Boltzmann Jaynes Inverse Problem, Maximum Entropy and Maximum Probability*

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Overview

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  - Conditional Limit Theorem
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  - Relative Entropy Maximization (REM/MaxEnt) method
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- Summary
Let there be a discrete random variable which can take on finite number $m$ of values $\mathcal{X} = [x_1, x_2, \ldots, x_m]$, with probabilities $q = [q_1, q_2, \ldots, q_m]$. $\mathcal{X}$ is called alphabet; its elements are letters. $q$ is called source, or generator.
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Let there be a random sample $x_1, x_2, \ldots, x_n$ of size $n$ drawn from $q$. Type $\nu^n$ which the sample induces is $\nu^n = [n_1, n_2, \ldots, n_m]/n$ just the vector of relative frequencies of the $m$ letters in the sample of size $n$. Usually $n$-type is used where it is necessary to highlight size of the underlying random sample.
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• Let there be a random sample \( x_1, x_2, \ldots, x_n \) of size \( n \) drawn from \( q \). Type \( \nu^n \) which the sample induces is \( \nu^n = [n_1, n_2, \ldots, n_m]/n \) just the vector of relative frequencies of the \( m \) letters in the sample of size \( n \). Usually \( n \)-type is used where it is necessary to highlight size of the underlying random sample.

• There are \( \Gamma(\nu^n) = \frac{n!}{\prod_{i=1}^{m} n_i!} \) sequences which induce the same type \( \nu^n \). The number \( \Gamma(\cdot) \) is called multiplicity of type.
Example

Let $\mathcal{X} = [1, 2, 3, 4]$.
Let $q = [0.13, 0.09, 0.42, 0.36]$.
Let $n = 10$
and the sample let be: $3, 4, 1, 1, 4, 3, 4, 3, 2, 4$.
The sample induces type $\nu^{10} = [2, 1, 3, 4] / 10$.

There is in total $\Gamma(\nu^n) = 1260$ sequences of length 10 which induce the same type.
What is the probability $\pi(\nu^n; q)$ that the source $q$ generates type $\nu^n$? Well,

$$\pi(\nu^n; q) = \Gamma(\nu^n) \prod_{i=1}^{m} \exp(n \sum_{i=1}^{m} \nu_i^n \log q_i).$$

For $q$ and $\nu^n$ from the Example it is $\pi(\nu^n; q) = 0.02384831$
How many $n$-types is there?

Let’s denote the set of all $n$-types (on the alphabet $\mathcal{X}$), for a fixed $n$, by $\mathcal{P}_n(\mathcal{X})$. It is useful to view $\mathcal{P}_n$ as a subset of the set $\mathcal{P}(\mathcal{X})$ of all possible probability distributions on $\mathcal{X}$.

The number of $n$-types in $\mathcal{P}_n$ is $J = \binom{n+m-1}{m-1}$.

For $m = 4$, $n = 10$ it is 286.
Imagine that you were told that the source $q$ generated \textit{SOME} $10$-type, from the set $\mathcal{P}_{10}$ of all $286$ possible $10$-types. Given the available information $\{X, q, n, \mathcal{P}_{10}\}$ you have to select a type.

This is an example of BJIP.
Simple Boltzmann Jaynes Inverse Problem

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How to proceed?
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**LLN:** for \( \epsilon > 0 \), as \( n \to \infty \),

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\pi(\nu^n \in B(q, \epsilon); q) = 1.
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So, for very large $n$ we know how to solve this instance of BJIP: select $\nu^n \approx q$. 
1) Note that among all probability distributions in $\mathcal{P}$, $q$ has the highest value of the relative entropy 

$$H(p \parallel q) = -\sum_{i=1}^{m} p_i \log(p_i/q_i),$$

with respect to $q$.

2) Let $\hat{\nu}^n$ denote a type in $\mathcal{P}_n$, for which $\pi(\nu^n; q)$ is maximal. Formally, $\hat{\nu}^n = \arg\sup_{\nu^n \in \mathcal{P}_n} \pi(\nu^n; q)$. It holds true that as $n \to \infty$, $\hat{\nu}^n$ converges to $q$. 
Finite n?

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Let’s try a harder BJIP. Now, we are given a set $\Pi \subset \mathcal{P}$ which does not contain $q$. Again, the only available information is $\{\mathcal{X}, q, n, \Pi\}$. How to select $n$-type from $\Pi$???
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Example (cont’d): Mean value for $q$ is 3.01. Let $\Pi = \{p : \sum_{i=1}^{\cdot} p_i x_i = 3.2\}$, so that $q$ is not in $\Pi$. Let $n = 10^{30}$. How to select $n$-type from $\Pi$?
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We would like to have a statement of the following form:
\[ \pi(\nu^n \in B(? , \epsilon) \mid \nu^n \in \Pi; q) = 1. \]
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The problem is to determine the point $?$ upon which the types conditionally concentrate.
Conditional Law of Large Numbers, aka Conditional Limit Theorem (CoLT):
Provided that $\Pi$ is convex, for $\epsilon > 0$, and $n \to \infty$, 
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This also solves the small $n$ problem!
In order to comply with CoLT, one thus has to solve BJIP by REM method; in other words: to select that $n$-type(s) from $\Pi_n$ for which the value of $H(\nu^n \mid \mid q)$ is maximal.
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... or by any other method which asymptotically obeys CoLT.
Set $\Pi_{10}$ of 10-types with mean value $\sum_{i=1}^{4} \nu_{i}^{10} x_i = 3.2$; and their probabilities (wrt $q$):

<table>
<thead>
<tr>
<th>Type</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 1 0 7</td>
<td>0.000429091</td>
</tr>
<tr>
<td>2 0 2 6</td>
<td>0.00817656</td>
</tr>
<tr>
<td>1 2 1 6</td>
<td>0.00242601</td>
</tr>
<tr>
<td>0 4 0 6</td>
<td>2.99919e-05</td>
</tr>
<tr>
<td>1 1 3 5</td>
<td>0.0264166</td>
</tr>
<tr>
<td>0 3 2 5</td>
<td>0.00195947</td>
</tr>
<tr>
<td>1 0 5 4</td>
<td>0.0359559*</td>
</tr>
<tr>
<td>0 2 4 4</td>
<td>0.0133353</td>
</tr>
<tr>
<td>0 1 6 3</td>
<td>0.0193609</td>
</tr>
<tr>
<td>0 0 8 2</td>
<td>0.00564692</td>
</tr>
</tbody>
</table>

Type denoted by asterisk has the highest $\pi(\nu^n; q)$ in this subset.
Table 1: MaxProb and REM/MaxEnt

<table>
<thead>
<tr>
<th>$n$</th>
<th>$J$</th>
<th>$\hat{p}^n; q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
<td>0.1000 0.0000 0.5000 0.4000</td>
</tr>
<tr>
<td>50</td>
<td>154</td>
<td>0.0800 0.0600 0.4400 0.4200</td>
</tr>
<tr>
<td>100</td>
<td>574</td>
<td>0.0800 0.0700 0.4200 0.4300</td>
</tr>
<tr>
<td>500</td>
<td>13534</td>
<td>0.0820 0.0700 0.4140 0.4340</td>
</tr>
<tr>
<td>1000</td>
<td>53734</td>
<td>0.0830 0.0700 0.4110 0.4360</td>
</tr>
<tr>
<td>$\hat{p}$</td>
<td></td>
<td>0.0826 0.0709 0.4103 0.4361</td>
</tr>
</tbody>
</table>

As $n \to \infty$ MaxProb type(s) converges to REM distribution.
MaxProb and REM

There are two options:
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1) REM can be viewed as an asymptotic instance of MaxProb.
2) REM is a self-standing method (i.e., when $n$ is finite choose the type(s) with highest value of relative entropy). REM and MaxProb asymptotically coincide.
There is yet another method which asymptotically converges to REM distribution: Expected Type method (EType). The method selects:

$$\tilde{\nu}^n = \frac{\sum_{j=1}^J \pi(\nu_j^n; q) \nu_j^n}{\sum_{j=1}^J \pi(\nu_j^n; q)}.$$
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It can be shown that for set II with unique REM distribution, EType converges to REM.
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It can be shown that for set $\Pi$ with unique REM distribution, EType converges to REM.

The asymptotic identity of EType and REM however breaks up when there are several REM distributions!
Summary

- Questions are more important than answers. Problems are more important than their solutions. When thinking about MaxEnt/REM, it is worth formulating the problem which is to be solved.
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- Boltzmann Jaynes Inverse Problem. Application of REM/MaxEnt for solving BJIP is justified by CoLT. No need to rely upon axiomatic arguments, etc. CoLT implies that BJIP can be solved by selecting REM type also when $n$ is finite.
Summary

- Questions are more important than answers. Problems are more important than their solutions. When thinking about MaxEnt/REM, it is worth formulating the problem which is to be solved.
- Boltzmann Jaynes Inverse Problem. Application of REM/MaxEnt for solving BJIP is justified by CoLT. No need to rely upon axiomatic arguments, etc. CoLT implies that BJIP can be solved by selecting REM type also when $n$ is finite.
- Maximum Probability method can be as well justified by CoLT. MaxProb converges to REM/MaxEnt. When $n$ is finite, it is thus also possible to select MaxProb type.
BJIP is not the only problem where one can ask a question of the form: ‘what is the most probable type...’ . There is also more complicated Jeffreys Inverse Problem, where MaxProb asymptotically converges to Jeffreys Entropy Maximization method. And one could take inspiration from Robert Niven and think also about Fermi Dirac Inverse Problem and Bose Einstein Inverse Problem.
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Though everything is possible, not everything is allowed (Roger Bacon). There are many entropies flying around. Some of them are even maximized! However, not each of the entropy maximization method can be given a shelter, in form of an associated inverse problem and CoLT. Whether there is such a chance for Renyi-Tsallis Entropy Maximization method is still an open problem¹.

¹However, check Bercher, arXiv:math-ph/0609077
Some references

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