw(G) = \frac{n^r - 1}{n - 1} cw(G). \end{aligned}$$

□

7.3.1 Optimal colinear layout

In this section we provide the algorithm for constructing the optimal colinear layout of the graph G with cutwidth $cw(G)$. In fact, this problem was solved in several papers, i.e. [25], [63], [45] for hypercube, [61] for complete graph. We provide a different way to obtain optimal colinear layout of any graph G with known cutwidth.

Lemma 7.3.2. *For any graph G there exists a colinear layout of the height $cw(G) + \Delta(G)$.*

Proof. We start the construction with optimal ordering of vertices of graph G having cutwidth $cw(G)$. We draw the edges as an arches between two end vertices. For this drawing we construct interval graph IG so that the edges are considered as intervals. The interval graph IG has vertices which represent the intervals (edges) from our graph G . Two vertices in IG are connected if, and only if, the corresponding intervals overlap. Then the minimum number of tracks necessary for routing of the edges of graph G and edge-track assignment is obtained as minimal vertex-coloring of IG . Observe that $cw(G)$ equals the maximal clique of IG . As IG is a perfect graph its chromatic number equals the size of the maximal clique i.e., the cutwidth of G . Note that interval graph can be colored in linear time[23].

7.3.2 One-active-layer model

Theorem 7.3.1. *Let G^r be the homogenous product of factor graph G and let r be divisible by 2. Then for the volume of the layout in one-active-layer model we have the following.*

$$V_{1-AL}(G^r) = \Omega \left(n^{\frac{r}{2}} r \Delta(G) cw(G^r) \right),$$

$$V_{1-AL}(G^r) = O \left(n^r r \Delta(G) cw(G^{\frac{r}{2}}) \right).$$

Proof. The lower bound comes from Theorem 7.1.1.

For the upper bound we provide the following construction. We arrange the vertices into the area with $n^{\frac{r}{2}}$ columns and $n^{\frac{r}{2}}$ rows. In every row and every column is a graph $G^{\frac{r}{2}}$. Find its optimal colinear layout with number of tracks equal to $cw(G^{\frac{r}{2}})$ (Lemma 7.3.2). All vertices of G are of degree at most $\Delta(G^r)$ and of area $\Delta^2(G^r)$. The total vertices area is of size $n(\Delta(G^r) + 1)^2$. We create the one-active-layer layout of graph G^r in three steps (see Fig. 7.6).

In the first step we route the edges as in colinear layouts of $G^{\frac{r}{2}}$. In the second step we turn the plane containing the edges into the space so that it is perpendicular to the basic plane. The height of such a layout is defined by $cw(G^{\frac{r}{2}})$.

In the third step we decrease the layout $\Delta(G^r)$ -times by spreading the edges into the routing canal which has width of the node side. This can be done in the same way as for the hypercube from previous section. The final height is $\frac{cw(G^{\frac{r}{2}})}{\Delta(G^r)}$ which after multiplying by the vertices area of $n(\Delta(G^r) + 1)^2$ and with use of $\Delta(G^r) = r\Delta(G)$ gives the claimed volume. \square

Observation 7.3.1. *The construction of one-active-layer layout from the proof of Theorem 7.3.1 is asymptotically optimal if*

$$cw(G^r) = \Theta \left(n^{r-1} cw(G) \right).$$

Proof. From Lemma 7.3.1 we have

$$V_{1-AL} = O \left(n^r r \Delta(G) cw(G^{\frac{r}{2}}) \right) = O \left(n^{\frac{3r}{2}-1} r \Delta(G) cw(G) \right).$$

Moreover, if we assume that $cw(G^r) = \Theta(n^{r-1} cw(G))$ we obtain the following claim for the lower bound:

$$V_{1-AL} = \Omega \left(n^{\frac{r}{2}} r \Delta(G) n^{r-1} cw(G) \right) = \Omega \left(n^{\frac{3r}{2}-1} r \Delta(G) cw(G) \right).$$

\square

7.3.3 General model

In this section we consider the volume layouts of homogenous products in general model.

Theorem 7.3.2. *Let G^r be the homogenous product of factor graph G and let r be divisible by 3. Then for the volume of the layout of graph G in general model we have*

$$VOL(G^r) = \Omega\left(cw^{\frac{3}{2}}(G^r)\right),$$

$$VOL(G^r) = O\left(n^r(\Delta(G) + \sqrt{cw(G^{\frac{r}{3}})})^3\right).$$

Proof. From Theorem 7.1.2 we have the following lower bound

$$VOL(G^r) \geq (cw(G^r) - \sqrt{2cw(G^r)})^{\frac{3}{2}},$$

$$VOL(G^r) = \Omega\left(cw^{\frac{3}{2}}(G^r)\right).$$

For the upper bound we provide the following construction.

We represent a node in general model by a cube with length of one side equal to its degree (denoted by Δ). We arrange the nodes of $G^{\frac{r}{3}}$ graphs in the 3D mesh so that their colinear layouts are in directions of x, y, z axes. The edges lead only in directions of x, y, z . The illustration of this type of layout (two-dimensional variant) is depicted in the Figure 7.5.

Then we enlarge the node volume by the volume necessary for routing of the edges. The necessary volume for accommodating the edges around one side of the cube is $cw(G^{\frac{r}{3}})$ since this is the number of edges leading in one of the directions x, y, z .

We increase the node side length to $\left(\Delta(G^r) + \sqrt{cw(G^{\frac{r}{3}})}\right)$ what is sufficient. By this operation we obtain a cube which contains original node inside. The volume of this cube is $\left(\Delta(G^r) + \sqrt{cw(G^{\frac{r}{3}})}\right)^3$. We put n^r such a cubes into the cubic box so that in every direction x, y or z are exactly $n^{\frac{r}{3}}$ cubes. Moreover, in every direction x, y, z these cubes have ordering which allows to form factors $G^{\frac{r}{3}}$ in the same way. Whole layout has volume of

$$O\left(n^r(\Delta(G^r) + \sqrt{cw(G^{\frac{r}{3}})})^3\right).$$

With use of $\Delta(G^r) = r\Delta(G)$ we get the claimed formula. \square

Observation 7.3.2. *The construction of the layout in general model from the proof of Theorem 7.3.2 is asymptotically optimal if*

$$cw(G^r) = \Theta\left(n^{r-1}cw(G)\right)$$

and

$$\Delta^2(G^r) \ll cw(G^{\frac{r}{2}}).$$

Proof. Assume that $cw(G^r) = \Theta\left(n^{r-1}cw(G)\right)$ and we get

$$VOL(G^r) = \Omega\left(n^{\frac{3r}{2}-\frac{3}{2}}cw^{\frac{3}{2}}(G)\right).$$

If for the factor graph holds $\Delta^2(G^r) \ll cw(G^{\frac{r}{3}})$ we can write the upper bound formula as

$$VOL(G^r) = O\left(n^r n^{\frac{r}{2} - \frac{3}{2}} cw^{\frac{3}{2}}(G)\right) = O\left(n^{\frac{3r}{2} - \frac{3}{2}} cw^{\frac{3}{2}}(G)\right),$$

and the lower and the upper bound are asymptotically equal. \square

7.3.4 Layout volumes of some product graphs

In this section we use previous results to obtain volume bounds for several known product graphs. The key parameters for our framework are cutwidths of the factor graphs and their products. For the cutwidths of product graphs we use an approximation through the lower bound for bisection width. Fernandez and Efe [21] made the lower bounds for bisection widths of product graphs based on the so called edge forwarding index (denoted by $\pi(G)$). We use Lemma 3.2.1 for obtaining the lower bound of bisection widths of product graphs. The resulting system of inequalities looks as follows.

$$\frac{n^{2r} - 1}{2\pi(G)n^{r-1}} \leq bw(G^r) \leq cw(G^r) \leq n^{r-1}cw(G) \quad (7.10)$$

In the following subsections we show that the volumes of considered graphs are asymptotically optimal. It is sufficient to prove that the optimality conditions are satisfied.

For every graph we show in the first step that

$$\frac{n^{2r} - 1}{2\pi(G)n^{r-1}} = \Theta(n^{r-1}cw(G))$$

and in the second step that

$$\Delta^2(G^r) \ll cw(G^{\frac{r}{2}}).$$

Note that this two steps verify the both optimality conditions from Observation 7.3.1 and Observation 7.3.2. In fact, it would be sufficient to show that

$$n^{r-1} \cdot cw(G) = \left(\frac{n^{2r} - 1}{2\pi(G)n^{r-1}} \right) \quad (7.11)$$

Table 7.1 contains the parameters of the factor graphs used in the following proofs.

Product of de Bruijn graphs

Denote de Bruijn graph on $n = 2^m$ vertices by DB_m . First, we verify the equation 7.11. The left side of the equation is equal to

$$n^{r-1}cw(DB_m) = \Theta\left(\frac{2^{mr}}{m}\right).$$

The right side of the equation is equal to

$$\begin{aligned} \frac{n^{2r} - 1}{2\pi(DB_m)n^{r-1}} &= \frac{2^{2rm} - 1}{2\Theta(m2^{m-1})2^{m(r-1)}} = \\ \Theta\left(\frac{2^{2mr}}{m2^m2^{m(r-1)}}\right) &= \Theta\left(\frac{2^{mr}}{m}\right). \end{aligned}$$

Factor graph	n	$\Delta(G)$	$cw(G)$	$\pi(G)$
deBruijn	2^m	4	$\Theta(\frac{2^{m+1}}{m})$ [47]	$\Theta(m2^{m-1})$ [51]
Star graph	$m!$	$m - 1$	$\Theta(m!)$ [61]	$\Theta(m!)$ [25]
Complete transposition graph	$m!$	$m - 1$	$\Theta(mm!)$ [52]	$\Theta((m - 1)!)$ [25]
Butterfly graph	$m2^m$	4	$\Theta(2^m)$	$\Theta(m^22^{m-1})$ [51]
Complete graph	m	$m - 1$	$\frac{m^2}{4}$ [61]	2
CCC graph	$m2^m$	m	$\Theta(2^m)$	$\Theta(m^22^m)$ [51]
Linear array	m	2	1	$\Theta(m^2)$ [31]

Table 7.1: Input parameters of considered factor graphs

As we showed the both sides of the equation 7.11 are equal what means that our one-active-layer construction of the product of de Bruijn graphs is asymptotically optimal. Now, we verify the optimality condition for the layout under the general model.

$$\begin{aligned} \Delta^2(DB_m^r) &<< cw(DB_{\frac{r}{2}}^{\frac{r}{2}}) \\ (4r)^2 &<< \frac{2^{mr}}{2m2^{m-1}2^{\frac{mr}{2}-1}} \\ 16r^2 &<< \frac{2^{\frac{mr}{2}}}{m \cdot 2^{m-1}} = 2^{\frac{m(r-2)+2}{2}}. \end{aligned}$$

The right side of the resulting inequality is significantly greater than the left side and the original inequality is therefore true. Since the both optimality conditions are satisfied the layout under the general model is asymptotically optimal.

Product of star graphs

Denote the star graph on $n = m!$ vertices by S_n . Again, we verify the optimality condition for one-active-layer layout first. The left side of the equation follows.

$$n^{r-1}cw(S_n) = (m!)^{r-1}\Theta(m!) = \Theta((m!)^r).$$

The right side:

$$\frac{n^{2r} - 1}{2\pi(S_n)n^{r-1}} = \frac{(m!)^{2r} - 1}{2\Theta(m!)(m!)^{r-1}} = \Theta((m!)^r).$$

We see that the both side of the optimality condition are equal. The one-active-layer layout of the product of star graphs S_n is therefore asymptotically optimal. We continue with verification of the optimality condition for the layout under the general model.

$$\begin{aligned} r^2(m - 1)^2 &<< \frac{(m!)^r}{2m!(m!)^{\frac{r}{2}-1}} \\ r^2(m - 1)^2 &<< \frac{(m!)^{\frac{r}{2}}}{2}. \end{aligned}$$

It is obvious that the right side of the inequality grows significantly faster than the left side. With both conditions satisfied the volume of general layout of the product of star graphs is asymptotically optimal.

Product of complete transposition graphs

Denote the complete transposition graph on $n = m!$ vertices by CTG_n . As in previous cases we verify the optimality condition for the volume of one-active-layer layout of product of CTG_n 's. The left side:

$$n^{r-1}cw(CTG_n) = (m!)^{r-1}\Theta(m \cdot m!) = \Theta(m(m!)^r).$$

The right side:

$$\begin{aligned} \frac{n^{2r} - 1}{2\pi(CTG_n)n^{r-1}} &= \frac{(m!)^{2r} - 1}{2\Theta((m-1)!(m!)^{r-1})} = \\ \Theta\left(\frac{(m!)^{2r}}{(m-1)!(m!)^{r-1}}\right) &= \Theta\left(\frac{(m!)^{r+1}}{(m-1)!}\right) = \\ \Theta\left(\frac{m(m!)^{r+1}}{m!}\right) &= \Theta(m(m!)^r). \end{aligned}$$

We see that both sides of equality are asymptotically equal. The volume of one-active-layer layout of the product of complete transposition graphs is therefore asymptotically optimal. We continue with verification of the optimality condition for layout under the general model.

$$\begin{aligned} (r(m-1))^2 &\ll \frac{(m!)^{r+1}}{2(m-1)!(m!)^{r-1}} \\ r^2(m-1)^2 &\ll \frac{(m!)^{\frac{r}{2}}m}{2} \end{aligned}$$

The right side of this inequality grows significantly faster than the left side. The second condition for general layout optimality is therefore fulfilled and the volume of layout under the general layout is asymptotically optimal.

Product of butterfly graphs

Denote the butterfly graph on $n = m2^m$ vertices by B_m .

Note that approximation of cutwidth of Butterfly graph from Table 7.1 can be obtained by placing the graph on the line recursively. The lower bound comes from bisection. From both, the construction and the lower bound, it comes that $cw(B_m) = \Theta(2^m)$.

The left side of the optimality condition for layout volume under one-active-layer layout follows.

$$n^{r-1}cw(B_m) = (m2^m)^{r-1}\Theta(2^m) = \Theta(m^{r-1}2^{mr})$$

The right side:

$$\begin{aligned} \frac{n^{2r} - 1}{2\pi(B_m)n^{r-1}} &= \frac{(m2^m)^{2r} - 1}{2\Theta(m^22^{m-1})(m2^m)^{r-1}} = \\ \Theta\left(\frac{m^{2r}2^{2mr}}{m^22^m m^{r-2}2^{mr-m}}\right) &= \Theta\left(\frac{m^{2r}2^{2mr}}{m^{r+1}2^{mr}}\right) = \Theta(m^{r-1}2^{mr}). \end{aligned}$$

As we can see both sides of the condition are equal. The volume of one-active-layer layout is therefore asymptotically optimal. Now, we check the optimality condition for layout volume under general model.

$$16r^2 \ll m^{\frac{r}{2}-1} 2^{\frac{mr}{2}}$$

Directly after the substitution one can see that the right side grows significantly faster than the left side. Therefore, the layout volume under the general model is asymptotically optimal.

Product of complete graphs

Denote the complete graph on $n = m$ vertices by K_m . Again, the verification of the one-active-layer layout optimality condition follows. The left side:

$$n^{r-1} cw(K_m) = m^{r-1} \frac{m^2}{4} = \Theta(m^{r+1}).$$

The right side:

$$\frac{n^{2r} - 1}{2\pi(K_m)n^{r-1}} = \frac{m^{2r} - 1}{2^m 2m^{r-1}} = \Theta(m^{r+1}).$$

We verified that both side of condition are equal. Therefore, the volume of one-active-layer layout of product of complete graphs is asymptotically optimal. We continue with the verification of the condition for layout under the general model:

$$r^2(m-1)^2 \ll \frac{m^r}{4m^{\frac{r}{2}-1}} = \frac{m^{\frac{r}{2}+1}}{4}.$$

It is obvious that the right side grows significantly faster than the left side. Therefore, the volume of the layout of product of complete graphs K_m under the general model is asymptotically optimal.

Product of CCC graphs

Denote the cube connected cycles graph on $n = m2^m$ vertices by CCC_m .

Note that approximation of cutwidth of complete transposition graph from Table 7.1 comes from approximation for the cutwidth of the hypercube.

First, we verify the optimality condition for layout under the one-active-layer model. The left side:

$$n^{r-1} cw(CCC_m) = (m2^m)^{r-1} \Theta(2^m) = \Theta(m^{r-1} 2^{mr}).$$

The right side:

$$\begin{aligned} \frac{n^{2r} - 1}{2\pi(CCC_m)n^{r-1}} &= \frac{(m2^m)^{2r} - 1}{2\Theta(m^2 2^m)(m2^m)^{r-1}} = \\ \Theta\left(\frac{m^{2r} 2^{2mr}}{m^2 2^m m^{r-1} 2^{m(r-1)}}\right) &= \Theta\left(\frac{m^{2r} 2^{2mr}}{m^{r+1} 2^{mr}}\right) = \Theta(m^{r-1} 2^{mr}). \end{aligned}$$

The both sides of the condition are equal so the volume of the layout of product of CCC_m 's under the one-active-layer model is asymptotically optimal. We continue with the optimality condition for layout under the general model:

$$\begin{aligned} (mr)^2 &<< \frac{m^r 2^{mr}}{m^2 2^m m^{\frac{r}{2}-1} 2^{m(\frac{r}{2}-1)}} \\ (mr)^2 &<< m^{\frac{r}{2}-1} 2^{\frac{mr}{2}}. \end{aligned}$$

The right side of the inequality grows significantly faster than the left side. Since the both optimality conditions are satisfied the volume of the layout of cube connected cycles graph under the general model is asymptotically optimal.

Product of linear arrays

We use the term "linear array" just to follow the terminology in [21] because this part of our work is analogical to their paper discussing the two dimensional case. In fact, the linear array on n vertices is the same as path on n vertices. We denote the linear array on $n = m$ vertices by P_m .

The left side of the optimality condition for one-active-layer layout follows:

$$n^{r-1} cw(P_m) = m^{r-1} \cdot 1 = \Theta(m^{r-1}).$$

The right side:

$$\frac{n^{2r} - 1}{2\pi(P_m)n^{r-1}} = \frac{m^{2r} - 1}{2\Theta(m^2)m^{r-1}} = \Theta\left(\frac{m^{2r}}{m^{r+1}}\right) = \Theta(m^{r-1}).$$

Both sides of the optimality condition are equal so the volume of the one-active-layer layout of the product of the linear arrays is asymptotically optimal. We continue with the condition for layout under the general model:

$$\begin{aligned} 4r^2 &<< \frac{m^r}{m^2 m^{\frac{r}{2}-1}} \\ 4r^2 &<< m^{\frac{r}{2}-1}. \end{aligned}$$

The right side of the inequality grows significantly faster than the left side. Then, the volume of the layout of product of linear arrays under the general model is asymptotically optimal.

Results review

We verified and proved that our constructions for both one-active-layer and general model are asymptotically optimal for all discussed products of typical networks. Table 7.2 contains a summary of results gained by substitution of graph parameters from Table 7.1 into formulas from Theorem 7.3.1 and Theorem 7.3.2.

Graph	1-AL model	General model
Complete transposition graph product	$\Theta\left((m!)^{\frac{3r}{2}} r m(m-1)\right)$	$\Theta\left((m(m!)^r)^{\frac{3}{2}}\right)$
deBruijn product	$\Theta\left(\frac{rn^{\frac{3r}{2}}}{\log n}\right)$	$\Theta\left(\frac{n^{\frac{3}{2}(r+1)}}{\log^{\frac{3}{2}} n}\right)$
Star graph product	$\Theta\left((m!)^{\frac{3r}{2}} r(m-1)\right)$	$\Theta\left((m!)^{\frac{3r}{2}}\right)$
Butterfly product	$\Theta\left(m^{\frac{3r}{2}-1} 2^{\frac{3mr}{2}} r\right)$	$\Theta\left(m^{\frac{3r}{2}-\frac{3}{2}} 2^{\frac{3mr}{2}}\right)$
Product of complete graphs	$\Theta\left(m^{\frac{3r}{2}+1} r(m-1)\right)$	$\Theta\left(m^{\frac{3r}{2}+\frac{3}{2}}\right)$
CCC graph product	$\Theta\left(m^{\frac{3r}{2}} r 2^{\frac{3mr}{2}}\right)$	$\Theta\left(m^{\frac{3r}{2}-\frac{3}{2}} 2^{\frac{3mr}{2}}\right)$
Linear array product	$\Theta\left(m^{\frac{3r}{2}-1} r\right)$	$\Theta\left(m^{\frac{3r}{2}-\frac{3}{2}}\right)$
Hypercube	$\Theta\left(2^{\frac{3m}{2}} m\right)$	$\Theta\left(2^{\frac{3m}{2}}\right)$

Table 7.2: Optimal layout volumes of several product graphs

Chapter 8

Conclusion

The layouts of graphs are very general-use concepts in computer science and graph theory allowing wide area of applications. This thesis was dedicated to two specific problems - the linear layout problem called antibandwidth and three-dimensional layout problem where one is minimizing the volume of the layout. However, our results in three-dimensional layouts are based on some one-dimensional concepts (cutwidth, colinear layout) what shows a strong connection between linear (one) and more dimensional layouts. The following sections conclude both discussed problems and provide some interesting open problems for further research in both areas.

8.1 The antibandwidth problem

Our research here extends the known results for the antibandwidth problem. We solved open problems concerning the antibandwidth value for meshes, tori, three-dimensional meshes, hypercubes and complete k-ary trees. We also extend the research and discuss a few other variations of the problem. Namely we provide a new results for cyclic case (paths, cycles, 2D meshes, tori and hypercubes). We also start a research in the edge modification of the problem based on some real-life application. All trends in our research were inspired by existing trends in dual problem - the bandwidth problem.

The antibandwidth problem is a problem with nice motivation behind. However, it is an interesting problem itself. It still offers a lot of interesting open problems. Our effort was to evaluate the antibandwidth invariant for some typical networks studied in computer science and to show its relation to the other graph parameters. As a conclusion to this part of the thesis we provide some interesting open problems to direct the possible future research in this area:

- evaluation of this invariant for other typical networks not discussed in this thesis (de Bruijn, Butterfly, Cube-connected cycles and others). Non-bipartite graphs are of special interest,
- research of others than linear variations of this problem. An introduction to cyclic and edge variation is discussed in this thesis and can be an inspiration for further research,
- general antibandwidth problem, where the host graph can be a general graph (mesh, hypercube, ...), especially the variation with the hypercube

as a host graph is in strong interest because of a connection of this variation to the coding theory. If the 2D or 3D mesh is used as a host graph this variation can model some obnoxious facility location problems in the plane or space respectively,

- this problem is quite intuitive and some other real-life applications can be probably found.

8.2 Three-dimensional layouts of graphs

Volume-optimal three-dimensional layouts of graphs are strongly motivated by computer science. They have an application in the three-dimensional chip design. In this work the volume-optimal three-dimensional layout of cartesian product graphs is discussed. This class of graphs has an interesting property allowing us to use the concepts from linear layouts of graphs in the layout construction. This property is a recursive structure given by the cartesian product construction of resulting network. Using the concepts of cutwidth and colinear layouts from the area of linear layout problems we were able to prove exact or asymptotically optimal results for cartesian product graphs. We focus on homogenous products of some known interconnection networks.

Since the question of volume-optimal three-dimensional layout is quite general the research in this area is based on some models and rules given by them. In this thesis a two most common of them are discussed: the one-active-layer model and the general model. We provide the results for both of them. Especially, we solved the open problem posted in Graph Drawing 2004 conference by Calamoneri and Massini and provide the optimal volume of the hypercube in the general model and one- and two-active-layers model. In the literature the most results were proved for models with bounded degree of a node. We extended the set of papers discussing the models where the degree of the node is arbitrarily high. Moreover, we provided the general-use lower bounds on volume of the layouts in both models based on graph parameter cutwidth. However, some problems still remain open.

- Our general model has one drawback - the node is represented by a cube with length of one side $deg(v)$. This is given by the use of the construction in the proof of lower bounds where we replace whole cubic node by a straight-line segment. Natural way to represent a node with arbitrarily high degree is a cube but the length of its side can be smaller than ours. This will decrease the volume of the layout as a consequence.
- This thesis provides the results for homogenous cartesian products graphs only. The next open problem is to study a three-dimensional layouts of a general graphs. This a more difficult problem because the nice recursive structure of cartesian products can not be used anymore.
- The volume-optimal three-dimensional layout of a graph is just a one point of view at the layout optimality. The second aspect is a total wire length of particular layout which we are not discussing at all in this thesis.

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