Univerzita Mateja Bela v Banskej Bystrici
Fakulta prírodných vied

Discrete Group Actions: Algorithms and Applications

Habilitačná práca
4e8b086b-7a82-461b-80da-e94688f06554

Odbor: 9.1.1 Matematika
Pracovisko: Inštitút matematiky a informatiky, CVV UMB

Banská Bystrica, 2014 Ján Karabáš

Acknowledgements

I wish to express many thanks to Roman Nedela for many stimulating discussions, for his leading, and personal help in all years we know each other. Extra thanks are due to Domenico Catalano, who carefully read the first version of the text and helped me to fix up swarms of syntactic and semantic errors; our discussions on the text and related topics were really pleasant and useful. I am also very indebted to many colleagues for their, often implicit, support. I would like to thank, last but not least, my wife and children for their permanent support and understanding.

Abstrakt

Zoznamy diskrétnych akcií grúp majú množstvo aplikácií v rôznorodých oblastiach matematiky, počínajúc geometriou, cez reálnu a komplexnú analýzu, až po diskrétnu matematiku. Samotný problém klasifikácie akcií konečných grúp na plochách patrí už zhruba 150 rokov medzi centrálne problémy teórie grúp a súvisí s mnohými otvorenými problémami. V teórii máp môžeme vďaka takýmto zoznamom pomerne „lacno" konštruovať a analyzovať vysoko symetrické mapy daného rodu: regulárne, vrcholovo alebo hranovo tranzitívne, Cayleyho mapy s danou grupou atď. Rozpoznanie akcií cyklických grúp na plochách vysokého rodu hrá kľúčovú úlohu v enumeráciách kombinatorických objektov, t.j. máp, grafov, vektorových priestorov a podobne.

Klasifikácia diskrétnych akcií grúp na sfére je klasickým výsledkom, ktorého korene môžeme nájsť až v antike. Samozrejme, v reči modernej matematiky bola klasifikácia sformulovaná v 19. storočí. Akcie grúp na tóre sú tak isto dobre známe, hoci v tomto prípade máme do činenia so 17-timi nekonečnými famíliami kryštalografických grúp. Vďaka tzv. Hurwitzovej hornej hranici (Hurwitz bound) vieme, že pre daný rod $g \geq 2$ existuje len konečne veľa akcií konečného počtu grúp na plochách daného rodu. Publikované zoznamy akcií – Brougton, 1990, Kuribayashi a Kimura, 1991-1993, Bogopoľskij, 1992 – siahali len po plochy orientovateľného rodu 5. Vďaka významnému rozvoju výpočtovej techniky a systémov výpočtovej algebry v poslednom čase sa nám podarilo túto klasifikáciu (s istými výhradami) rozšíriť po rod 21 v prípade všeobecných grúp a po rod v rádoch stoviek v prípade cyklických grúp. Poznamenajme, že v prípade tzv. veľkých (nie nutne komutatívnych) grúp existuje čiastočná klasifikácia M. Condera pre orientovateľné plochy do rodu 301.

Tento text je zamýšľaný ako stručný úvod do problematiky a stručný prehľad výsledkov autora. Každý z výsledkov, ktorým sa v práci venujeme, je postavený na našich znalostiach o diskrétnych akciách konečných grúp na orientovateľných plochách. Niekoľko kapitol obsahuje rozšírený úvod k článkom pripojeným k textu. Navyše, na rozdiel od podobných textov, opisujeme aj algoritmické a implementačné aspekty problémov, ktorým sa v texte venujeme.

Abstract

Discrete actions of finite groups on surfaces appears in many situations in numerous branches of mathematics, cryptography, quantum physics, and many other fields of science. In topological graph theory they can be used to derive lists of highly symmetrical maps of fixed genus: regular maps, vertex-transitive maps, Cayley maps, or edge-transitive maps. For example, the classification of actions of cyclic groups is essential for solving enumeration problems of combinatorial objects, i.e. maps, graphs and others. Lists of discrete group actions are used as an experimental material for further research.

The problem of classification of discrete actions of groups on orientable surfaces of genus $g \geq 2$ is nowadays challenge. The classification of groups acting on the sphere is a classical part of crystallography. In case of torus the situation is in principle known, though there are infinitely many group actions. Thanks to Riemann-Hurwitz formula we know that for higher genera there are just finitely many finite groups acting on a surface of a given genus. Published lists of actions go up to genus five (Broughton; Bogopolskij; Kuribayashi and Kimura). For small genera, the classification can be done with help of computer algebra systems. For example, using Magma we derived the list of actions of discrete groups on surfaces of genus $2 \leq g \leq 21$.

This text is intended as a brief introduction to the problematic and a summary of results achieved by the author (in collaboration with other researchers); the extensive use of discrete group actions is emphasised. We also deal with algorithmic and implementation details, extend and comment results contained in attached reprints of author's papers.

Contents

List of Algorithms

1. Introduction

For instance, a race of hyperintelligent pan-dimensional beings once built themselves a gigantic supercomputer called Deep Thought to calculate once and for all the Answer to the Ultimate Question of Life, the Universe, and Everything.

For seven and a half million years, Deep Thought computed and calculated, and in the end announced that the answer was in fact Forty-two - and so another, even bigger, computer had to be built to find out what the actual question was†.

– *Douglas Adams,* The Restaurant at the End of Universe

The objective of *topological graph theory* is, from rough point of view, the problem of graph drawing on a surface such that no pair of edges of the graph cross. This intuitive geometric problem has many varieties; we can specify various side conditions on that problem. For example, we can be interested in maps possessing as much symmetries as possible or we can ask, whether it is even possible to draw a fixed graph into a fixed surface, or which surfaces admit maps with given underlying graph, and so on.

The systematic study of (regular) maps can be traced back to the paper of Felix Klein (1878) who described a regular map of type $\{3, 7\}$ on an orientable surface of genus 3 [66], related with the solutions of certain algebraic equation. From the very beginning, the study of (regular) maps was tightly connected with group theory as one can see in the Burnside's monograph [18], works by Brahana in 20ties of the last century [10, 11, 12], and more recently in Coxeter and Moser's monograph [34, Chapter 8], published in 1936. Breaking point for topological graph theory was the solution of Heawood map colouring problem done by Ringel and Youngs [86] in 1968. A fundamental monograph on topological graph theory, by Gross and Tucker [43], appeared in the first edition in 1978, in the same year as the essential paper by Jones and Singerman [54], which settled algebraic approach and interconnections with classical fields of mathematics‡. The present-time interest in theory of maps extends to their connection to Dyck's triangle groups, Riemann surfaces [91, 92], algebraic curves [1, 102], Galois groups [56, 52] and many other fields of mathematics and other fields such as quantum physics, cryptology, hardware design, material sciences and so on. Massive development of computers and methods of computational algebra [49, 88] gave us powerful tools to classify and construct maps of higher genera, maps on non-orientable surfaces, or even with boundary. Regular maps on orientable surfaces up to genus $g \leq 301$ were classified completely in last decade [21, 20]. The attention turned also to maps with high degree of symmetry, but not necessarily regular, including Cayley maps, vertex and edge-transitive maps [14, 63, 84, 85]. We should also mention results concerning various enumerations of classes of maps [95, 71, 72, 79], classification of infinite families of maps [16, 25, 26, 24], or estimates of asymptotic behaviour of families of maps [2, 36], to name but a few. The area of theory of maps is very active, with hundreds of published papers and tens new ones published yearly.

Essentially, we recognise three main streams of studies in theory of maps, concerning in principle with following problems:

1) Classify (symmetric) maps with fixed underlying surface.
2) Which maps can be constructed, given a group of symmetries of a map?
3) How do look like all (symmetric) embeddings of a given fixed graph?

† D. Adams, The Hitchhiker Trilogy Boxset, Picador, London, 2004. ‡ as usual, the common knowledge has been shared as a 'folklore' in the community

This text is devoted to the studies in the first of the aforementioned directions. We address also the computational part of the problems we deal with. As it is explained by Jones and Singerman [54] and many other authors, (hyper)maps can be regarded as subgroups of certain infinite group, which is universal for the considered family of (hyper)maps. A direct consequence of this fact is that methods (algorithms, programs) of computational algebra can be used to determine and study maps and their invariants, including their internal or external symmetries. In what follows, we present several fundamental algorithms based on this, algebraic in the meaning, approach to maps. All these algorithms can be easily implemented either directly or by using computer algebra systems as Magma [9] or GAP [38]. In examples done in this text we use Magma implementation, done by author.

The first chapter of the text is the summary† of mathematical background, serving as a unifying preliminary text for the following chapters and the included papers as well. Large part of this chapter is based on essential sections of the survey "Regular maps – combinatorial objects relating different fields of mathematics" [81] by Roman Nedela. In contrast with this survey, we slightly relaxed the rôle of regular maps. The second difference is that we paid more attention to topics related with Riemanian maps and discrete group actions on orientable surfaces, a core idea used in the rest of the text and in included papers. We also added several examples and algorithms together with discussions about computational complexity, implementations, and their usage. The category of maps is broad, including infinite maps on rather strange surfaces, e.g. non-compact ones or those with non-empty boundary. The general theory of maps covers also those situations. However, maps we will deal mostly with, will be oriented, on closed (orientable) surfaces. We will not avoid more general categories of maps and the related mathematical models – simply, we cannot avoid them without loss of mathematical correctness. Graphs will be finite and connected, and surfaces will be closed and orientable, unless otherwise explicitly stated.

Chapters 2, 3, and 4 summarise briefly the results and methods used to obtain results presented in the papers attached to the thesis. Some features, methods and explanations were modernised and treated more carefully as it was done in the papers. The reasons, why the papers do not treat some things put into this text are either historical (in certain moments, the theory was not developed completely), or simply because not everything can be put into the published version of a paper. We added some notes, which was not explicitly discussed in the papers; specifically those, concerning with algorithmic an implementation problems.

Three of the papers included into the thesis [62, 63, 64] deal with a particular instance of the following problem.

Problem 1.1. *Classify all maps with high degree of symmetry on an orientable surface of genus g.*

By the term 'high degree of symmetry' we mean that the automorphism group of a map of our interest is transitive on vertices, edges, darts or flags. Objects of this category are sometimes called *symmetric maps*. A solution of an instance of Problem 1.1 can be divided into independent subproblems:

1) given a surface $\mathcal{S}_g$ of genus g, classify groups of self-homeomorphisms of $\mathcal{S}_g$, i.e. discrete group actions on $\mathcal{S}_g$,
2) recognise and classify the corresponding quotient maps on a quotient surface $\mathcal{S}_{g'}$ (a quotient orbifold of genus g'),
3) given a group and a quotient map, reconstruct the map of desired family by employing T-reduced voltage assignments on darts of the quotient map.

† rather brief than comprehensive

All those three papers give a answer to the particular questions the the fashion that the three subproblems are solved. The delicacy and the mathematical beauty of the solution evolve and nowadays it seems that we have powerful tools for theoretical considerations and also computations for solving problems of this sort.

The fourth paper [14] is devoted to a solution of a more general problem, naturally enlightened when classification of Archimeean maps of higher genera [63] has been done. The study in this direction is fresh and still not yet complete (the essential paper [15] is going to be published) and it includes and generalises some older results in that direction [41, 89, 90].

When all papers in the thesis are put together, the essential motif emerges: symmetric maps understood as Riemanian maps can be efficiently and with elegance treated through inspection of properties of discrete group actions on surfaces. Numerous problems can be then solved with help of computational algebra systems.

Banská Bystrica/Aveiro *Ján Karabáš*

2. Foundations

2.1 Topological and combinatorial maps

2.1.1. Graphs. There are several ways how to formalise the notion of a graph. Essentially, we need to emphasise algebraic properties of graphs instead their structural properties. However, the structure of a graph must be easily determined, if needed. With respect to this demand we define a (combinatorial) *graph* as a quadruple $G = (D, V; I, L)$ where $D = D(G)$ and $V = V(G)$ are disjoint non-empty finite sets, $I \colon D \to V$ is a surjective mapping, and $L = L_G$ is an involutory permutation on D. The elements of D and V are *darts* and *vertices* respectively, I is the incidence function assigning its *initial vertex* to every dart , and L is the *dart-reversing involution*. Darts from the set, called the *neighbourhood of a vertex* v, defined as $n(v) = \{x \colon x \in D \mid I(x) = v\}$ give rise to the set of *adjacent edges*. The order of $n(v)$, is the *valency of the vertex* $v^\dagger$, usually denoted as $\mathrm{val}(v)$. The orbits of the group $\langle L \rangle$ on D are *edges* of the graph G. If a dart x is a fixed point of L, then the corresponding edge is a *semi-edge*. If $IL(x) = I(x)$ but $(x)L \neq x$, then the edge is a *loop*. Edges are called *links* in the remaining case. As follows, a link is incident with two distinct vertices while a loop or a semi-edge is incident with a single vertex. A link or a loop gives rise to two oppositely directed darts, each of them is reverse to the other. A semi-edge incident to a vertex v gives rise to a single dart initiating at v that is opposite to itself. From the topological point of view, a semi-edge is equivalent to a pendant edge except that its pendant end-point is not listed as a vertex. The usual graph-theoretical concepts such as cycles, connectedness, etc., easily translate to our present formalism.

Let us make the short excursion from the main line of the text. In map theory we strongly distinguish between two different actions of groups (algebraic structures) on the sets (objects of interests). The right action is usually called a *monodromy action* and it is responsible for the structure of the examined object. The left action can be called a *morphism action* (a covering, a automorphism, an isomorphism) and it is a morphism in the category of examined object. This exact distinguishing came from the theory of Cayley graphs (see Section 4.2) and it is widely accepted in the community. In the literature (and also here) the parethesis notation for mappings is often used‡. In that situation, say M be a monodromy and α be a morphism, the expression $\alpha M(x)$, or αM reads such that the right-most mapping (before parenthesis) is applied first, from the proper side on the object it has action on. The aforementioned mappings I and L are certainly monodromy actions.

In fact, a graph can be seen as a finite 1-dimensional cell complex. Let us remark that the same type of graphs are considered in Jones and Singerman [54] (see also [76, 82]).

2.1.2. Topological maps. A *map* on a surface is a cellular decomposition of a closed surface into 0-cells called *vertices*, 1-cells called *edges* and 2-cells called *faces*. The vertices and edges of a map form its *underlying graph*, which correspond to the notion of allowed graph introduced in Section 2 of Jones and Singerman's fundamental paper [54]. According to [54, Lemma 2.1a, b, c] it is easy to determine a one-to-one correspondence of vertices, edges and faces of the topological map with distinguished points $\mathcal{V} \in \mathcal{S}$, homeomorphic images of the interval $[0, 1] \cong \mathcal{E} \subset \mathcal{S}$ and discs $\mathcal{D} \cong \mathcal{F} \subseteq \mathcal{S}$, respectively. Idea of 'cutting and patching' of sub-spaces, widely used in topology, then directly gives us a surface with the graph embedded into – the map.

† degree of v ‡ in fact, mostly

The only condition we have to satisfy is that any pair of sub-spaces, corresponding either to edges or faces, of the resulting surface have either empty intersection or have only boundary points in common. Here we force faces (2-cells as above) to be homeomorphic to (closed) discs, hence the resulting surface will be always orientable. However, we can relax this condition† obtaining more broader class of maps. Map theory in this case is similar to the theory of maps on orientable closed surfaces, but it is slightly beyond the scope of this text. A map is said to be *orientable* if the supporting surface is orientable, and is *oriented* if one of two possible orientations of the surface has been specified; otherwise a map is *unoriented*. The following result, relating numerical invariants of maps with the Euler characteristic $\chi(\mathcal{S})$ of the supporting orientable surface, is well-known.

Theorem 2.1 (Euler-Poincarè formula)**.** *Let* $\mathbf{M}$ *be a map on a closed orientable surface* $\mathcal{S}_g$ *of genus* g *with* v *vertices,* e *edges,* s *semi-edges and* f *faces. Then* $\chi(\mathbf{M}) = \chi(\mathcal{S}_g) = v - e + s + f = 2 - 2g$.

Let us note that the *genus of the map* $\mathbf{M}$ is introduced in the terms of Theorem 2.1 in its meaning. Certainly, given map $\mathbf{M}$, the genus can be expressed as an algebraic invariant of $\mathbf{M}$ instead of meaning that it is a genus of the underlying surface of $\mathbf{M}$.

2.1.3. Algebraic maps. In principle there are two essential approaches to the combinatorial description of graph embeddings into surfaces. The first approach, is based on the pair of permutation of the given degree acting on the dart set of the order equal to the degree of these permutations. This definition automatically involves the orientation of the supporting surface and so is suitable only for maps on orientable surfaces [5, 54]. The corresponding combinatorial structure is called a combinatorial (or, sometimes, algebraic) oriented map. The latter approach, involving three involutions of the same degree acting on mutually incident (vertex, edge, face)-triples called flags, is orientation insensitive and allows us to represent maps on non-orientable surfaces as well [58]. The resulting combinatorial structure will be called a combinatorial unoriented map. However, both approaches are equivalent, when we deal with maps on orientable surfaces.

We will focus onto maps on orientable surfaces. The following definition of *algebraic oriented map* [54] gives also very convenient computer representation for maps on orientable surfaces. By a (combinatorial, algebraic) oriented map we mean a triple $(D; R, L)$ where $D = D(\mathbf{M})$ is a non-empty finite set of *darts*, and R and L are two permutations of D such that L is an involution and the group $\mathrm{Mon}(\mathbf{M}) = \langle R, L \rangle$ acts transitively on D from right. The group $\mathrm{Mon}(\mathbf{M})$ is called the oriented *monodromy group* of $\mathbf{M}$. The permutation R is called the *rotation* of $\mathbf{M}$. The orbits of the group $\langle R \rangle$ are the *vertices* of $\mathbf{M}$, and elements of an orbit v of $\langle R \rangle$ are the darts *radiating* (or *emanating*) from v, that is, v is their initial vertex. The cycle of R permuting the darts emanating from v is the *local rotation* R_v at v. The permutation L is the *dart-reversing involution* of $\mathbf{M}$, and the orbits of $\langle L \rangle$ are the *edges* of $\mathbf{M}$. The orbits of $\langle RL \rangle$ define the face-boundaries of $\mathbf{M}$‡. The incidence between vertices, edges and faces is given by nontrivial set intersection. The vertices, darts and the incidence function provide the representation of the underlying graph of map $\mathbf{M}$, which is always connected due to the transitive action of the monodromy group. Vice-versa, an oriented map can be equivalently described as a pair $(G; R)$ where $G = (D, V; I, L)$ is a connected graph and R is a permutation of the dart-set of G cyclically permuting darts with the same initial vertex, that is, $IR(x) = I(x)$ for every dart x of G.

† we just need to ask for interiors of sub-spaces corresponding to vertices, edges and faces to be simply connected ‡ cycles of RL are read in reverse direction to the chosen orientation of the surface. It is just matter of taste whether we define faces to be cycles of $(RL)^{-1}$ or cycles of RL.

Although we will not deal with maps on non-orientable surfaces nor with maps on surfaces with boundary, we shall at least mention the general concept of the definition of a map. Sometimes it is useful even in considerations employing just oriented maps. Combinatorial unoriented maps are build from three involutions acting on a non-empty finite set F of *flags* [58]. A (combinatorial, algebraic) *unoriented map* is a quadruple $(F;\ \lambda, \rho, \tau)$ where λ, ρ and τ are fixed-point free involutory permutations of $F = F(\mathbf{M})$ called the *longitudinal*, the *rotary* and the *transversal involution*, respectively, which satisfy the following conditions:

1) $\lambda\tau = \tau\lambda$; and
2) the group $\langle \lambda, \rho, \tau \rangle$ acts transitively on F.

This group is the unoriented *monodromy group* $\mathrm{Mon}(\mathbf{M})$ of $\mathbf{M}$.

We define the *vertices* of $\mathbf{M}$ to be the orbits of the subgroup $\langle \rho, \tau \rangle$, the *edges* of $\mathbf{M}$ to be the orbits of $\langle \lambda, \tau \rangle$, and the *face-boundaries* to be the orbits of $\langle \rho, \lambda \rangle$ under the action on F, the incidence being given by nontrivial set intersection. Note that each orbit z of $\langle \lambda, \tau \rangle$ has cardinality 2 or 4 according to whether z is a semi-edge or not. Moreover, this representation allows us to introduce maps on surfaces with boundary, if we assume that ϱ, λ, and τ may have fixed points. Clearly, the even-word subgroup $\langle \rho\tau, \tau\lambda \rangle$ of $\mathrm{Mon}(\mathbf{M})$ has always index at most two. If the index of the aforementioned subgroup is two, then $\mathbf{M}$ is said to be orientable and there exist also an algorithm to obtain the representation of the corresponding oriented map† in terms of $(D; R, L)$. Furthermore, given an oriented algebraic map $\mathbf{M} = (D;\ R, L)$ we can easily derive the corresponding unoriented algebraic map $\mathbf{M}^\natural = (F^\natural;\ \lambda^\natural, \rho^\natural, \tau^\natural)$ such that $F^\natural = D \times \{1, -1\}$ and for a flag $(x, j) \in D \times \{1, -1\}$ we define the corresponding involutions as follows

$$\lambda^\natural(x, j) = (L(x), -j), \quad \rho^\natural(x, j) = (R^j(x), -j), \quad \text{and} \quad \tau^\natural(x, j) = (x, -j).$$

From the previous, it is straightforward that the underlying graph of an algebraic map is a graph *with ordered neighbourhoods of vertices*, and faces of $\mathbf{M}$ can be identified with set of disks. On the other hand, given a drawing of a graph we can easily determine the corresponding algebraic map. Hence, there is a functor between categories of algebraic and topological maps [54, Proposition 5.5]. As a result, we shall usually employ the same notation for a topological map and for the corresponding combinatorial structure. For the sake of technical convenience, we shall usually replace topological graphs and maps by their combinatorial counterparts. In the following we will not deal with topological maps, however any claim can be translated from combinatorial/algebraic language to topological terms.

Example 2.2. Let us examine oriented maps – the17 embeddings of the Petersen graph P into orientable surfaces. We know, that P has 10 vertices, 15 links, and certainly, the respective maps will have 30 darts. Substituting to Euler-Poincarè formula we get

$$2 - 2g = -5 + f \implies 2g + f \leq 7.$$

Immediately we know that $0 \leq g \leq 3$. But what about planar embeddings of P? The Petersen graph has girth 5 and this give bound for number of faces, $f = 6$; pentagonal faces are the shortest possible for any embedding of P and we have 30 darts. But a planar embedding has to have 7 faces. It is clear that it does not exist. Other examples of embeddings for genera $g = 1, 2, 3$ are displayed below.

† also its mirror image

$g = 1$	$R =$	$(1,11,6)(2,12,7)(3,13,8)(4,9,14)(5,10,15)(16,26,21)(17,27,22)$ $(18,28,23)(19,24,29)(20,25,30)$,
	$L =$	$(1,7)(2,9)(3,6)(4,10)(5,8)(11,16)(12,17)(13,18)(14,19)(15,20)$ $(21,29)(22,30)(23,27)(24,28)(25,26)$
$g = 2$	$R =$	$(1,11,6)(2,7,12)(3,8,13)(4,9,14)(5,10,15)(16,21,26)(17,22,27)$ $(18,23,28)(19,24,29)(20,25,30)$,
	$L =$	$(1,7)(2,9)(3,6)(4,10)(5,8)(11,16)(12,17)(13,18)(14,19)(15,20)$ $(21,29)(22,30)(23,27)(24,28)(25,26)$
$g = 3$	$R =$	$(1,11,6)(2,12,7)(3,8,13)(4,14,9)(5,10,15)(16,26,21)(17,27,22)$ $(18,23,28)(19,24,29)(20,30,25)$,
	$L =$	$(1,7)(2,9)(3,6)(4,10)(5,8)(11,16)(12,17)(13,18)(14,19)(15,20)$ $(21,29)(22,30)(23,27)(24,28)(25,26)$

Let us look to the computer representation of one of aforementioned maps. It is done in Magma script† and we will use that representation of maps in examples throughout whole text. The map in the display the one-face embedding of P of genus 3. The data members of the record are in our opinion self-explanatory.

```
> R := Sym(30)!(1, 11, 6)(2, 12, 7)(3, 8, 13)(4, 14, 9)(5, 10, 15)\
        (16, 26, 21)(17, 27, 22)(18, 23,28)(19, 24, 29)(20, 30, 25);
> L := Sym(30)!(1, 7)(2, 9)(3, 6)(4, 10)(5, 8)(11, 16)(12, 17)(13,18)\
        (14, 19)(15, 20)(21, 29)(22, 30)(23, 27)(24, 28)(25, 26);
> M:=OrmapByPermutations(R,L);
> M;
rec<OrmapType |
R := (1, 11, 6)(2, 12, 7)(3, 8, 13)(4, 14, 9)(5, 10, 15)(16, 26, 21)(17,\
        27, 22)(18, 23, 28)(19, 24, 29)(20, 30, 25),
L := (1, 7)(2, 9)(3, 6)(4, 10)(5, 8)(11, 16)(12, 17)(13, 18)(14, 19)(15,\
        20)(21, 29)(22, 30)(23, 27)(24, 28)(25, 26),
F := (1, 16, 25, 15, 8, 18, 27, 30, 26, 29, 14, 2, 17, 23, 24, 21, 11, 3,\
        5, 4, 19, 28, 13, 6, 7, 9, 10, 20, 22, 12),
d := 30,
v := 10,
e := 15,
s := 0,
f := 1,
xi := -4,
g := 3>
>
```

2.1.4. Homomorphisms of maps. Let $\mathbf{M}_1 = (D_1; R_1, L_1)$ and $\mathbf{M}_2 = (D_2; R_2, L_2)$ be two oriented maps. A *homomorphism* $\varphi\colon \mathbf{M}_1 \to \mathbf{M}_2$ of oriented maps is a mapping $\varphi\colon D_1 \to D_2$ such that

$$\varphi R_1 = R_2 \varphi \quad \text{and} \quad \varphi L_1 = L_2 \varphi.$$

Analogously, a *homomorphism* $\varphi\colon \mathbf{M}_1 \to \mathbf{M}_2$ of unoriented maps $\mathbf{M}_1 = (F_1; \lambda_1, \rho_1, \tau_1)$ and $\mathbf{M}_2 = (F_2; \lambda_2, \rho_2, \tau_2)$ is a mapping $\psi\colon F_1 \to F_2$ such that

$$\psi\lambda_1 = \lambda_2\psi, \quad \psi\rho_1 = \rho_2\psi \quad \text{and} \quad \psi\tau_1 = \tau_2\psi.$$

The properties of homomorphisms of both varieties of maps are similar except that homomorphisms of unoriented maps ignore orientation. Every map homomorphism induces an epimorphism of the corresponding monodromy groups. Indeed, it is not

† the characters '\' are soft breaks in Magma console output

difficult to see that if $\psi\colon (F_1;\ \lambda_1, \rho_1, \tau_1) \to (F_2;\ \lambda_2, \rho_2, \tau_2)$ is a map homomorphism then the assignment $\lambda_1 \mapsto \lambda_2$, $\rho_1 \mapsto \rho_2$, $\tau_1 \mapsto \tau_2$ extends to an epimorphism ψ^* called the *induced epimorphism* of the corresponding monodromy groups. Furthermore, transitive actions of the monodromy groups ensure that every map homomorphism is surjective and that it also induces an epimorphism of the underlying graphs. Moreover, the transitive action of monodromy in both maps (preimage an image) ensures that any homomorphism between maps is fully determined by the image of one dart or flag, respectively. This fact is strongly used further (see Algorithm 1).

Topologically speaking, a map homomorphism is a graph preserving branched covering projection of the supporting surfaces with branch points possibly at vertices, face centres or free ends of semi-edges. Therefore we can say that a map $\mathbf{M}$ *covers* $\bar{\mathbf{M}}$ if there is a homomorphism $\mathbf{M} \to \bar{\mathbf{M}}$ which induce the corresponding (possibly branched) covering of the underlying surfaces of maps. A map homomorphism is *smooth* if it preserves the valencies of vertices, the lengths of faces and does not send a link or a loop onto a semi-edge and it is a *branched covering* otherwise.

With map homomorphisms, it is natural to introduce map isomorphisms and automorphisms. The automorphism group $\mathrm{Aut}^+(\mathbf{M})$ of an oriented map $\mathbf{M} = (D;\ R, L)$ consists of all permutations in the symmetry group $\mathrm{Sym}(D)$ which commute with both R and L. Similarly, the automorphism group $\mathrm{Aut}(\mathbf{M})$ of an unoriented map $\mathbf{M} = (F;\ \lambda, \rho, \tau)$ is formed by all permutations in the symmetry group $\mathrm{Sym}(F)$ which commute with each of λ, ρ and τ. Hence, in both cases the automorphism group is nothing but the centraliser of the monodromy group in the corresponding symmetry group of the supporting set of the map (cf. [54, Proposition 3.3(i)]).

Since the action of the monodromy group $\mathrm{Mon}(\mathbf{M})$ is transitive, $|\mathrm{Mon}(\mathbf{M})| \geq |D(\mathbf{M})|$ for every oriented map $\mathbf{M}$, and $|\mathrm{Mon}(\mathbf{M})| \geq |F(\mathbf{M})|$ for every unoriented map $\mathbf{M}$. If the equality is attained, then the monodromy group acts regularly on the supporting set, and therefore the map is called orientably-regular or *regular*, respectively. The automorphism group of an orientably regular map $\mathbf{M}$ acts regularly on darts of $\mathbf{M}$, and similarly $\mathrm{Aut}(\mathbf{M})$ of a regular map $\mathbf{M}$ acts regularly on flags of $\mathbf{M}$. We use of the term regular map in the same manner as it is used in Gardiner et al. [40] and Wilson [101]. For instance, Jones and Thornton [58] uses the term 'reflexible', and White [100] calls such maps 'reflexible symmetrical'. On the other hand, our orientably regular maps are called 'regular' in [34], 'symmetrical' in [6] and [100], and 'rotary' in [101].

For each homomorphism $\varphi\colon \mathbf{M}_1 \to \mathbf{M}_2$ of oriented maps there is the corresponding homomorphism $\varphi^\natural\colon {\mathbf{M}_1}^\natural \to {\mathbf{M}_2}^\natural$ defined by $\varphi^\natural(x, i) = (\varphi(x), i)$. If $\mathbf{M}_1 = \mathbf{M}_2 = \mathbf{M}$, that is, φ is an automorphism, then this definition and the assignment $\varphi \mapsto \varphi^\natural$ yield the isomorphic embedding of $\mathrm{Aut}^+(\mathbf{M}) \to \mathrm{Aut}(\mathbf{M}^\natural)$. It is easy to see that the index $|\mathrm{Aut}(\mathbf{M}^\natural) : \mathrm{Aut}^+(\mathbf{M})|$ is at most two. If it is two, then the map $\mathbf{M}$ is said to be *reflexible*, otherwise it is *chiral*. In the former case, there is an isomorphism $\psi^\natural$ of the map $\mathbf{M} = (D;\ R, L)$ with its mirror image $\mathbf{M}^{-1} = (D;\ R^{-1}, L)$ called a *reflection* of $\mathbf{M}$. Clearly, $\psi^\natural$ is a homomorphism† that extends $\mathrm{Aut}^+(\mathbf{M})$ to $\mathrm{Aut}(\mathbf{M}^\natural)$. Topologically speaking, automorphisms of an oriented map preserve the chosen orientation of the supporting surface whereas reflections reverse it.

Let us clarify the symbolic concerning automorphism groups as it is used in the following text. In both cases, oriented and unoriented maps, the automorphism group is a centraliser of a monodromy group in the corresponding symmetric group; in the first case the base set is set of darts, while in the latter case it is the set of flags. The group of automorphisms of a map $\mathbf{M}$ is denoted, as usual, by the symbol $\mathrm{Aut}(\mathbf{M})$. In the case we have an oriented map $\mathbf{M} = (D; R, L)$ its automorphism group is in fact

† an outer automorphism

a *group of orientation-preserving automorphisms*. We emphasise this setting by using symbol $\mathrm{Aut}^+(\mathbf{M})$. However, we often need to count the map $\mathbf{M}$ and its mirror image $\mathbf{M}^{-1}$ under some reflection as the same object. Reflections have been introduced as automorphisms in the category unoriented maps, but we need to use them in the category of oriented maps as well. The solution is the following: a reflection $\varrho\colon D \to D$ is a permutation $\varrho \in \mathrm{Sym}(D)$ and it is easy to observe that it is an outer automorphism of $\mathbf{M}$ with respect to $\mathrm{Aut}^+(\mathbf{M})$. Hence we introduce $\mathrm{Aut}(\mathbf{M}) = \langle \mathrm{Aut}^+(\mathbf{M}), \varrho \rangle$, a *full group of automorphisms* of $\mathbf{M}$. Transitivity of $\mathrm{Mon}(\mathbf{M})$ on D implies that a extension of $\mathrm{Aut}^+(\mathbf{M})$ by *any* reflection give the same full automorphism group of $\mathbf{M}$.

If $\mathbf{M} = (F; \varrho, \lambda, \tau)$ is unoriented map, then the centraliser of monodromy group in symmetric group over flags, $\mathrm{Aut}(\mathbf{M})$, denotes the full group of automorphisms and $\mathrm{Aut}^+(\mathbf{M}) \cong \langle \varrho\tau, \tau\lambda \rangle$ is even-word subgroup of $\mathrm{Aut}(\mathbf{M})$.

Algorithm 1: Rooted mutual conjugation of lists of permutations

```
Input: x_i, x_j: starting pair of darts – the root; d_1, C_1: the list of permutations
       all of degree d_1; d_2, C_2: the list of permutations all of degree d_2
Output: true, if x_i → x_j extends to a conjugation mapping C_2 = C_1^φ and φ;
        false otherwise
// permutations are arrays of integers of lengths equal to
   their degrees
if d_1 ≠ d_2 then return false, _
l_1 ←Length(C_1)
l_2 ←Length(C_2)
if l_1 ≠ l_2 then return false, _
for i ← 1 to l_1 do
    if Passport(C_1[i])≠ Passport(C_2[i]) then return false, _
// 'Passport' give the multiset of lengths of cycles of a
   permutation C_j[i], j ∈ {1,2}
not_visited ← {i: i = 1,2...,d_1}
phi ← [−1 : i = 1,2...,d_1]
phi[x_i] ← x_j
while Length(not_visited) > 0 do
    dart ← −1
    for c ← 1 to d_1 do
        if phi[c] ≠ −1 and c in not_visited then
            dart ← c
            break
    dart_phi ← phi[dart]
    Remove(~not_visited, dart)
    for i ← 1 to l_1 do
        dart ← C_1[i][dart]          // expression of dart ← dart^C[i]
        dart_phi ← C_2[i][dart_phi]
        if phi[dart] ≠ −1 then
            if phi[dart] ≠ dart_phi then
                return false, _
        else
            phi[dart] ← dart_phi
// the array phi defines the isomorphism M_1 → M_2!
return true, phi
```

2.1.5. Map isomorphism problem. The recognition and the construction of isomorphisms of maps on n darts or the construction of their automorphism groups[†] is one of the essential tasks we meet in the study of those objects. From the first look it seems to be a very difficult problem: we have to find a permutation in certain symmetric group of degree n. It seems like a looking for a needle in a haystack: how to find one permutation among $n!$ of others? The search for an automorphism group seems also to be intractable for maps with hundreds or thousands of darts. Although there exist effective algorithms for computing centralisers of permutation groups in symmetric groups [49, Section 4.6.5], the direct use of them with e.g. Sym(500) simply stuck any computer.

[†] the same principle is used in case of unoriented maps

The isomorphism problem be surprisingly easily solved using combinatorial maps, since we are strongly restricted by definition of that category. Given two maps $\mathbf{M}_1 = (D; R_1, L_1)$ and $\mathbf{M}_2 = (D; R_2, L_2)$ we just need to find a permutation φ which sends the ordered list of permutations $[R_1, L_1]$ to $[R_2, L_2]$. In other words, the map isomorphism problem essentially reduces to the problem to find a mutual conjugation of lists of permutations, see Algorithm 1. A program implementing it tries to extend the prescribed mapping $x_i \mapsto x_j$ to a permutation $\varphi\colon D \to D$ and answers, negatively or positively, in approximate time $O(n^2)$, where n is degree of the permutation representation of $\mathbf{M}_1$, $\mathbf{M}_2$, and φ. This algorithm is a simple implementation of well-known 'union-find' strategy (see e.g. [31]).

Algorithm 2 computes a map isomorphism employing the previous routine essentially, the only thing we have to check is whether pairs of permutations generate transitive groups on respective set of darts. The complexity of isomorphism test can be estimated by $O(n^3)$. The algorithm obtaining the automorphism group of a map also uses Algorithm 1, the only difference is that we have to traverse through all possible combinations of roots†, record all obtained permutations and eventually construct group over the set of obtained permutations. The complexity of the construction of a automorphism group can roughly be estimated by $O(n^4)$. Algorithms treating these problems for unoriented maps work in similar fashion, using appropriate inputs, i.e. lists of involutions, but they are of the same complexity.

Algorithm 2: Isomorphism of maps

Input: d_1, R_1, L_1: the first map, on d_1 darts; d_2, R_2, L_2: the second map, on d_2 darts

Output: $true$ if maps are isomorphic and the isomorphism φ; $false$ otherwise

// permutations are arrays of integers of lengths equal to their degrees

// $|R_i| = |L_i| = d_i, \ \ i \in \{1, 2\}$

if $d_1 \neq d_2$ **then return** $false$, _
if not `IsTransitive(Group(`R_1,L_1`)`,$[1..d_1]$`)` **or**
  not `IsTransitive(Group(`R_2,L_2`)`,$[1..d_1]$`)` **then**
    return $false$, _
for $i \leftarrow 1$ **to** d_1 **do**
    $phi, success \leftarrow$ `ListConjugate(`1, i, d_1, $[R_1,L_1]$, d_2, $[R_2,L_2]$`)` // c.f. Algorithm 1
    if *success* **then return** $true$, phi
return $false$, _

Let us remark that in this section we touched the more general problem – the graph isomorphism problem. Graph isomorphism problem (GIP) is one of the central open problems in the theory of complexity of algorithms. Many authors attributed the development of computational group theory to the efforts to find the solution of graph isomorphism problem‡. The state-of-art is that the complexity of a solution of graph isomorphism problem lies somewhere between polynomial (P) and near-polynomial (NP) algorithms. It is usually accepted, that a solution if GIP has rather near-polynomial solution. In contrast, we have shown that a problem to decide whether two *embeddings of graphs* are isomorphic has polynomial complexity.

† as referred in Algorithm 1 ‡ or, at least to decide, whether it has a polynomial or a near-polynomial solution

2.1.6. Quotient maps. As follows, quotient maps will play a crucial rôle in our considerations. A concise definition of a quotient map is given in the paper by Malnič, Nedela and Škoviera [77, Section 3]. We will rephrase it here. Let $\varphi : \mathbf{M} \to \bar{\mathbf{M}}$, where $\mathbf{M} = (D; R, L)$ and $\bar{\mathbf{M}} = (\bar{D}; \bar{R}, \bar{L})$, be a covering of maps. The fibre transformation group $\mathrm{FT}(\varphi)$ of φ is the group of all automorphisms τ of $\mathbf{M}$ such that the diagram (2.1) commutes.

$$\begin{tikzcd} \mathbf{M}^{\tau} \arrow[r] \arrow[d, "\varphi"'] & \mathbf{M}^{\varphi} \arrow[d] \\ \bar{\mathbf{M}}^{id} \arrow[r] & \bar{\mathbf{M}} \end{tikzcd} \tag{2.1}$$

The covering $\varphi : \mathbf{M} \to \bar{\mathbf{M}}$ is said to be a *regular covering* if $\mathrm{FT}(\varphi)$ acts transitively[†] on each fibre over a dart. As a result we have that a map covering is regular if and only if the size of the fibre transformation group equals to the number of sheets of the covering. Since the monodromy group acts transitively on darts of the map, it is sufficient to check whether the action of $\mathrm{FT}(\varphi)$ is transitive on a single fibre.

Let $\mathbf{M} = (D; R, L)$ be a map. Take a subgroup $\mathrm{G} \le \mathrm{Aut}(\mathbf{M})$ and let $\bar{D} = \{[x]_{\mathrm{G}} \mid x \in D\}$ be the set of orbits of G on D. We then define the *quotient map* $\mathbf{M}/\mathrm{G} = \bar{\mathbf{M}} = (\bar{D}; \bar{R}, \bar{L})$ with set of darts $\bar{D}$ setting $\bar{R}([x]_{\mathrm{G}}) = [R(x)]_{\mathrm{G}}$ and $\bar{L}([x]_{\mathrm{G}}) = [L(x)]_{\mathrm{G}}$. If $[x]_{\mathrm{G}} = [y]_{\mathrm{G}}$, then there is $g \in \mathrm{G}$ such that $g(x) = y$. Since G is a group of automorphisms of $\mathbf{M}$, it follows that $R(y) = R(g(x)) = R \circ g(x) = g \circ R(x) = g(R(x))$. From the definition of $\bar{R}$ we have $\bar{R}([y]_{\mathrm{G}}) = [R(y)]_{\mathrm{G}} = [g(R(x))]_{\mathrm{G}} = [R(x)]_{\mathrm{G}} = \bar{R}([x]_{\mathrm{G}})$. The equality $\bar{L}([y]_{\mathrm{G}}) = \bar{L}([x]_{\mathrm{G}})$ is checked in the same manner. Hence $\bar{R}$ and $\bar{L}$ are well-defined permutations of $\bar{D}$. The permutation $\bar{L}$ is clearly an involution and the group $\langle \bar{R}, \bar{L} \rangle$ acts transitively on $\bar{D}$ (from right). Hence $\bar{\mathbf{M}}$ is a well-defined map. It is easy to see that the natural projection $\pi_{\mathrm{G}} = \pi : \mathbf{M} \to \bar{\mathbf{M}}$, $x \mapsto [x]_{\mathrm{G}}$ is a map homomorphism, and moreover a regular map covering. To see this, it is sufficient to observe that any fibre over a dart $[x]_{\mathrm{G}}$ is the orbit $\{g(x) : g \in \mathrm{G}\}$ in the action of G over D and that $\mathrm{FT}(\pi) = \mathrm{G}$. We have the following theorem

Theorem 2.3 ([77, Theorem 3.1]). *A map homomorphism $\varphi : \mathbf{M} \to \bar{\mathbf{M}}$ is regular if and only if it is equivalent to the natural projection $\pi : \mathbf{M} \to \mathbf{M}/\mathrm{FT}(\varphi)$.*

Let us note that in the following we will always consider $\mathrm{FT}(\varphi) = \mathrm{G} \le \mathrm{Aut}^{+}(\mathbf{M})$, i.e. we will factorise maps only by orientation-preserving automorphisms.

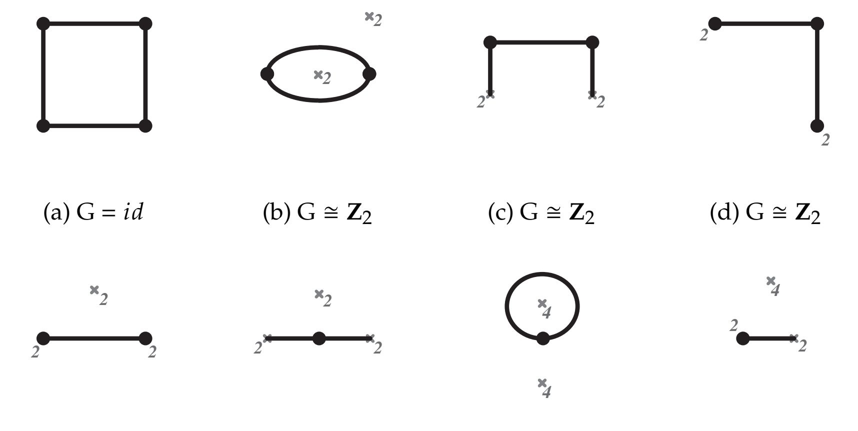
Algorithm 3 summarises the ideas given in this section and constructs a quotient map $\bar{\mathbf{M}} = \mathbf{M}/\mathrm{G}$ from given oriented map $\mathbf{M} = (D; R, L)$ and a subgroup G of its orientation-preserving group of automorphism group. The essence of the algorithm is on creating orbits of darts $[x]_{\mathrm{G}}$ in the left action of G on the dart set D. The orbits represent darts of the quotient map and using definition of quotient map we 'record' the action of R and L obtaining the rotation and the dart-reversing involution of $\bar{\mathbf{M}}$. The algorithm itself consider a map as a combinatorial structure, no information about topological properties are not provided and computed. Any extension considering also topology has to employ this essential routine. We will see the use of Algorithm 3 in the following sections as well.

[†] the action of an automorphism is semi-regular on a fibre, hence the action of $\mathrm{FT}(\varphi)$ is regular on a fibre if it is transitive.

Algorithm 3: Regular quotient map

```
Input: d, R, L: the map M on d darts; G: the subgroup of Aut+(M)
Output: qd, qR, qL: the quotient map
// permutations are arrays of integers of lengths equal to
   their degrees
orbits ←Orbits(G, [1..d])
qd ←Length(orbits)
qR ← [−1 : i = 1, 2, . . . qd]
qL ← [−1 : i = 1, 2, . . . qd]
for i ← 1 to qd do
    qR[i] ←Position(orbits[i]^R, orbits)
    // orbits[i]^R is the action of R on the set - the orbit
    // 'Position' give the index of the element in the list
    qL[i] ←Position(orbits[i]^L, orbits)
return qd, qR, qL
```

2.1.7. Quotient maps and quotient surfaces. We begin with an example; we will examine all quotients of the spherical embedding of the 4-cycle **M**. The map **M** is reflexible, and hence $|\operatorname{Aut}(\mathbf{M}) : \operatorname{Aut}^+(\mathbf{M})| = 1$, so we have to examine all possible automorphisms of **M**, including reflections. It is clear from the Figure 2.1 that automorphisms correspond to well-understood geometrical operations. For example, the quotient map of Figure 2.1b arises as a quotient by 180°-rotation along the centre of the square; the quotient map on Figure 2.1d comes from the factorisation by the diagonal reflection; the quotient map on Figure 2.1e is done by factorising by composition of the two diagonal reflections, and so on.



(a) $G = id$ (b) $G \cong \mathbf{Z}_2$ (c) $G \cong \mathbf{Z}_2$ (d) $G \cong \mathbf{Z}_2$

(e) $G \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ (f) $G \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ (g) $G = \mathbf{Z}_4$ (h) $G = D_4$

Figure 2.1: Quotients of $\mathbf{M} = C_4 \hookrightarrow \mathcal{S}_0$ through all subgroups of $\operatorname{Aut}^+(\mathbf{M})$

As reader already observed, the quotient maps need not to have simple underlying graph, moreover they might have semi-edges. There is another important information displayed in Figure 2.1. On every subfigure we have a set of distinguished points in the sphere (sometimes marked by crosses), endowed with integer parameters. Free ends of semi-edges are always of that kind, and some vertices are also marked in such a way – so that they are of that sort.

From the previous we know that a combinatorial map is in one-to-one correspondence with a topological map [54], in fact this correspondence is a functor between those categories. Automorphisms of combinatorial maps have natural topological counterparts. They are, as expected, self-homeomorphisms of the corresponding surface to which the map is embedded into. A regular quotient of a map $\mathbf{M}$ by a group $\mathrm{G} \leq \mathrm{Aut}^+(\mathbf{M})$, denoted by $\bar{\mathbf{M}} = \mathbf{M}/\mathrm{G}$, is an embedding of the quotient graph into the particular *quotient surface*. These surfaces are called *orientable orbifolds* [29, 28], the name is attributed to W. Thurston and came from 3-dimensional topology [94]. The distinguished points in the orbifold† are called *branch points*. The integer associated with a branch point is called *branch index* of the branch point.

The c-sheeted regular covering of maps, $\kappa\colon \mathbf{M} \to \bar{\mathbf{M}} = \mathbf{M}/\mathrm{G}$ may have fixed points; these points are fixed by automorphisms in G. We usually denote such kind of homomorphisms as (regular) *branched coverings* [43, 55]. Fixed-point-free automorphisms give rise to *smooth coverings* in this setting (cf. Figure 2.1a). The important property of branched coverings over orientable closed surfaces is that the set of branch points in the quotient orbifold O is *always* discrete and finite. The expression $\sigma(O) = (\gamma; \{m_1, m_2, \ldots, m_r\})$ is called the *signature* of the orbifold O, where γ denotes the genus of closed surface O and the second member of the pair $\sigma(O)$ is nothing but the multiset of corresponding branch indices of branch points. For example the orbifold on Figure 2.1a is $(0; -)$, the signature in Figure 2.1g is $(0; \{4, 4\})$, further on Figure 2.1h is $(0; \{2, 2, 4\})$ and so on.

Which quality is marked by values of branch points? Let $\kappa\colon \mathbf{M} \to \bar{\mathbf{M}}$ be a regular c-sheeted cover of maps (and the corresponding surfaces). If κ is smooth, the the order of any fibre $\mathrm{fib}_\kappa(\bar{x})$ over a particular point $\bar{x} \in O$ is equal to c. Let us look at the Figure 2.1f, $\bar{\mathbf{M}} = \mathbf{M}/\mathbf{Z}_2 \times \mathbf{Z}_2$, the corresponding orbifold has signature $(0; \{2, 2, 2\})$ and the regular covering has 4 sheets. The unique vertex $\bar{v}$ of $\bar{\mathbf{M}}$ is covered smoothly so that $|\,\mathrm{fib}_\kappa(\bar{v})| = 4$. This is something expected, all four vertices of C_4 were mapped by κ to $\bar{v}$. Any internal point of a semi-edge, $\bar{x}$ is smoothly covered, hence $\forall \bar{x} \in E(\bar{\mathbf{M}})\colon |\,\mathrm{fib}_\kappa(\bar{x})| = 4$. The situation changes when we look at free ends of semi-edges. Let $\bar{e}$ be the free end of a semi-edge in $\bar{\mathbf{M}}$. It is not so difficult to see that the pre-image of the point $\bar{e}$ contains only 2 distinguished points the centres of opposite edges in C_4. At this moment branch index enters the scenes. If a point $\bar{p}$ in an orbifold (obtained by c-sheeted regular branched covering) is a branch point with non-trivial branch index m, then $|\,\mathrm{fib}_\kappa(\bar{p})| = \frac{c}{m}$. Hence $|\,\mathrm{fib}_\kappa(\bar{e})| = \frac{4}{2} = 2$. The same holds for the remaining free end. The situation is just a little bit more difficult if we consider branch points in faces of $\bar{\mathbf{M}}$. It follows from the theory of Riemann surfaces‡, that any two points in the same face of the map are conformally equivalent. Thus, if one point in a face is a branch point, the every point is a branch point of the same branch index. This is the reason, why we put only one distinguished branch point into the interior of the face. The theory of branched coverings give the same; if r is the internal point of the face with branch point of index n, then $|\,\mathrm{fib}_\kappa(\bar{r})| = \frac{c}{n}$. Hence we have $\bar{r} \in F(\mathbf{M})$, $|\,\mathrm{fib}_\kappa(\bar{r})| = \frac{4}{2} = 2$. From geometrical point of view, it is clear that inner and outer face of $\mathbf{M}\colon C_4 \hookrightarrow \mathcal{S}_0$ are mapped to the unique face of $\bar{\mathbf{M}}$ and every point $\bar{r}$ has two different pre-images in κ. All remaining cases in Figure 2.1 are be examined in the same manner.

At the end, let us show the record of Magma session which gave us all possible quotient maps displayed on Figure 2.1. Branch points (and branch indexes) are certainly not considered in this program, but in is not very difficult to improve the program to count with them.

```
> R:=Sym(8)!(1,2)(3,4)(5,6)(7,8);
> L:=Sym(8)!(1,8)(2,3)(4,5)(6,7);
```

† does not matter, whether they are discrete points, vertexes, or free ends of semi-edges. ‡ it is something taught in a course of complex analysis

```
> M:=OrmapByPermutations(R,L);
> A:=OrmapAutomorphismGroup(M);
> A;  // D_4, as expected

Permutation group A acting on a set of cardinality 8
Order = 8 = 2^3
(1, 2)(3, 8)(4, 7)(5, 6)
(1, 3, 5, 7)(2, 4, 6, 8)

> S:=Subgroups(A);
> for elt in S do
for> qM:=OrmapRegularQuotient(M,elt`subgroup);
for> "+++";
for> "R = ",qM`R;
for> "L = ",qM`L;
for> "|A:G| = ",Index(A,elt`subgroup)," G = ",IdentifyGroup(elt`subgroup);
for> end for;
+++
R =  (1, 2)(3, 4)(5, 6)(7, 8)
L =  (1, 8)(2, 3)(4, 5)(6, 7)
|A:G| =  8  G =  <1, 1>     // G ~ Id, Fig. 2.1a
+++
R =  (1, 2)(3, 4)
L =  (1, 4)(2, 3)
|A:G| =  4  G =  <2, 1>     // G ~ Z_2, Fig. 2.1b
+++
R =  (1, 2)(3, 4)
L =  (1, 3)
|A:G| =  4  G =  <2, 1>     // G ~ Z_2, Fig. 2.1c
+++
R =  (2, 3)
L =  (1, 2)(3, 4)
|A:G| =  4  G =  <2, 1>     // G ~ Z_2, Fig. 2.1d
+++
R =  Id($)
L =  (1, 2)
|A:G| =  2  G =  <4, 2>     // G ~ Z_2 x Z_2, Fig. 2.1e
+++
R =  (1, 2)
L =  Id($)
|A:G| =  2  G =  <4, 2>     // G ~ Z_2 x Z_2, Fig. 2.1f
+++
R =  (1, 2)
L =  (1, 2)
|A:G| =  2  G =  <4, 1>     // G ~ Z_4, Fig. 2.1g
+++
R =  Id($)
L =  Id($)
|A:G| =  1  G =  <8, 3>     // G ~ D_4, Fig. 2.1h
```

2.1.8. Euler-Poincaré equation for quotient maps. Let us have a c-sheeted regular covering of maps $\kappa\colon \mathbf{M} \to \bar{\mathbf{M}} = \mathbf{M}/\mathrm{G}$, $\mathrm{G} \leq \mathrm{Aut}^{+}(\mathbf{M})$, $c = |\mathrm{G}|$. As it was mentioned before, κ is a branched covering†. It is easy to derive the relationship between the Euler characteristic of $\mathbf{M}$ and the Euler characteristic of $\bar{\mathbf{M}}$. First, recall that the size of any fibre over a dart, $[x]_{\mathrm{G}} = \bar{x} \in \bar{D}$, is equal to c. The cycle of R‡, call it $v(x)$, corresponding to the initial vertex of x, is mapped onto the cycle of $\bar{R}$, namely $v(\bar{x})$. Since π is a branched covering then for any dart $x \in D$, $|v(x)| \geq |(v(\bar{x})|$, but since π is also a regular mapping of permutations, we have

$$|v(x)| = m.|v(\bar{x})|, \quad \text{where } m \mid c, \text{ see e.g [87].} \tag{2.2}$$

† smooth coverings are considered as a special case of branched coverings. ‡ the orbit of a dart x by R

The definition of $\bar{\mathbf{M}}$ says that $\bar{R}([x]_G) = [R(x)]_G$. Hence the fibre over the vertex $\bar{v}$ of the quotient map consists of disjoint union of cycles of R of the same length such that the equation (2.2) holds. Hence, we can express the number of vertices of the map $\mathbf{M}$ as

$$V(\mathbf{M}) = c\left[V(\bar{\mathbf{M}}) - \mathfrak{v} + \sum_{i=1}^{\mathfrak{v}} \frac{1}{m_i}\right], \tag{2.3}$$

where $\mathfrak{v}$ is the number of vertices in $\bar{\mathbf{M}}$ which are not smoothly-covered. Since the faces of a map $\mathbf{M} = (D; R, L)$ correspond to the cycles of RL, using similar argument as before we have

$$F(\mathbf{M}) = c\left[F(\bar{\mathbf{M}}) - \mathfrak{f} + \sum_{j=1}^{\mathfrak{f}} \frac{1}{m_j}\right], \tag{2.4}$$

where $\mathfrak{f}$ is the number of non-smoothly covered faces in $\bar{\mathbf{M}}$.

Some edges of the map $\mathbf{M}$ may be 'folded' in the projection π; hence they are mapped onto semi-edges. Let us have $\mathfrak{s}$ semi-edges in $\bar{\mathbf{M}}$ such that their preimages in π are edges[†] in $\mathbf{M}$. The number of edges of $\mathbf{M}$ is then expressed as

$$E(\mathbf{M}) = c\left[E(\bar{\mathbf{M}}) + \frac{\mathfrak{s}}{2}\right]. \tag{2.5}$$

From Theorem 2.1 and the equations (2.3), (2.4), and (2.5) we have

$$\begin{aligned}
\chi(\mathbf{M}) &= V(\mathbf{M}) - E(\mathbf{M}) + F(\mathbf{M}) \\
&= c\left[V(\bar{\mathbf{M}}) - E(\bar{\mathbf{M}}) + F(\bar{\mathbf{M}}) - \mathfrak{v} - \mathfrak{f} + \sum_{i=1}^{\mathfrak{v}} \frac{1}{m_i} - \frac{\mathfrak{s}}{2} + \sum_{j=1}^{\mathfrak{f}} \frac{1}{m_j}\right] \\
&= c\left[\chi(\bar{\mathbf{M}}) - \mathfrak{v} - \mathfrak{s} - \mathfrak{f} + \sum_{i=1}^{\mathfrak{v}} \frac{1}{m_i} + \sum_{k=1}^{\mathfrak{s}} \frac{1}{2} + \sum_{j=1}^{\mathfrak{f}} \frac{1}{m_j}\right].
\end{aligned}$$

Recall that $\pi\colon \mathbf{M} \to \bar{\mathbf{M}} = \mathbf{M}/G$ is a regular map covering with $c = |G|$ sheets. Setting $r = \mathfrak{v} + \mathfrak{s} + \mathfrak{f}$ and regrouping the sums of 'reciprocals' we can rewrite the previous equation to the form

$$\chi(\mathbf{M}) = |G|\left[\chi(\bar{\mathbf{M}}) - \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right]. \tag{2.6}$$

2.1.9. Maps on orbifolds. Let G be a finite group acting by homeomorphisms on an orientable closed surface $\mathcal{S}_g$ of genus g. Then we can form a quotient surface $\mathcal{S}_g/G$ which is known to be homeomorphic to an orbifold O with signature $(\gamma; \{m_1, m_2, \dots, m_r\})$ of genus $\gamma \le g$. For more complete information we refer to [37, 55].

Let G be a finite group acting on a surface of genus $g \ge 2$. Let $O = \mathcal{S}_g/G$ be the quotient orbifold. By a theorem of Koebe [102] there is a universal orbifold $\widetilde{\mathcal{S}} = \mathbb{H}$, the upper-half complex plane, such that $\Gamma \cong \pi_1(O)$ acts on $\mathbb{H}$ as a discrete group of automorphisms (self-homeomorphisms). Moreover, there is a regular covering $\mathbb{H} \to \mathcal{S}_g$ with a group of covering transformations $K \trianglelefteq \Gamma$ such that $G = \Gamma/K$. Hence the group G is a quotient of $\pi_1(O)$ by some torsion-free normal subgroup K of finite index $|G|$.

A map $\mathbf{M}$ on an orbifold $O(g'; \{m_1, m_2, \dots, m_r\})$ is a map on $S_{g'}$ such that neither a face nor an edge contains more than one branch point. The free end of a semi-edge

[†] links or loops

is either non-singular, or a branch point of index two. A vertex may, or may not, be a singular point. It follows that a map on an orbifold gives rise to a function

$$b : V(\mathbf{M}) \cup E(\mathbf{M}) \cup F(\mathbf{M}) \to \{1, m_1, m_2, \dots, m_r\},$$

taking values $m_i \geq 2$ for some $x \in V(\mathbf{M}) \cup E(\mathbf{M}) \cup F(\mathbf{M})$ if either x is a vertex of branch index m_i; or x is a face containing a branch point of index m_i; or x is a semi-edge and its free end is a branch point of index $m_i = 2$. In all other cases $b(x) = 1$. Vice-versa, a pair $(\mathbf{M}, b)$, where $\mathbf{M}$ is a map on $S_{g'}$ and $b\colon V(\mathbf{M}) \cup E(\mathbf{M}) \cup F(\mathbf{M}) \to \{1, m_1, m_2, \dots, m_r\}$, is a function satisfying the above conditions, determines a map on the orbifold $O = (g'; \{m_1, m_2, \dots, m_r\})$.

2.1.10. Homomorphisms of maps on orbifolds. Given a pair of maps $\mathbf{M}$, $\mathbf{N}$ on an orbifold, we define a homomorphism of the maps $\phi : \mathbf{M} \to \mathbf{N}$ as follows. The restriction of the mapping ϕ on a map $\mathbf{M}$ is the map homomorphism $\phi|_{\mathbf{M}} \colon \mathbf{M} \to \mathbf{N}$. Recall that ϕ is a homeomorphism of the underlying surface of both maps, $\mathbf{M}$ and $\mathbf{N}$. The image of a branch point is a branch point and ϕ preserves the associated branch indexes, i.e. $\phi \circ b = b \circ \phi$. The general situation, i.e. coverings of maps on (possibly different) orbifolds, is much more complicated and its definition is not needed for our purposes. Just note that such covering must be a branched covering of maps which can be extended to a branched covering of the respective orbifolds.

2.1.11. Lifting of maps. Birth of topological graph theory is closely related with the effort to find a solution of the Heawood map colouring problem. The solution is done due to Ringel and Youngs [86]. In particular, the solution is obtained first by reducing the problem to the problem of determining of genus† of the complete graphs. To describe all triangular or near-triangular embedding of complete graphs one needs to develop a combinatorial theory of graph covers. The ideas of Ringel and Youngs were distilled and generalised in the well-known monograph by Gross and Tucker [43]. Unfortunately, the classical theory of graph and map covers does not count with the case of branched map coverings [76, 77, 98] with branch points in vertices. The following definition [64] fill in this gap, incorporating known techniques of voltage-assignments on angles of a map [77].

A walk W in a map is a sequence

$$W := v_0^{e_0} x_0 v_1^{e_1} x_1 \dots v_k^{e_k} x_k v_{k+1}^{e_{k+1}},$$

where the darts x_i and $L(x_{i-1})$ originate at the same vertex v_i, for $i = 1, 2, \dots k$, x_0 originates at v_0 and the exponents e_i are integers for $i = 0, 1, \dots, k$. The image $\psi(W)$ of W in an orientation preserving map automorphism ψ is a walk $w_0^{e_0} y_0 w_1^{e_1} y_1 \dots w_k^{e_k} y_k w_{k+1}^{e_{k+1}}$, where $w_i = \psi(v_i)$ and $y_i = \psi(x_i)$ for $i = 0, 1, \dots, k+1$. If ψ is an orientation reversing automorphism, then all the exponents e_i are in the image multiplied by -1. In what follows we make an agreement that v^0 will be omitted from a sequence determining a walk, and v^1 for a vertex v will be replaced just by v.

Given a quotient map $\bar{\mathbf{M}} = (D, R, L)$ and a group G we can reconstruct each map $\mathbf{M}$ such that $\bar{\mathbf{M}} \cong \mathbf{M}/\mathrm{G}$ employing the idea of voltage assignments as follows (see [76, 77, 98]). Let T be a spanning tree of the underlying graph $\mathfrak{G}$ of $\bar{\mathbf{M}}$ with one distinguished dart x_0, based at a vertex v_0, which will be called the *root*. Clearly, for every vertex $v \in \mathfrak{G}$ there exist a unique dart based at v on a shortest path in T joining $v \neq v_0$ to v_0. Form a set $D^+(T)$ as follows. By definition set $x_0 \in D^+(T)$. For a dart x set $x \in D^+(T)$ if x is a dart on the unique shortest path on T joining a vertex v to the root. Observe that for each vertex v there is exactly one dart in $D^+(T)$ originating at v.

By a *T-reduced voltage assignment* on $\mathbf{N}$ we mean a mapping $\xi : D \cup V \to \mathrm{G}$ taking values in a group G satisfying the following conditions:

† genera of all orientable embeddings

1) all darts in the rooted spanning tree $D^+(T) = (T, x_0)$ receive trivial voltages,
2) $\xi_{xL} = \xi_x^{-1}$ for all $x \in D$,
3) $\mathrm{G} = \langle\{\xi_x : x \in D \cup V\}\rangle$.

The *derived map* $\mathbf{M} = \mathbf{N}^\xi = (D^\xi, R^\xi, L^\xi)$ is defined as follows: $D^\xi = D \times \mathrm{G}$ and

$$(x,g)R^\xi = \begin{cases} (xR, g \cdot \xi_v), & x \in D^+(T), \\ \\ (xR, g), & \text{otherwise} \end{cases} \tag{2.7}$$

$$(x,g)L^\xi = (xL, g \cdot \xi_x) \tag{2.8}$$

If $\xi|_V = id$, then $\mathbf{N}^\xi$ coincides with the classic construction by Gross & Tucker [43]. It is easy to see that the *natural projection* $\pi_\xi : \mathbf{N}^\xi \to \mathbf{N}$, erasing the second coordinate, is a map covering. Observe that for each element $a \in \mathrm{G}$ the mapping $\psi_a : (x,g) \mapsto (x, ag)$ is a fibre-preserving automorphism of $\mathbf{N}^\xi$ and that the group $\widetilde{\mathrm{G}} = \{\psi_a;\ a \in \mathrm{G}\}$ is isomorphic to G. Moreover, the projection $\pi_{\widetilde{\mathrm{G}}} : \mathbf{M} \to \mathbf{M}/\widetilde{\mathrm{G}} = \bar{\mathbf{M}}$ is clearly equivalent to π_ξ. Therefore, π_ξ is a regular map homomorphism. The converse holds as well, see [77, Theorem 5.1] for the special case when $\xi(v) = id$ for each vertex v of $\bar{\mathbf{M}}$. Observe that a T-reduced voltage assignment defined on $\mathbf{N}$ naturally extends from vertices and darts onto walks by setting

$$\xi_W = \xi(v_0^{e_0} x_0 v_1^{e_1} x_1 \dots v_k^{e_k} x_k v_{k+1}^{e_{k+1}}) = \xi_{v_0}^{e_0} \prod_{i=0}^{k} \xi_{x_i} \xi_{v_{i+1}}^{e_{i+1}}.$$

Given a T-reduced voltage assignment on a map $\mathbf{N}$ on a surface $\mathcal{S}_{g'}$ implicitly determines branch points at vertices and centres of faces. The index of a vertex v is the order of an element assigned to v and the index of a face is the order of the voltage of the boundary walk.

Example 2.4. Let us construct the embedding of the Petersen graph of genus 2 through voltage assignment in $\mathbf{Z}_5$ on the spherical embedding of the 3-dipole. The vertices obtain the trivial voltages in the voltage group $\mathbf{Z}_5$, the spanning tree is the edge corresponding to the orbit $\{3,4\}$ of $\bar{L}$ and obtain the the trivial voltage (see Figure 2.2). The dart 1 obtain the voltage equal to the generator of $\mathbf{Z}_5$, while the dart 2 obtain the voltage equal to the square of voltage of the dart 1. The corresponding map and its underlying graph is displayed in the following Magma snippet.

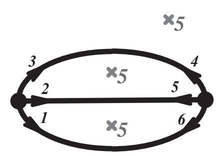


Figure 2.2: The quotient of the Petersen graph on $(0; \{5,5,5\})$

```
> C5 := PermutationGroup<5 | Sym(5)!(1,2,3,4,5)>;
        // the covering transformations group
> e := Id(C5);x := C5.1;
        // the identity and the generator of C5
> RQ := Sym(6)!(1,2,3)(4,5,6);
> LQ:= Sym(6)!(1,2)(3,4)(5,6);
```

```
> MQ := OrmapByPermutations(RQ,LQ);
> va := [x,x^-1,e,e,x^2,x^-2];
       // the voltage assignment to the darts of MQ
> _,L := OrmapLift(MQ,C5,va : Dplus := [3,4]);
> L;
  // the derived map MQ^va
rec<OrmapType |
    R := (1, 6, 11)(2, 7, 12)(3, 8, 13)(4, 9, 14)(5, 10, 15)(16, 21, 26)
        (17, 22, 27)(18, 23, 28)(19, 24, 29)(20, 25, 30),
    L := (1, 7)(2, 9)(3, 6)(4, 10)(5, 8)(11, 16)(12, 17)(13, 18)(14, 19)
        (15, 20)(21, 29)(22, 30)(23, 27)(24, 28)(25, 26),
    F := (1, 3, 5, 4, 2)(6, 16, 29, 14, 10, 20, 26, 11, 7, 17, 30, 15, 8,
        18, 27, 12, 9, 19, 28, 13)(21, 25, 22, 23, 24),
    d := 30,
    v := 10,
    e := 15,
    s := 0,
    f := 3,
    xi := -2,
    g := 2>

> GG := OrmapUnderlyingGph(L);
> GG;
  // ... and its underlying graph
Multigraph
Vertex  Neighbours

1       6 3 2 ;
2       7 4 1 ;
3       8 5 1 ;
4       9 5 2 ;
5       10 3 4 ;
6       10 9 1 ;
7       8 10 2 ;
8       9 7 3 ;
9       8 6 4 ;
10      6 7 5 ;
```

As the second example let us consider the toroidal embedding of $K_{3,3}$ lifted from the spherical embedding of K_2 through voltages from $\mathbf{Z}_3 \times \mathbf{Z}_3 = \mathrm{G}$. The unique edge is the spanning tree of $\bar{\mathbf{M}}$, hence all darts obtain the trivial voltage – identity of G. The vertices have non-trivial voltages, the generators of factors of the direct product. The following Magma snippet contains the result, including the underlying graph.

```
> MQ:=OrmapByPermutations(Id(Sym(2)),Sym(2)!(1,2));
> G:=PermutationGroup<6 | Sym(6)!(1,2,3), Sym(6)!(4,5,6)>;
> x:=G.1;y:=G.2; e:=Id(G);
> _,M:=OrmapLift(MQ,G,[e,e] : vva := [x,y], Dplus := [1,2]);
> M;
rec<OrmapType |
    R := (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 13, 16)(11, 14, 17)(12, 15, 18),
    L := (1, 10)(2, 11)(3, 12)(4, 13)(5, 14)(6, 15)(7, 16)(8, 17)(9, 18),
    F := (1, 11, 5, 15, 9, 16)(2, 12, 6, 13, 7, 17)(3, 10, 4, 14, 8, 18),
    d := 18,
    v := 6,
    e := 9,
    s := 0,
    f := 3,
    xi := 0,
    g := 1>

> L:=OrmapUnderlyingGph(M);
```

```
> L;
Multigraph
Vertex  Neighbours

1       6 5 4 ;
2       6 5 4 ;
3       6 5 4 ;
4       3 2 1 ;
5       3 2 1 ;
6       3 2 1 ;
```

2.2 Generalisation to hypermaps, Walsh map of a hypermap

2.2.1. Hypermaps. A *topological hypermap* **H** is a cellular embedding of a connected trivalent graph X into a closed surface $\mathcal{S}$ such that the 2-cells are 3-coloured (say by black, grey and white colours) with adjacent cells having different colours. Let us denote colours by numbers 0, 1 and 2, and label the edges of X with the missing adjacent cell number. We can define 3 fixed-point-free involutory permutations r_i, $i = 0$, 1, 2, on the set F of vertices of X; each r_i switches the pairs of vertices connected by i-edges (edges labelled i). The elements of F are called *flags* and the group G generated by r_0, r_1 and r_2 will be called the *monodromy group*[†] Mon(**H**) of the hypermap **H**. The cells of **H** coloured 0, 1, and 2 are called the *hypervertices*, *hyperedges*, and *hyperfaces*, respectively. Since the graph X is connected, the monodromy group acts transitively on F and orbits of $\langle r_0, r_1\rangle$, $\langle r_1, r_2\rangle$ or $\langle r_0, r_2\rangle$ on F determine hyperfaces, hypervertices and hyperedges, respectively. The orders of the elements $k = |r_0r_1|$, $m = |r_1r_2|$, and $n = |r_2r_0|$ are least common multiples of the valencies of hyperfaces, hypervertices, and hyperedges, respectively. The triple (k, m, n) is called the *type* of the hypermap.

Maps correspond to hypermaps satisfying condition $(r_0r_2)^2 = 1$, or in other words, maps are hypermaps of type $(p, q, 2)$ or of type $(p, p, 1)$. Thus we can view the category of maps as a subcategory of the category of hypermaps. A *hypermap* can be combinatorially (algebraically) described in terms of a 4-tuple $(F;\ r_0, r_1, r_2)$, where r_i $(i = 0, 1, 2)$ are the fixed-points-free involutory permutations generating the monodromy group Mon(**H**) acting transitively on F. Similarly, the category of oriented hypermaps arises by relaxing the condition $L^2 = 1$ in the definition of an oriented map. More precisely, an *oriented hypermap* is a 3-tuple $(D;\ R, L)$, where R and L are permutations acting on D such that the oriented monodromy group is transitive on D. The notions defined in the previous section extend from maps to hypermaps in an obvious way. For more information on hypermaps the reader is referred to [30].

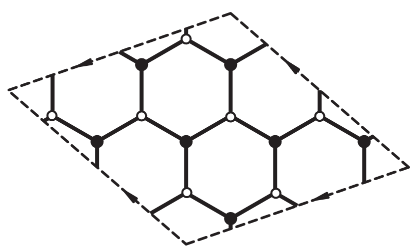


Figure 2.3: Walsh map of the Fano plane embedding in torus

[†] This group has been called the monodromy group [57, 82], the connection group [101], the Ω-group [13], and the cartographic group [69].

*2.2.2. **Walsh representation of hypermaps.*** An important and convenient way to visualise hypermaps is done by employing bipartite maps and it is introduced by Walsh in [99]. Topologically, a map can be seen as a cellular embedding of a graph in a closed surface and a hypermap as a cellular embedding of a hypergarph in a closed surface. Since hypergarphs are in a sense bipartite graphs[†] a hypermap can be viewed as a bipartite map. In fact, for any topological hypermap **H** we can construct a topological bipartite map $W(\mathbf{H})$, called the Walsh bipartite map associated to **H** by taking first the dual of the underlying 3-valent map and then deleting the vertices (together with the edges attached to them) lying inside the hyperfaces of **H**. The resulting map is bipartite with one monochromatic set of vertices lying on the faces coloured black, representing the hypervertices of **H**, and the other monochromatic set lying on the faces coloured grey, representing the hyperedges. As an example see the toroidal embedding of the Fano plane in Figure 2.3

This construction can be reversed: given any topological bipartite map **B**, where the vertices are bi-partitioned in black and grey, we construct an associated topological hypermap $W^{-1}(\mathbf{B})$=Trunc($\mathbf{B}^*$) by truncating the dual map $\mathbf{B}^*$; the faces of the resulting 3-valent map Trunc($\mathbf{B}^*$) contains the vertices and the face-centres of the original map and are henceforth 3-colourable black, grey and white, with all these colours meeting at each vertex of Trunc($\mathbf{B}^*$). If $\mathbf{B} = W(\mathbf{H})$ is the Walsh bipartite map of an oriented hypermap $\mathbf{H} = (D; R, L)$ then R and L are the respective rotations on the two bipartition sets of the dart set of **B**, so the rotation of **B** is $RL = LR$.

*2.2.3. **External symmetries of hypermaps, dualities.*** In Section 2.1.4 we dealt with reflections of maps and this notion naturally transfers into theory of hypermaps in the same setting. We noticed that an oriented map (hypermap) **M** may be reflexible or chiral and in the latter case the reflection ϱ is an external automorphism giving to rise the (full) group of automorphisms Aut(**M**), such that $|\operatorname{Aut}(\mathbf{M}) : \operatorname{Aut}^+(\mathbf{M})| = 2$. However, a hypermap **M** may have also other symmetries which are not elements of Aut(**M**).

The dual of a map **M** is another map which may, or may not, be isomorphic to **M**. Hence, dual is a functor in category of maps. If a map **M** is self-dual (i.e., isomorphic to its dual), the dual operation can be seen as an external symmetry of **M** swapping faces with vertices.

In the category of hypermaps we (may) have six kinds of *cell-operations* [50, 74], distinguished by the respective permutations of their *cells*, namely the sets of hypervertices (marked by 0), hyperedges (marked by 1), and hyperfaces (marked by ∞). A cell-operation is a functor (operation) on the category of oriented hypermaps, preserving the underlying surface, which is induced by a particular outer automorphism of the free group $\langle r, \ell\rangle$ of rank 2. Cell-operations preserve sets of darts and act on monodromy groups as group automorphisms. As a consequence, each cell-operation can be orientation-preserving or orientation-reversing. Thus, we have 12 kinds of cell-operations in the category of oriented hypermaps.

Let $\sigma^-_{(0,1)}$ be the involutory cell-operation (duality operation) $r \mapsto \ell^{-1}$, $\ell \mapsto r^{-1}$ that transforms the hypermap $\mathbf{H} = (D; R, L)$ to the hypermap $\mathbf{H}^-_{(0,1)} = (D; L^{-1}, R^{-1})$, and let $\sigma^+_{(0,\infty)}$ be the involutory cell-operation sending **H** to $\mathbf{H}^+_{(0,\infty)} = (D; L^{-1}R^{-1}, L)$ corresponding to the outer automorphism determined by $r \mapsto \ell^{-1}r^{-1}$ and $\ell \mapsto \ell$. It is easily seen that the group of cell-operations generated by the dualities $\sigma^-_{(0,1)}$ and $\sigma^+_{(0,\infty)}$ is the dihedral group of order 12; the central involution is the mirror-image operation $(\sigma^-_{(0,1)}\sigma^+_{(0,\infty)})^3$. A permutation of cells and a sign (where '+' means orientation-

[†] with one monochromatic set of vertices representing the hypervertices and the other monochromatic set of vertices representing the hyperedges

preserving) uniquely determines one of the twelve cell-operations. For instance, $\sigma^{-}_{(0,1)}$ is the cell-operation that fixes faces and transposes vertices with edges while the global orientation is reversed.

Let $\alpha \in \mathrm{Sym}(\{0,1,\infty\})$ and $e \in \{+,-\}$. We say that a hypermap is *α^e-self-dual* if $\mathbf{H} \cong \sigma^e_\alpha(\mathbf{H}) = (D; \sigma^e_\alpha(R), \sigma^e_\alpha(L))$. The following table describes all possible cell-operations σ^e_α acting on a hypermap $\mathbf{H} = (D; R, L)$.

α	e	σ^e_α	e	σ^e_α
id		$R \mapsto R,\ L \mapsto L$		$R \mapsto R^{-1},\ L \mapsto L^{-1}$
$(0,1)$		$R \mapsto L,\ L \mapsto R$		$R \mapsto L^{-1},\ L \mapsto R^{-1}$
$(0,\infty)$		$R \mapsto L^{-1}R^{-1},\ L \mapsto L$		$R \mapsto LR,\ L \mapsto L^{-1}$
$(1,\infty)$	$+$	$R \mapsto R,\ L \mapsto L^{-1}R^{-1}$	$-$	$R \mapsto R^{-1},\ L \mapsto LR$
$(0,1,\infty)$		$R \mapsto L,\ L \mapsto R^{-1}L^{-1}$		$R \mapsto L^{-1},\ L \mapsto RL$
$(0,\infty,1)$		$R \mapsto R^{-1}L^{-1},\ L \mapsto R$		$R \mapsto RL,\ L \mapsto R^{-1}$

Table 2.1: The 12 possible cell-operations.

Let $\mathbf{H} = (D; R, L)$ be an oriented hypermap with $\mathrm{G} = \mathrm{Aut}^+(\mathbf{H}) \cong \langle R, L\rangle$. Clearly, every α^e-self-duality of $\mathbf{H}$ induces an automorphism of G. Therefore we can form an *extended group of automorphisms* $\mathrm{Aut}^*(\mathbf{H})$ of the hypermap $\mathbf{H}$, generated by G and all reflections and self-dualities of $\mathbf{H}$. There is a natural homomorphism from $\mathrm{Aut}^*(\mathbf{H})$ into the dihedral group of order 12 with kernel G. For more details on maps and hypermaps we refer to [32, 54].

2.3 Maps, hypermaps and groups

2.3.1. Schreier representations. In the previous text we have seen that maps and hypermaps can be represented by means of two or three permutations satisfying some conditions. Our aim now is to show that one can study (hyper)maps as purely group-theoretical objects. The idea emerges from the fact that every transitive permutation group is equivalent to a group acting on cosets by translation. Following [96, 97], we call these representations Schreier representations.

Schreier representations of oriented maps appear implicitly in Jones and Singerman [54]. Vince [96] developed a theory of Schreier representations of (hyper)maps on closed surfaces described by three involutions. Here we introduce Schreier representations of oriented hypermaps.

Let G be a finite group generated by two elements r and ℓ. In other words, G is a finite quotient of some *triangle group* $\Delta^+(k,m,n) = \langle r, \ell \mid \ell^n = r^m = (r\ell)^k = 1\rangle$, k,m and n being positive integers. Further, let S be a subgroup of G. The action of G on the set $C = \mathrm{G}/\mathrm{S}$ of right cosets of S in G by the right translation determines a hypermap $\mathbf{A}(\mathrm{G}/\mathrm{S}; r, \ell)$ whose monodromy group is a homomorphic image of G and the local monodromy group is a homomorphic image of S. We take the cosets as darts of the hypermap and define the rotation R and the dart reversing involution L by setting

$$\begin{aligned} R(\mathrm{S}h) &= \mathrm{S}hr, \\ L(\mathrm{S}h) &= \mathrm{S}h\ell, \end{aligned} \tag{2.9}$$

respectively, $\mathrm{S}h$ being an arbitrary element of C. For the resulting hypermap $(C; R, L) = \mathbf{A}(\mathrm{G}/\mathrm{S}; r, \ell)$ we easily check that the assignment $r \mapsto R$, $\ell \mapsto L$ extends to a homomorphism $\Delta^+(k,m,n) \to \mathrm{Mon}(\mathbf{A}(\mathrm{G}/\mathrm{S}; r, \ell))$.

A *Schreier representation* of an oriented hypermap $\mathbf{H}$ is an isomorphism $\mathbf{H} \to \mathbf{A}(\mathrm{G}/\mathrm{S}; r, \ell)$ for an appropriate group $\mathrm{G} = \langle r, \ell \rangle$ and a subgroup $\mathrm{S} \leq \mathrm{G}$, or simply the hypermap $\mathbf{A}(\mathrm{G}/\mathrm{S}; r, \ell)$ itself. Given any hypermap $\mathbf{H} = (D; R, L)$, it is not difficult to find a Schreier representation for $\mathbf{H}$. Indeed, we first fix a dart a of $\mathbf{H}$ and set $\mathrm{G} = \mathrm{Mon}(\mathbf{H}) = \langle R, L \rangle$ and $\mathrm{S} = \mathrm{Mon}(\mathbf{H}, a)$, to be the stabiliser of the dart a. Then, for an arbitrary dart x we take any element $h \in \mathrm{Mon}(\mathbf{H})$ with $h(a) = x$ and label x by the coset $\mathrm{S}h \in C$, thereby obtaining a labelling $\alpha(x) = \mathrm{S}h$. Observe that α is well-defined since for any two elements h and h' of $\mathrm{Mon}(\mathbf{M})$ with $h(a) = x = h'(a)$ we have $\mathrm{S}h = \mathrm{S}h'$. In fact, α is a bijection of $D(\mathbf{H})$ onto C. Clearly, $\alpha(R(x)) = R\alpha(x)$ and $\alpha(L(x)) = L\alpha(x)$ which means that $\alpha\colon \mathbf{H} \to \mathbf{A}(\mathrm{G}/\mathrm{S}; R, L)$ is the required isomorphism.

If we start from a given hypermap $\mathbf{H}$, the Schreier representation we have just described is in some sense best possible because the monodromy group $\mathrm{Mon}(\mathbf{H})$ is not merely a homomorphic image of G but is actually isomorphic to it. In this case we say that the Schreier representation is *effective*. In general, a Schreier representation $\mathbf{A}(\mathrm{G}/\mathrm{S}; r, \ell)$ is effective if and only if G acts faithfully on C, i.e. when the translation by every non-identity element of G is a non-identity permutation of C. Elementary theory of group actions or straightforward computations yield that the latter occurs precisely when the subgroup $\bigcap_{h \in \mathrm{G}} h^{-1}\mathrm{S}h$, the *core* of S in G, is trivial [87].

For an arbitrary Schreier representation $\mathbf{A}(\mathrm{G}/\mathrm{S}; r, \ell)$ of a hypermap $\mathbf{H}$ we have $\mathrm{Aut}(\mathbf{H}) \cong N_{\mathrm{G}}(\mathrm{S})/\mathrm{S}$, where $N_{\mathrm{G}}(\mathrm{S})$ is the normaliser of S in G (see [32, Proposition 3.7] and also [35, Theorem 4.1A]). In particular, if $\mathbf{H}$ is orientably regular we can take $\mathrm{G} = \mathrm{Mon}(\mathbf{H})$ and $\mathrm{S} = id$. Then $\mathrm{Aut}((\mathbf{A}(\mathrm{G}/id;\, r, \ell)) \cong \mathrm{G} \cong \mathrm{Mon}(\mathrm{A}(\mathrm{G}/id;\, r, \ell))$, implying that $\mathrm{Aut}(\mathbf{H}) \cong \mathrm{Mon}(\mathbf{H})$. Let us remark that the above isomorphism assigns the left translation by an element $h \in \mathrm{G}$ (representing a monodromy of $\mathbf{H}$) to the right translation ξ_h (representing an automorphism of $\mathbf{H}$). Summing up we get the following theorem.

Theorem 2.5. *Let* $\mathbf{H} = (D; R, L)$ *be an oriented hypermap. Then* $|\mathrm{Aut}(\mathbf{H})| \leq |D| \leq |\mathrm{Mon}(\mathbf{H})|$ *and the following conditions are equivalent:*

- $\mathbf{H}$ *is regular,*
- $\mathrm{Mon}(\mathbf{H}) \cong \mathrm{Aut}(\mathbf{H})$,
- *the action of* $\mathrm{Aut}(\mathbf{H})$ *on* D *is regular.*

The characterisation of regular unoriented hypermaps can be obtained in similar manner; by replacing darts by flags in the claim of Theorem 2.5.

Schreier representations provide a convenient tool to deal not only with automorphisms but also with homomorphisms between hypermaps. If

$$\mathrm{G} = \langle r, \ell \mid \ell^n = r^m = (r\ell)^k = 1, \dots \rangle$$

is a finite quotient of the triangle group $\Delta^+(k, m, n)$ and $\mathrm{S} \leq \mathrm{S}' \leq \mathrm{G}$ are two subgroups then the natural projection $\pi\colon \mathrm{G}/\mathrm{S} \to \mathrm{G}/\mathrm{S}'$, $\mathrm{S}h \mapsto \mathrm{S}'h$ for $h \in \mathrm{G}$, is a homomorphism $\mathbf{A}(\mathrm{G}/\mathrm{S}; r, \ell) \to \mathbf{A}(\mathrm{G}/\mathrm{S}'; r, \ell)$. In fact, every hypermap homomorphism $\varphi\colon \mathbf{H}_1 \to \mathbf{H}_2$, where $\mathbf{H}_i = (D_i;\, R_i, L_i)$, is in the usual sense equivalent to an appropriate natural projection.

Example 2.6. In what follows, we construct Schreier representations of all oriented maps on 12 darts. Certainly, we will not show all the possible maps – we shall refer just to the number of non-isomorphic oriented maps in line 4 of the following Magma display. We want to exhibit the method how the problem of construction of maps with given number of darts (flags) can be solved using a computer algebra system. The finitely presented group U with generators r and ℓ, isomorphic to $\mathbf{Z} * \mathbf{Z}_2$ is the *universal oriented map* (see the following text) in our example. Then we simply use search for low-index subgroups of index 12 through all possible subgroups of U giving rise to non-isomorphic maps on 12 darts. Every subgroup S (cf. line 7) is then used to derive Schreier representation of the particular map. The intrinsic command `CosetAction` give two results, the first one is an action homomorphism – in fact the Schreier representation `phi` (cf. lines 8,9) of the map and the second is the monodromy group of the corresponding map (we silently thrown away this result). The Schreier representation is then used to derive the oriented map in terms of $(D; R, L)$.

The rest of the program sorts the maps into two categories: regular (when $S \triangleleft U$) and others. We see that there are 6 regular maps on 12 darts and it is easy to understand what we have obtained: the planar embedding of 6-dipole, one-face embedding of bouquet of circles (loops) of genus 3, the regular map `R2.5` of genus 2 [22], the tetrahedron, the 6-cycle in the sphere and the 12-semi-star.

```
1 > U<r,l> :=Group<r,l | l^2>;
2 > L:=LowIndexSubgroups(U,<12,12>);
3 > #L,"\n";
4 90033
5
6 > maps:=[];regmaps:=[];
7 > for S in L do
8 for> phi:=CosetAction(U,S);
9 for> M:=OrmapByPermutations(phi(r),phi(l));
10 for> if IsNormal(U,S) then
11 for|if> Append(~regmaps,M);
12 for|if> else
13 for|if> Append(~maps,M);
14 for|if> end if;
15 for> end for;
16 > #regmaps,"\n";
17 6
18
19 > i:=1;
20 > for m in regmaps do
21 for> "+++ Map no. ",i,"+++";
22 for> "R = ",m`R;
23 for> "L = ",m`L;
24 for> "g = ",m`g;
25 for> "v = ",m`v,"e =",m`e,"f = ",m`f;
26 for> i+:=1;
27 for> end for;
28
29 +++ Map no.  1 +++
30 R =  (1, 2, 5, 9, 8, 4)(3, 7, 11, 12, 10, 6)
31 L =  (1, 3)(2, 6)(4, 7)(5, 10)(8, 11)(9, 12)
32 g =  0
33 v =  2 e = 6 f =  6
34 +++ Map no.  2 +++
35 R =  (1, 2, 5, 9, 11, 7, 3, 6, 10, 12, 8, 4)
36 L =  (1, 3)(2, 6)(4, 7)(5, 10)(8, 11)(9, 12)
37 g =  3
38 v =  1 e = 6 f =  1
39 +++ Map no.  3 +++
40 R =  (1, 2, 5, 9, 8, 4)(3, 6, 10, 12, 11, 7)
41 L =  (1, 3)(2, 6)(4, 7)(5, 10)(8, 11)(9, 12)
42 g =  2
43 v =  2 e = 6 f =  2
44 +++ Map no.  4 +++
45 R =  (1, 2, 4)(3, 6, 7)(5, 9, 10)(8, 12, 11)
46 L =  (1, 3)(2, 5)(4, 8)(6, 11)(7, 9)(10, 12)
47 g =  0
```

```
v =  4 e = 6 f =  4
+++ Map no.  5 +++
R =  (1, 2)(3, 5)(4, 6)(7, 9)(8, 10)(11, 12)
L =  (1, 3)(2, 4)(5, 7)(6, 8)(9, 11)(10, 12)
g =  0
v =  6 e = 6 f =  2
+++ Map no.  6 +++
R =  (1, 2, 4, 6, 8, 10, 12, 11, 9, 7, 5, 3)
L =  Id($)
g =  0
v =  1 e = 0 f =  1
>
```

2.3.2. Generic hypermap. One consequence of these considerations is that every oriented hypermap is a finite quotient of an oriented regular hypermap. In fact, for every oriented hypermap $\mathbf{H} = (D;\, R, L)$ there exists a regular hypermap $\mathbf{H}^{\#}$ and a homomorphism $\pi\colon \mathbf{H}^{\#} \to \mathbf{H}$ with the following universal property: for every hypermap $\widetilde{\mathbf{H}}$ and every homomorphism $\varphi\colon \widetilde{\mathbf{H}} \to \mathbf{H}$ there is a homomorphism $\varphi'\colon \widetilde{\mathbf{H}} \to \mathbf{H}^{\#}$ such that $\varphi = \pi\varphi'$. In terms of Schreier representations, the homomorphism π is equivalent to the natural projection $\mathbf{A}(\mathrm{G}/id;\, R, L) \to \mathbf{A}(\mathrm{G}/S; R, L) \cong \mathbf{H}$ where $\mathrm{G} = \mathrm{Mon}(\mathbf{H})$ and $\mathrm{S} = \mathrm{Mon}(\mathbf{H}, a)$ is the stabiliser of some dart $a \in D(\mathbf{H})$. We shall call the hypermap $\mathbf{H}^{\#}$ the *generic regular hypermap* over $\mathbf{H}$ and $\pi\colon \mathbf{H}^{\#} \to \mathbf{H}$ the *generic homomorphism*. It is obvious that the induced homomorphism $\pi^*\colon\ \mathrm{Mon}(\mathbf{H}^{\#}) \to \mathrm{Mon}(\mathbf{H})$ is an isomorphism and that $\mathbf{H}^{\#}$ and $\mathbf{H}$ have the same type. Observe that $\mathbf{H}^{\#}$ is the smallest regular hypermap covering $\mathbf{H}$.

2.3.3. Maps and hypermaps from triangle groups. The above theory of Schreier representations apply without any problem to infinite hypermaps as well. It follows that oriented maps and hypermaps of given type (k, m, n) can be described as quotients of the *universal oriented hypermap* of type (k, m, n) which (oriented) monodromy group is $\Delta^+(k, m, n)$. This is the even-word subgroup of the triangle group

$$\Delta(k, m, n) = \langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = r_2^2 = (r_0 r_1)^k = (r_1 r_2)^m = (r_2 r_0)^n = 1\rangle,$$

which is the monodromy group of the *universal hypermap* $\mathbf{A}(\Delta(k, m, n); r_0, r_1, r_2)$ for the category of (unoriented) hypermaps of type (k, m, n). Note that the universal maps of type (k, m) with the monodromy group $\Delta(k, m, 2)$ are the well known tessellations of the sphere, plane or hyperbolic plane by k-gons (m of them meeting at each vertex) provided the expression $\frac{1}{k} + \frac{1}{m}$ is greater, equal or less than $\frac{1}{2}$, respectively.

2.3.4. Hypermap subgroups. We can go even one step further. Let us denote by

$$\Delta = \Delta(\infty, \infty, \infty) = \langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = r_2^2 = 1\rangle,$$

the free product of three two-element groups. Since the monodromy group of any hypermap $\mathbf{H}$ is a finite quotient of Δ we can identify every hypermap with the algebraic hypermap $\mathbf{A}(\Delta/\mathrm{S}; r_0, r_1, r_2)$ for some $\mathrm{S} \le \Delta$ of finite index. The subgroup S is called the *hypermap subgroup*. Consequently, one can study hypermaps via the subgroups of Δ of finite index. The facts listed in the following statement are well-known between map- and hypermap experts (see [30, 32]).

Theorem 2.7. *Let* $\mathbf{H}$*,* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *be hypermaps, and let* $\Delta = \langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = r_2^2 = 1\rangle$*.*

1) $\mathbf{H}_1$ *covers* $\mathbf{H}_2$ *if and only if there are* $\mathrm{S}_1 \le \mathrm{S}_2 \le \Delta$ *such that* $\mathbf{H}_1 \cong \mathbf{A}(\Delta/\mathrm{S}_1;\, r_0, r_1, r_2)$ *and* $\mathbf{H}_2 \cong \mathbf{A}(\Delta/\mathrm{S}_2;\, r_0, r_1, r_2)$*,*
2) $\mathbf{H}_1 \cong \mathbf{H}_2$ *if and only if the corresponding hypermap subgroups are conjugate in* G*,*

3) **H** *is orientable if and only if its hypermap subgroup is contained in the even-word subgroup* $\Delta^+ \leq \Delta$,
4) *the hypermap subgroup of the unoriented generic hypermap for a hypermap given by hypermap subgroup* $\mathrm{S} \leq \Delta$ *is the largest normal subgroup contained in* S. *In particular, regular hypermaps correspond to normal subgroups of* Δ.

Using the algebraic representation via hypermap subgroups one can handle many problems. For instance, it is straightforward that given two hypermaps $\mathbf{H}_1$, $\mathbf{H}_2$ with the respective subgroups S_1, S_2, the intersection $\mathrm{S}_1 \cap \mathrm{S}_2$ defines the *smallest common cover* for both $\mathbf{H}_1$ and $\mathbf{H}_2$. Because of many advantages the investigation of maps and hypermaps via the corresponding hypermap subgroups possesses, sometimes a hypermap itself is identified with its hypermap subgroup (see [3, 39]).

2.4 Maps, hypermaps and Riemann surfaces

The aim of this section is to explain briefly the relationship between hypermaps and Riemann surfaces. Following the approach of Jones [57] we firstly show that the underlying surface of any hypermap can be endowed with the structure of a Riemann surface. A natural question arises: What kind of Riemann surfaces are associated with hypermaps? Surprisingly beautiful answer [57] is a consequence of the theorem of Belyĭ [1]. In what follows we have extracted some ideas from [57] where one can find more detailed information as well as an exhaustive list of relevant references.

2.4.1. Riemann surfaces. A *Riemann surface* $\mathcal{S}$ is a second-countable, connected Hausdorff space with an atlas of charts, $\phi\colon U \to V$, here ϕ is a homeomorphism from the open subset $U \subseteq \mathcal{S}$ to V, an open subset of $\mathbb{C}$. For two charts $\phi\colon U \to V$ and $\phi'\colon U' \to V'$, with $U \cap U' \neq \emptyset$ we require that the coordinate transition function $\phi' \circ \phi^{-1}\colon \phi(U \cap U') \to \phi'(U \cap U')$, $z \to z'$ is an analytic function†. The conformality of coordinate transition function means that choice of charts in a neighbourhood of a point $x \in \mathcal{S}$ does not affect the local analysis we do around x. The positive orientation of the complex plane is transferred via charts to the Riemann surface $\mathcal{S}$, hence any Riemann surface is orientable. Two atlases of charts are compatible if the coordinate transition functions from each to the other are analytic. This is an equivalence relation of atlases and the *complex structure* on $\mathcal{S}$ is an equivalence class of compatible atlases on $\mathcal{S}$. A Riemann surface is thus a tuple consisting of a connected Hausdorff space $\mathcal{S}$ and a complex structure on $\mathcal{S}$. In fact, $\mathcal{S}$ may be endowed with many (uncountable many) complex structures. The charts give local coordinates on $\mathcal{S}$, allowing to measure lengths of curves, amplitudes of angles and so on‡. Therefore, any Riemann surface is a Hausdorff space endowed with *geometry*.

An isomorphism $f : \mathcal{S} \to \mathcal{X}$ of Riemann surfaces is a bijection transforming local coordinates analytically. If f is analytic, injective and surjective, the two Riemann surfaces $\mathcal{S}$ and $\mathcal{X}$ are said to be *conformally equivalent*. That is, the geometry on $\mathcal{X}$ is essentially the same as that defined on $\mathcal{S}$. An automorphism of a Riemann surface $\mathcal{S}$ is an isomorphism $\mathcal{S} \to \mathcal{S}$. Automorphisms of Riemann surface $\mathcal{S}$ are orientation-preserving self-homeomorphisms of $\mathcal{S}$ and they form the *automorphism group* $\mathrm{Aut}(\mathcal{S})$ of that surface.

2.4.2. Uniformisation. Every path-connected topological space $\mathcal{S}$ has a simply connected *universal covering space* $\widetilde{\mathcal{S}}$, see e.g. commentary following Theorem 1.38 in Allen Hatcher's book [48]. Moreover, the surface $\mathcal{S}$ can be constructed as the quotient

† analytic functions are infinitely differentiable in a neighbourhood of every point in its domain and equal to its own Taylor series. The name 'holomorphic function' is used often as a synonym. ‡ in fact, we can do complex analysis

surface $\mathcal{S} = \widetilde{\mathcal{S}}/\mathrm{H}$ where H is a discontinuous group acting on $\widetilde{\mathcal{S}}$, isomorphic to the fundamental group $\pi_1(\mathcal{S})$ [78, §8 in Chapter V]. It follows that to classify (closed and connected) topological surfaces it is sufficient to understand

1) the structure of universal covering spaces;
2) the structure of fundamental groups;
3) the quotients $\mathcal{S} = \widetilde{\mathcal{S}}/\mathrm{H}$, where the structure of $\mathcal{S}$ depends on the action of group H on the universal covering space $\widetilde{\mathcal{S}}$.

All three items (questions) are solved problems. Surfaces, as they are 2-manifolds, have just two possible universal covering spaces – the sphere $\mathbb{S}^2$ and the plane $\mathbb{R}^2$ [78, Chapter V]. The structure of fundamental groups are well known [78, Chapter I]. There are two quotients of $\mathbb{S}^2$: namely the sphere $\mathbb{S}^2$ and the projective plane $\mathbb{P}^2$, with the corresponding groups Id and $\mathbf{Z}_2$, respectively. All the other surfaces come as quotients of $\mathbb{R}^2$. Their fundamental groups are $2g$-generated one-relator groups in orientable case, or $\hat{g}$-generated one-relator groups in non-orientable case [78, Theorem 5.1]. By famous Poincarè Theorem, the fundamental group completely determines the surface up to homeomorphisms.

We will employ the same ideas to classify Riemann surfaces. As a gift we will obtain that a (local) geometry, i.e. its Riemann structure, defined on the universal Riemann surface transfers into the (local) geometry of a quotient Riemann surface. Again, it is sufficient to deal with simply connected Riemann surfaces and their automorphism groups. As follows, the first step is miraculously reduced due to Poincarè-Koebe Uniformisation Theorem

Theorem 2.8 ([55, Theorem 4.17.2])**.** *Every simply connected Riemann surface is conformally equivalent to just one of:*

1) *the Riemann sphere* Σ,
2) *the complex plane* $\mathbb{C}$, *or*
3) *the open unit disk* $\mathcal{D} = \{z\colon z \in \mathbb{C} \mid |z| < 1\}$, *homeomorphic to the upper half complex plane* $\mathbb{H}$.

Just note that for sake of convenience we often freely interchange the disk $\mathcal{D}$ by upper-half complex plane $\mathbb{H} = \{z\colon z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. The construction of isomorphism between $\mathcal{D}$ and $\mathbb{H}$ is straightforward. The argument is given in Section 2.8 in [55] and more explicit construction is given in [53, Section 1.3].

It follows from Theorem 2.8 that the classification of (compact) Riemann surfaces reduces to the classification of discontinuous subgroups of the groups of automorphisms of the universal Riemann surfaces. Group of automorphisms of universal Riemann surfaces are: $\mathrm{PSL}(2,\mathbb{C})$ for the Riemann sphere, $\mathrm{AGL}(1,\mathbb{C})$, the affine general linear group, in case of the complex plane $\mathbb{C}$, and $\mathrm{PSL}(2,\mathbb{R})$ for the upper half plane $\mathbb{H}$.

Example 2.9. The universal cover itself give us an information about the genus of the studied surfaces: surfaces of genus $g = 0$ are covered by the Riemann sphere, while toroidal Riemann surfaces ($g = 1$) are covered by the complex plane. The situation is well-understood in both cases.

Let us describe all groups of isometries of the Riemann sphere Σ, acting discontinuously on Σ. They are A_4, S_4, A_5, cyclic, or dihedral groups, both of arbitrarily large order. Full description of this situation can be found e.g. in [37, IV.9.1–IV.9.3] and also in nicely written monograph by Conway et al. [27]. The result confirms and clarifies the ancient Greek classification of Platonic solids.

The quotients of the complex plane by discrete group of isometries of $\mathbb{C}$ give rise to infinitely many quotient surfaces. It is surprising that many information about them were known already in medieval ages[†], but for more precise classification we refer e.g. to [37, IV.9.4–IV.9.6]. All actions of groups on the torus are related to the 17 wallpaper groups [27]. We remark that automorphism groups may be even infinite in toroidal case.

The remaining case, i.e. classification of Riemann surfaces of genus ≥ 2, is a real challenge.

2.4.3. Riemanian maps. Following [54, Section 5], we will introduce a Riemann structure corresponding to a hypermap. Given a hypermap $\mathbf{M}$, it is possible to introduce a Riemann structure (of the underlying surface of $\mathbf{M}$) in two 'equivalent' ways. We can construct a quotient hypermap $\mathbf{M}$ of a universal hypermap $\widetilde{\mathbf{M}}$ (see later), hence the Riemann structure of $\mathbf{M}$ is inherited from the Riemann structure corresponding to $\widetilde{\mathbf{M}}$, defined on the upper-half plane $\mathbb{H}$. The Riemann structure corresponding to $\mathbf{M}$ can be also defined as a lift of the Riemann structure of the Riemann sphere Σ[‡] along a meromorphic function ω, called Belyĭ function (see later). Both approaches has advantages and disadvantages. For example the quotient method is convenient constructing maps, as we did in [62, 63, 64], while the examination of a Belyĭ function give us a tool for detailed structural analysis of the Riemann surface. The result of the paper [14] is based on this idea – we are strongly using Belyĭ functions corresponding to hypermaps.

The upper half-plane is a model of hyperbolic geometry, the geodesics being the Euclidean lines and semi-circles which meet the real line $\mathbb{R}$ at right-angles. The *modular group* $\Gamma = PSL(2, \mathbf{Z})$, consisting of the Möbius transformations

$$T\colon z \mapsto \frac{az+b}{cz+d} \qquad (a, b, c, d \in \mathbf{Z}, \quad ad - bc = 1), \tag{2.10}$$

acts on $\mathbb{H}$ as a group of orientation-preserving hyperbolic isometries. It is well-known (see for instance [57]) that the modular group is of rank 2. Hence it can serve as a universal covering group for the set of monodromy groups of oriented hypermaps which are two-generator groups. In order to get a representation of a hypermap by means of a Riemann surface one has to embed the underlying graph of the certain infinite Walsh bipartite map into $\mathbb{H}$ in the 'right way'. This can be done provided we extend $\mathbb{H}$ as follows. Observe that Γ acts (transitively) on the rational projective line $P^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$, and hence it acts on the extended hyperbolic plane

$$\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}\,.$$

Let us denote by

$$\begin{aligned}
[0] &= \{\,\frac{p}{q} \in \mathbb{Q} \cup \{\infty\} \mid p \text{ is even and } q \text{ is odd}\},\\
[1] &= \{\,\frac{p}{q} \in \mathbb{Q} \cup \{\infty\} \mid p \text{ and } q \text{ are both odd}\},\\
[\infty] &= \{\,\frac{p}{q} \in \mathbb{Q} \cup \{\infty\} \mid p \text{ is odd and } q \text{ is even}\}.
\end{aligned}$$

The subgroup stabilising both [0] and [1] (see equation (2.10)) is

$$\Gamma(2) = \{\, T \in \Gamma \mid b \equiv c \equiv 0 \mod (2)\,\}\,.$$

[†] Alhambra wallpapers from 14th century, also many paintings of M. C. Escher [‡] which is unique

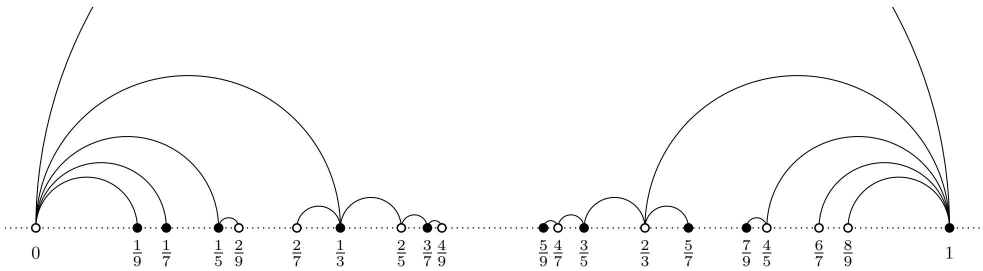


Figure 2.4: Universal Walsh map on $\overline{\mathbb{H}}$.

The *universal (Walsh) bipartite map* $\hat{\mathbf{B}}$ on $\overline{\mathbb{H}}$ has [0] and [1] as its sets of black and white vertices†, and its edges are the hyperbolic geodesics between vertices $\frac{a}{b}$ and $\frac{c}{d}$ where $ad - bc = \pm 1$; this implies that a and c have opposite parity, so the map $\hat{\mathbf{B}}$ is indeed bipartite.

The automorphism group of $\hat{\mathbf{B}}$ preserves the orientation and colours is denoted by the symbol $\Gamma(2)$. It is a free group of rank 2 generated by

$$\hat{R} : z \mapsto \frac{z}{-2z+1} \quad \text{and} \quad \hat{L} : z \mapsto \frac{z-2}{2z-3}\,.$$

It follows that if $\mathbf{B}$ is any bipartite map, representing an oriented hypermap $\mathbf{H}$ with monodromy group $\mathrm{G} = \langle R, L\rangle$, then there is an epimorphism

$$\Gamma(2) \to \mathrm{G}, \quad \hat{R} \mapsto R, \quad \hat{L} \mapsto L\,,$$

giving a transitive action of $\Gamma(2)$ on the set E of edges of $\mathbf{B}$. The map $\mathbf{B}$ is then nothing but the Walsh bipartite map of the hypermap $\mathbf{H}$. The stabiliser of an edge in this action is a subgroup B of index $N = |E|$ in $\Gamma(2)$, called the *map subgroup* corresponding to $\mathbf{B}$ (different choices of an edge lead to conjugate subgroups). Since $\mathrm{B} \leq \Gamma(2) = \mathrm{Aut}(\hat{\mathbf{B}})$, one can form the quotient map $\hat{\mathbf{B}}/\mathrm{B}$, isomorphic to $\mathbf{B}$. In particular, the oriented hypermap $\mathbf{H}$ is regular if and only if B is normal in $\Gamma(2)$, in which case $\mathrm{Aut}(\mathbf{H}) \cong \Gamma(2)/\mathrm{B} \cong \mathrm{G}$. Note that the above map subgroup B can be viewed as an oriented version of the hypermap subgroup introduced above.

Using the above factorisation process we end with an isomorphic copy of our original Walsh bipartite map $\mathbf{B}$, endowed with some extra structure. The underlying surface is now a compact Riemann surface $\mathcal{X} = \overline{\mathbb{H}}/\mathrm{B}$, in which the underlying graph is very rigidly embedded: the edges are all geodesics, the angles between successive edges around a vertex are all equal, and the automorphisms of $\mathbf{B}$ are all conformal automorphisms of $\overline{\mathbb{H}}/\mathrm{B}$ (induced by the action of $N_{\Gamma(2)}(\mathrm{B})$ on $\hat{\mathbf{B}}$).

Clearly, we can represent the trivial bipartite map on the Riemann surface Σ with one black vertex at 0, one white vertex at 1, and the unique edge joining them as the interval [0,1]. Let $\mathbf{B}_1$ denotes this representation. Since any bipartite map covers $\mathbf{B}_1$, alternatively, one can derive the structure of a Riemann surface associated with $\mathbf{B}$ by considering the branched regular covering $\mathbf{B} \cong \hat{\mathbf{B}}/\mathrm{B} \to \mathbf{B}_1 \cong \hat{\mathbf{B}}/\Gamma(2)$ given by the inclusion $\mathrm{B} \leq \Gamma(2)$. This conclusion can be viewed as the reformulation of the famous Koebe Theorem [102] in the particular setting of Fuchsian groups with signature $(0; \{k, m, n\})$ (see also [37, 55]).

2.4.4. Riemann surfaces and algebraic curves. If $A(x, y) \in \mathbb{C}[x, y]$ is a polynomial in x and y with complex coefficients, then the equation $A(x, y) = 0$ defines the complex variable y as an c-valued function of the complex variable x, where c is the degree of A in y. Consequently, the Riemann surface $\mathcal{X}_A$ of this equation can be constructed by

† [∞] are centres of the faces

taking c copies of the Riemann sphere Σ (one for each branch of the function), cutting them between the branch-points, and then rejoining the sheets across these cuts.

A Riemann surface is called *algebraic* if it is isomorphic to $\mathcal{X}_A$ for such a polynomial A. The following major result was known to Riemann:

Theorem 2.10 (Riemann)**.** *A Riemann surface is compact if and only if it is algebraic.*

If K is a subfield of $\mathbb{C}$, then we say that a compact Riemann surface $\mathcal{X}$ is *defined over* K if $\mathcal{X} \cong \mathcal{X}_A$ for some polynomial $A(x, y) \in K[x, y]$. Let $\overline{\mathbb{Q}}$ denotes the field of algebraic numbers. A Belyĭ function is a meromorphic function $\mathcal{X} \to \Sigma$ with no critical values outside $\{0, 1, \infty\}$. The following powerful result is due to Belyĭ [1]:

Theorem 2.11 (Belyĭ [1])**.** *A compact Riemann surface $\mathcal{X}$ is defined over $\overline{\mathbb{Q}}$ if and only if there is a Belyĭ function $\omega : \mathcal{X} \to \Sigma$.*

Belyĭ Theorem implies the following theorem (see Jones [57]) giving a correspondence between hypermaps, Riemann surfaces and algebraic curves. It shows that the Riemann surfaces associated with hypermaps are precisely those defined over the field of algebraic numbers.

Theorem 2.12. *If $\mathcal{X}$ is a compact Riemann surface then the following are equivalent:*

1) *$\mathcal{X}$ is defined over $\overline{\mathbb{Q}}$;*
2) *$\mathcal{X} \cong \overline{\mathbb{H}}/\mathrm{M}$ for some subgroup M of finite index in the modular group Γ;*
3) *$\mathcal{X} \cong \overline{\mathbb{H}}/\mathrm{B}$ for some subgroup B of finite index in $\Gamma(2)$;*
4) *$\mathcal{X} \cong \mathbb{H}/\mathrm{T}$ for some subgroup T of finite index in a hyperbolic triangle group $\Delta(k, m, n)$ where the integers k, m, n satisfy $1/k + 1/m + 1/n < 1$.*

2.4.5. Dessins d'enfants. For each hypermap $\mathbf{H} = (D; R, L)$, there is an associated bipartite map $w(\mathbf{H}) = (D_w; R_w, L_w)$ defined as follows:

$$\begin{aligned} D_w &= D \times \{\pm 1\}, \\ (x, 1)R_w &= (xR, 1), \\ (x, -1)R_w &= (xL, -1), \\ (x, i)L_w &= (x, -i). \end{aligned}$$

A *topological dessin* (or shortly dessin) $W(\mathbf{H})$ defined by the hypermap $\mathbf{H}$ is the topological map associated with $w(\mathbf{H})$ together with a fixed bi-colouring of the vertices. We use convention that black vertices of $W(\mathbf{H})$ represent hypervertices of $\mathbf{H}$ and white vertices of $W(\mathbf{H})$ represent the hyperedges of $\mathbf{H}$. By definition, dessins do not admit semi-edges. A homomorphism between dessins $W(\mathbf{H}_1) \to W(\mathbf{H}_2)$ is a colour-preserving covering of the maps. It is well-known that $W : \mathbf{H} \mapsto W(\mathbf{H})$ is an invertible functor between categories of orientable hypermaps and dessins.

Dessins are defined as topological objects. However, dessins and hypermaps can be viewed as geometric objects as well. Let $\mathbf{B}_1 = W(\mathbf{I})$ be the trivial dessin† embedded into the sphere $\mathcal{S}_0$. Then $\mathbf{B}_1$ can be viewed as a map embedded into the Riemann sphere Σ with the black vertex located at 0, the white vertex located at 1 and the unique edge being the unit interval $(0, 1)$. The point ∞ is the centre of the face of $\mathbf{B}_1$. Let $\mathbf{H}$ be a hypermap of genus g. By Belyĭ's theorem it is possible to introduce a structure of Riemann surface $\mathcal{R}(\mathbf{H})$ on $\mathcal{S}_g$ such that the canonical covering $\omega_{\mathbf{H}} : W(\mathbf{H}) \to \mathbf{B}_1$ extends to a meromorphic function $\omega^*_{\mathbf{H}}$ with exactly three singular values 0, 1, and ∞. The function $\omega^*_{\mathbf{H}}$ is called the Belyĭ function associated with $\mathbf{H}$. Conversely, any Belyĭ function β determines a dessin $\mathbf{H} = \beta^{-1}(\mathbf{B}_1)$ on a Riemann surface. It is well known that the Belyĭ function associated with a hypermap $\mathbf{H}$ is determined up to the action of the group of Möbius transformations of Σ.

† with two vertices, black and white, one edge and one face.

2.5 Actions of groups on Riemann surfaces

A group G acting by homeomorphisms on a topological space $\mathcal{S}$ acts *discontinuously* if every point $x \in \mathcal{S}$ has an open neighbourhood U such that $U \cap U^a = \emptyset$ for all nontrivial elements $a \in \mathrm{G}$. In other words, the action of the group G is fixed-point-free, i.e. $x^a = x$ if and only if a is the identity of G.

Let $\mathcal{S}_g$ be a Riemann surface of genus $g \geq 2$. The theory of coverings and Uniformisation Theorem (cf. Theorem 2.8) says that the Riemann structure on $\mathcal{S}_g$ is given by a regular smooth covering $s\colon \mathbb{H} \to \mathcal{S}_g = \mathbb{H}/\mathrm{K}$, where K is a surface group of genus g, acting discretely on $\mathbb{H}$. The universal covering space $\mathbb{H}$ possesses a natural complex structure projected onto that of $\mathcal{S}_g$. We say that $\mathcal{S}_g$ is *uniformised* by the group K [53]. Riemann surfaces of genus g are fully determined by their automorphism groups.

Theorem 2.13 ([53, Theorem 5.1])**.** *If a Riemann surface $\mathcal{S}_g$, of genus $g \geq 2$ is uniformised by a subgroup* $\mathrm{K} \leq \mathrm{PSL}(2,\mathbb{R})$*, then* $\mathrm{Aut}(\mathcal{S}_g) \cong N(\mathrm{K})/\mathrm{K}$.

The action of a surface group K is fixed-point-free on $\mathbb{H}$, while elements of $N(\mathrm{K})$ may have fixed points in $\mathbb{H}$. A weaker concept than discontinuity is therefore needed to describe their actions. A subgroup $\Theta \leq \mathrm{PSL}(2,\mathbb{R})$ acts *properly discontinuously* if any point $x \in \mathbb{H}$ has an open neighbourhood U such that if $U \cap U^a \neq \emptyset$ for some $a \in \mathrm{H}$, then $x^a = x$. As a result, fixed points are allowed, but an orbit of the action of Θ on $\mathbb{H}$ do not contain its limit points. If the action of a subgroup Θ is discontinuous, then it is also properly discontinuous, but the converse is not true.

If the group $\Theta \leq \mathrm{PSL}(2,\mathbb{R})$ have properly discontinuous action by homeomorphisms on the upper-half plane, some points $x \in \mathbb{H}$ may be fixed. Let us characterise non-identity elements of Θ giving to rise to fixed points. An element of $\mathrm{PSL}(2,\mathbb{R})$ is an automorphism f of $\mathbb{H}$ given by

$$f(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{R},\ ad-bc = 1.$$

The equation $f(z) = z$ can be rewritten into the quadratic equation $cz^2+(d-a)z-b = 0$ with real coefficients over $\mathbb{C}$. The discriminant is then $D = (a+d)^2 - 4$. We have the following possibilities:

1) If $D < 0$, then there are two complex conjugate roots yielding a single fixed point of f in $\mathbb{H}$. Such an element is called *elliptic*. As an example we can take $f\colon z \to -1/z$, fixing $\mathrm{i} \in \mathbb{H}$.
2) if $D = 0$, there is a single root at the boundary $\partial\mathbb{H} = \hat{\mathbb{R}}$, hence f has no fixed point in $\mathbb{H}$. This element is called *parabolic*. The element $f\colon z \to z+\lambda$, $\lambda \in \mathbb{R}$ is parabolic element fixing $\infty \in \hat{\mathbb{R}}$.
3) if $D > 0$, there are two distinct real roots and so f has no fixed point in $\mathbb{H}$. This element is called *hyperbolic*. Take $f\colon z \to \lambda z$, where $\lambda \in \mathbb{R}$, $1 \neq \lambda > 0$, fixing $0, \infty \in \hat{\mathbb{R}}$.

Note that the stabiliser of any point $x \in \mathbb{H}$ is a cyclic group generated by an elliptic element.

Important feature of properly discontinuous actions of subgroups of $\mathrm{PSL}(2,\mathbb{R})$ has been found by Poincarè[†].

Theorem 2.14 ([53, Theorem 5.2])**.** *A group* $\Theta \leq \mathrm{PSL}(2,\mathbb{R})$ *acts properly discontinuously on* $\mathbb{H}$ *if and only if it is discrete.*

[†] at that time he was studying a paper on differential equations by L. Fuchs

Here *discrete subgroup* of $\mathrm{PSL}(2,\mathbb{R})$ means that any infinite sequence of elements $g_n \in \Theta$ which converges in subspace topology to the identity, is eventually constant; there exist $N < \infty$ such that $g_n = id$ for $n \geq N$. Thus, no orbit of Θ on $\mathbb{H}$ contain its limiting points. Discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$ are called *Fuchsian groups*. If $\mathrm{K} \leq \mathrm{PSL}(2,\mathbb{R})$ is a Fuchsian group, then also its normaliser is a Fuchsian group [53, Theorem 5.3].

As a result we obtain the classification theorem for *compact* Riemann surfaces of genus $g \geq 2$.

Theorem 2.15 ([53, Theorem 5.4]). *If $\mathcal{S}$ is a compact Riemann surface which is uniformised by a subgroup $\mathrm{K} \leq \mathrm{PSL}(2,\mathbb{R})$, then the genus of $\mathcal{S}$ is greater or equal to 2 and $N(\mathrm{K})$ is a Fuchsian group.*

2.5.1. Riemann-Hurwitz equation. The discrete action of a group Θ induces a partition P of $\mathbb{H}$ by closed subsets, such that for any $\mathcal{F} \in P$ one has

1) $\bigcup_{g\in\Theta} \mathcal{F}^g = \mathbb{H}$,
2) $\mathrm{int}\,\mathcal{F} \cap (\mathrm{int}\,\mathcal{F})^g = \emptyset$ for all nontrivial $g \in \Theta$, where $\mathrm{int}\,\mathcal{F}$ stands for the interior of $\mathcal{F}$.

A set $\mathcal{F} \in P$ is called a *fundamental region* of the group Θ [55, Section 5.8]. Moreover, by Therorem 2.14, if a subgroup $\Theta \leq \mathrm{PSL}(2,\mathbb{R})$ has (properly) discontinuous action on $\mathbb{H}$, then there exist a fundamental region $\mathcal{F}(\Theta)$ such that Θ has fixed-point-free action on interior of $\mathcal{F}(\Theta)$.

Given a point $x \in \mathbb{H}$, the *Dirichlet region of* Θ centered at x, is given by

$$\mathcal{D}_x(\Theta) = \{z : z \in \mathbb{H} \mid \varrho(z, x) \leq \varrho(z, x^g), g \in \Theta\}, \tag{2.11}$$

where ϱ stands for metric on $\mathbb{H}$. A Dirichlet region $\mathcal{D}_x(\Theta)$ is a hyperbolically convex closed set and it is a connected fundamental region $\mathcal{F}(\Theta)$ if its central point $x \in \mathbb{H}$ is not fixed by the action of $\Theta \leq \mathrm{PSL}(2,\mathbb{R})$. Moreover, if Θ acts properly discontinuously on $\mathbb{H}$, then $\mathcal{D}_x(\Theta)$ is locally finite. A Dirichlet region $\mathcal{D}_x(\Theta)$ is bounded by geodesics in $\mathbb{H}$ and possibly by the segments of the real axis. Its boundary (curve) can be very complicated. The intersection of two boundary geodesics (or the real axis and a geodesic) is called a vertex of $\mathcal{D}_x(\Theta)$. It can be proved that vertices are isolated [65], $\mathcal{D}_x(\Theta)$ is thus a convex polygon in $\mathbb{H}$. A Fuchsian group Θ is *cofinite*, if the fundamental region $\mathcal{F}(\Theta)$ (a Dirichlet region), has finite area. These groups are certainly of our interest. In this case has $\mathcal{F}(\Theta)$ also finite number of sides and the corresponding side-pairing of the associated elements generate Θ.

Introducing Dirichlet regions, we are allowed to determine the area of quotient spaces of the upper half plane $\mathbb{H}$. Let Γ be a Fuchsian group acting (properly discontiuously) on $\mathbb{H}$. The area of the quotient space $\mathbb{H}/\Gamma$ is induced by hyperbolic area on $\mathbb{H}$[†], the hyperbolic area $\mu(\mathbb{H}/\Gamma)$ is well defined and equal to the area of a fundamental region $\mathcal{F}(\Gamma)$ [65, Chapter 3]. Since we have chosen our fundamental region to be a Dirichlet region, which is in our case a finitely-sided hyperbolic polygon, it is relatively easy to compute its area.

If a $\mathrm{K} \leq \mathrm{PSL}(2,\mathbb{R})$ act without fixed points on $\mathbb{H}$, then its fundamental region $\mathcal{F}(\mathrm{K})$ (a Dirichlet region) is a hyperbolic $4g$-gon with sides A'_1,B'_1,A_1,B_1,…,A'_g,B'_g,A_g,B_g in this prescribed order. The group K then has the presentation

$$K = \langle a_1, b_1, \dots, a_g, b_g \mid \prod_{j=1}^{g} [a_j, b_j] = 1\rangle, \tag{2.12}$$

[†] it is a Riemann surface with a metric

where a_i, b_i are hyperbolic elements in $\mathrm{PSL}(2,\mathbb{R})$ identified with sides of $\mathcal{F}(\mathrm{K})$ such that a_j is identified with A_j, b_j with B_j and their inverses are identified with A'_j, B'_j respectively. We see that K is nothing but $\pi_1(\mathcal{S})$, where $\mathcal{S}$ is a surface of genus g. Thus K is called a *surface group*. The area $\mu(\mathrm{K})$ is obtained by decomposing $\mathcal{F}(\mathrm{K})$ into triangles, and adding their areas. The are of $\mathcal{F}(\mathrm{K})$ is

$$\mu(\mathrm{K}) = \mu(\mathcal{F}(\mathrm{K})) = 2\pi(2g-2). \tag{2.13}$$

Denote by, $\Gamma = N(\mathrm{K})$ the Fuchsian group normalising the surface group K in $\mathrm{Aut}(\mathbb{H})$. The situation is analogous to the case of surface groups above. Given a cofinite Fuchsian group Γ, there is an algorithm due to Poincarè obtaining the (hyperbolic) polygonal representation of the fundamental region $\mathcal{F}(\Gamma)$. The action of the Fuchsian group Γ is properly discontinuous on $\mathbb{H}$, so Γ contains elliptic, parabolic, and hyperbolic elements. It is proved, that an elliptic element of a Fuchsian group must have finite order $m_i > 2$ (see Theorem 5.7.3 in [55]). Any elliptic element in a cofinite Fuchsian group is conjugate to one of the *standard elliptic elements* x_i [55, Theorem 5.7.2] and the set of standard elliptic elements has size r. The subgroups $\langle x_i\rangle$, $i \in \{1,2,\dots,r\}$ correspond to representatives of conjugacy classes of maximal elliptic subgroups in Γ. Cofiniteness of Γ also forces that there are finitely many conjugacy classes of maximal parabolic subgroups generated by standard parabolic generators y_l. Every subgroup $\langle y_l\rangle$, $l \in \{1,2,\dots,s\}$ has infinite order. The quotient space $\mathbb{H}/\Gamma$ is of genus g', hence we have $4g'$ standard hyperbolic generators a_j, b_j of infinite order.

A cofinite Fuchsian group in standard form is fully determined by its *signature*, which is the triple

$$(g'; \{m_1, m_2, \dots, m_r\}; s), \tag{2.14}$$

where g' stands for the genus of $\mathbb{H}/\Gamma$, m_i are orders of maximal elliptic elements in Γ and s is the number of maximal parabolic elements of Γ. The set $\{m_1, m_2, \dots, m_r\}$ in the signature may contain multiplicities. A surface group of genus g' has for example the signature $(g'; -; 0)$, while $\mathrm{PSL}(2,\mathbf{Z}) < \mathrm{PSL}(2,\mathbb{R})$ has signature $(0; \{2,3\}; 1)$. Given the signature we have the presentation of Γ in the form

$$\begin{aligned}\Gamma = \langle x_1, \dots, x_r, y_1, \dots, y_s, a_1, b_1, \dots, a_{g'}, b_{g'} \mid x_1^{m_1} = x_2^{m_2} = \ldots = x_r^{m_r} = 1,\\ \prod_{j=1}^{g'}[a_j, b_j]\prod_{i=1}^{r} x_i \prod_{l=1}^{s} y_i = 1\rangle.\end{aligned} \tag{2.15}$$

The area of the corresponding fundamental region $\mathcal{F}(\Gamma)$ with signature $(g'; \{m_1, m_2, \dots, m_r\}; s)$ can be derived using Gauss-Bonet formula and has value

$$\mu(\mathcal{F}(\Gamma)) = \mu(\Gamma) = 2\pi\left(2g' - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + s\right). \tag{2.16}$$

The existence of a Fuchsian group Γ depends on the fact whether $\mu(\Gamma)$ is a positive number. This result has been proved by Poincaré and we have

Theorem 2.16 ([55, Theorem 5.10.5])**.** *If $g' \geq 0$ and $m_i \geq 2$ are integers and if*

$$2g' - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + s > 0, \tag{2.17}$$

then there exist a Fuchsian group with signature $(g'; \{m_1, m_2, \dots, m_r\}; s)$.

The proof of that theorem is just sketched in [55], but contains important information about Poincareé's method to obtain a fundamental region in a standard form and certainly uses this standard form to obtain the result.

Having a Fuchsian group Γ with signature $(g'; \{m_1, m_2, \ldots, m_r\}; s)$ it can be proved that the quotient space $\mathbb{H}/\Gamma$ contains exactly s punctures (s points are removed out of the space), so it is compact if and only if $s = 0$ [55, Theorem 5.9.9]. A Fuchsian group Γ is called *cocompact*, if the quotient space $\mathcal{S} = \mathbb{H}/\Gamma$ is compact. We are interested in those Riemann surfaces, which underlie finite hypermaps, thus they are compact by definition.

One of the most important results in theory of Riemann surfaces is the Riemann-Hurwitz equation which reads as follows. Let $\mathbb{H} \to \mathbb{H}/\mathrm{K} = \mathcal{S}_g$, $g \geq 2$, be an c-sheeted regular cover. Taking Theorem 2.15 into account, the areas of hyperbolic fundamental regions given by K and its normaliser $N(\mathrm{K})$ are related by

$$\mu(\mathrm{K}) = c.\mu(N(\mathrm{K})). \tag{2.18}$$

The order of the automorphism group of a compact Riemann surface $\mathcal{S}_g$ of genus $g \geq 2$ is bounded, since the surface group K is discrete, thus cocompact, and its normaliser $\Gamma = N(\mathrm{K})$ is a cofinite, cocompact Fuchsian group. Using Riemann-Hurwitz formula (2.18) we have

$$|\operatorname{Aut}(\mathcal{S}_g)| = \frac{\mu(\mathrm{K})}{\mu(N(\mathrm{K}))}. \tag{2.19}$$

Moreover, employing signatures we can derive the following equation

$$2g - 2 = |\operatorname{Aut}(\mathcal{S}_g)| \left(2g' - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)\right); m_i \geq 2, m_i \mid |\operatorname{Aut}(\mathcal{S}_g)|. \tag{2.20}$$

It is obvious that all numbers in (2.20) are integers. Hence the order m_i of an elliptic element x_i must divide $|\operatorname{Aut}(\mathcal{S}_g)|$.

Now, observe the similarity between the equation (2.6) and the equation (2.20). Those equations were derived in the similar fashion (in wider sense), but they are completely different. The equation (2.6) has been derived in combinatorial fashion and a topology (in fact the corresponding Riemann structure) has been introduced later by involving the underlying surface of a map. The equation (2.20) comes in geometrical (topological) way, it do not need any combinatorial support. However, thanks to deep interconnections of the theory of maps and the theory of Riemann surfaces – by Belyĭ Theorem, a hypermap defines the Riemann structure of its underlying surface – we can use advantages of both approaches at once. This means, we can study (hyper)maps by studying discrete group actions and vice-versa.

The following result, due to Hurwitz, is of crucial importance. The area of a fundamental (Dirichlet) region $\mathcal{F}(\Gamma)$ of a Fuchsian group Γ cannot be arbitrarily small, for a cofinite Fuchsian group Γ. We have the following bound.

Theorem 2.17 ([55, Theorem 5.10.7])**.** *If $\mathcal{F}(\Gamma)$ is a Dirichlet region for a Fuchsian group Γ with $\mathbb{H}/\Gamma$ compact, then $\mu(\mathcal{F}(\Gamma)) \geq \frac{\pi}{21}$.*

The argument in the proof of Theorem 5.10.7 in [55] is done as the solution of integer optimisation† problem given by equation 2.17. As a result we obtain well-known Hurwitz bound.

Corollary 2.18 ([55, Theorem 5.11.1])**.** *Let $\mathcal{S}$ be a compact Riemann surface of genus $g \geq 2$. Then* $|\operatorname{Aut}(\mathcal{S})| \leq 84(g-1) = 42.(-\chi(\mathcal{S}))$.

† minimisation

Groups of automorphisms of Riemann surfaces of order $84(g-1)$ are called *Hurwitz groups* and were studied from very beginning by Hurwitz, Poincarè, Klein and others. The genus spectrum of Riemann surfaces admitting actions of Hurwitz groups is very sparse. Up to genus $g = 301$ only surfaces of genera $g = 3, 14, 118, 129, 146$ admits Hurwitz groups as their group of automorphisms [21]. On the other hand, there are infinitely many surfaces admitting actions of Hurwitz groups [70, Theorem 2, Theorem 3]. These actions correspond to regular maps of type $\{3, 7\}$.

3. Discrete group actions

Let $\mathcal{S}_g$ be a compact Riemann surface of genus g. A Fuchsian group[†] Γ with signature $(g';\{m_1, m_2, \ldots, m_r\})$ is a normaliser of the surface group K of genus g in $\mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}(2,\mathbb{R})$. The quotient space $\mathcal{O} = \mathbb{H}/\Gamma$ is a compact Riemann surface called the *orbifold* [27] with the signature $(g';\{m_1, m_2, \ldots, m_r\})$. The covering $\kappa\colon \mathbb{H} \to \mathcal{O}$ is regular and branched [102], provided that the signature $(g';\{m_1, m_2, \ldots, m_r\})$ is not trivial, i.e. Γ has fixed elliptic points on $\mathbb{H}$; the action of Γ on $\mathbb{H}$ is properly discontinuous, but not discontinuous. The orbifold $\mathcal{O}$ is homeomorphic to a Riemann surface of genus g', $0 \le g' \le g$ with a distinguished discrete set x_i of r points chosen – called *branch points*. Every branch point x_i is endowed with the *branch index* m_i, the order of the corresponding maximal elliptic subgroup $\langle x_i \rangle < \Gamma$. The covering κ is smooth outside the set of branch points. The branch index of a branch point b_i is $m_i = |\operatorname{fib}(p)|/|\operatorname{fib}(b_i)|$, where p is not a branch point and $|\operatorname{fib}(p)|$ is size of the fibre over a point p. The following diagram, corresponding to the claim of the famous Koebe Theorem [102], has to commute.

$$\begin{tikzcd}
{}^{K}(\mathbb{H},\Gamma) \arrow[dd, "\Gamma"'] \arrow[dr] & \\
& (\mathcal{S}_g, \mathrm{G})^{\mathrm{G}=\Gamma/K} \arrow[dl] \\
(\mathcal{O} = \mathcal{S}_g/\mathrm{G} = \mathbb{H}/\Gamma, 1) &
\end{tikzcd} \tag{3.1}$$

The action of $\mathrm{G} = \mathrm{Aut}(\mathcal{S}_g)$ is not fixed-point-free on $\mathcal{S}_g$ in general. The covering $f\colon \mathcal{S}_g \to \mathcal{O}$ is a regular $|\mathrm{G}|$-sheeted branched covering. The fibres $\varphi^{-1}(x)$, over each point $x \in \mathcal{S}_{g'}$ are the orbits of the action of G. The size of each fibre divides the order of G and except finitely many singular points $\{x_1, x_2, \ldots, x_r\} \in \mathcal{S}_{g'}$ we have $|\varphi^{-1}(x)| = |\mathrm{G}|$. The branch index m_i of a singular point x_i is a divisor of $|\mathrm{G}|$, $m_i \ge 2$. All the parameters are related by Riemann-Hurwitz equation

$$2g - 2 = |\mathrm{G}|\left(2g' - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right); m_i \ge 2, m_i \mid |\mathrm{G}. \tag{3.2}$$

It is convenient to assign the branch indexes to the distinguished set of points of $\mathcal{S}_{g'}$ thus forming an orbifold $\mathcal{O}$ with the signature $(g';\{m_1, m_2, \ldots, m_r\})$, corresponding to the quotient surface $\mathcal{S}_g/\mathrm{G}$. Two orbifolds with the same signature are homeomorphic and invariant under permutation of branch indexes. The signature of the quotient orbifold $\mathcal{O}$, $(g';\{m_1, m_2, \ldots, m_r\})$, determines its fundamental group $\pi_1(\mathcal{O})$, isomorphic to a finitely presented group with presentation

$$\Gamma = \langle x_1, \ldots, x_r, a_1, b_1, \ldots, a_{g'}, b_{g'} \mid x_1^{m_1} = x_2^{m_2} = \ldots = x_r^{m_r} = 1, \prod_{j=1}^{g'}[a_j, b_j]\prod_{i=1}^{r} x_i = 1\rangle. \tag{3.3}$$

Given genus g we know, that a finite group G acting on a Riemann surface $\mathcal{S}_g$ has order bounded by Hurwitz bound, see Corollary 2.18. Solving Riemann-Hurwitz

[†] we are exclusively interested in cofinite and cocompact groups here

equation we obtain numerical conditions on possible signatures. Given a signature $(g'; \{m_1, m_2, \ldots, m_r\})$, the corresponding finitely presented group Γ with presentation (3.3) is determined. If there exist a Riemann surface $\mathcal{S}_g$ with automorphism group $\mathrm{G} = \Gamma/\mathrm{K}$, then there must exist a normal subgroup K of index $|\mathrm{G}|$ in Γ such that K is torsion-free. If we find such a subgroup K, then it must be a surface group, isomorphic to $\mathbf{Z}^{2g}$ and Γ is a Fuchsian group, hence the diagram in the expression (3.1) commutes.

Censa of *all* actions of finite groups on Riemann surfaces of genus g were completed even for small genera quite recently. The classification of actions of *all* groups acting on surfaces of genera two and three appeared in paper by Broughton [17], of genus four in papers by Bogopolski [7] and also by A. Kuribayashi and I. Kuribayashi [68]. Genus five was classified by A. Kuribayashi and Kimura in [67] and genus 16 was treated in [68].

The most complete classification was done by author, see [60]. This classification is done with respect to the *abstract group structure* for genera $2 \le g \le 21$. Given a finite group G and a Riemann surface $\mathcal{S}_g$, different actions of G by homeomorphisms of $\mathcal{S}_g$ are possible. The groups acting on Riemann surfaces are known up to genus $g = 21$ in [60], however there are groups in genus $g = 11$ for which we were not able to list all their actions on that surface. Taking this into account, this classification is the census of *discrete group actions* up to genus $g = 10$ and up to chosen equivalence (see below). The hardest cases are actions of 2-groups and small Abelian groups. For example, the first case when we were able to find that a group has an action on a Riemann surface but we were not able to collect all possible actions appeared in genus $g = 11$, the action of $\mathbf{Z}_2 \times \mathbf{Z}_2$ and the signature $(2; \{2^6\})$.

However, the classification of actions of cyclic groups can be treated efficiently. The corresponding signature $(g'; \{m_1, m_2, \ldots, m_r\})$ of the action of a cyclic group $\mathbf{Z}_\ell$ has the following properties [47]

1) $\mathrm{lcm}(M) = \ell$, where $M = \{m_1, m_2, \ldots, m_r\}$,
2) $\forall i\colon\ \mathrm{lcm}(M \setminus \{m_i\}) = \mathrm{lcm}(M),\ i \in \{1, 2, \ldots, r\}$.

The second property is called the elimination property. Number of discrete actions of a cyclic group $\mathbf{Z}_\ell$ on a Riemann surface $\mathcal{S}_g$ can be determined by using so-called *epicyclic function* $\mathrm{Epi}_O(\mathcal{S}_g, \mathbf{Z}_\ell)$. This arithmetic function is known in additive form [79] as well as in multiplicative form [73]. As follows, epicyclic function serve as a kind of characteristic function, i.e. if there is no epimorphism of $\mathbf{Z}_\ell$ on a surface $\mathcal{S}_g$, then $\mathrm{Epi}_O(\mathcal{S}_g, \mathbf{Z}_\ell)$ yields zero. Moreover, epicyclic function plays crucial rôle in enumeration of various combinatorial structures, e.g. maps on a Riemann surface $\mathcal{S}_g$ of given genus [79]. It seems to be not difficult also to determine all actions of a particular cyclic group on a Riemann surface of given genus in the same fashion as it was done in [60].

The restricted problem is the classification of actions of so-called 'large groups of automorphisms', i.e. those with $|\mathrm{G}| > 12(g-1)$. The complete classification of actions with signatures $(0; \{k, m, n\})$, done by M. Conder [20, 21], ranges to genus $g = 301$. This list of actions was created due to classification of regular maps and regular hypermaps on orientable surfaces and it is important resource for many studies in the area.

3.1 Numerical solutions

Let G be a finite group which has an action by homeomorphisms on the Riemann surface $\mathcal{S}_g$ of genus $g \ge 2$. Then there exist a regular branched covering $\kappa\colon \mathcal{S}_g \to O = \mathcal{S}_g/\mathrm{G}$, where O is an orbifold with signature $(g'; \{m_1, m_2, \ldots, m_r\})$ and Riemann-Hurwitz equation (3.2) is satisfied. Then the signature of the quotient orbifold O is called *g-admissible* signature. We want to find all possible group actions on $\mathcal{S}_g$, but at this point we have only information about the genus of the surface. We have to numerically find all integer solutions of (3.2). The following criteria have to be satisfied.

1) $0 \le g' \le g$,
2) $\forall i : m_i$ is a non-trivial divisor of $|\mathrm{G}|$,
3) $\forall i : m_i \in \mathbf{Z},\ 2 \le m_i \le 4g+2$,
4) $r \le 2g+2$,
5) $|\mathrm{G}| \le 84(g-1)$.

The criterion 1) is natural; we are dealing with coverings of surfaces. The condition 2) must be satisfied, because right-hand side of (3.2) must be an integer.

The condition 3) follows from the fact that an automorphism of a Riemann surface cannot have order grater than $4g+2$ [47]. Each branch index is equal to the order of a maximal elliptic element in the Fuchsian group Γ with signature $(g'; \{m_1, m_2, \ldots, m_r\})$. The automorphism group of $\mathcal{S}_g$ is a quotient of Γ by a normal torsion-free subgroup K, hence the elliptic elements are mapped onto elements of the same order. No elliptic element has order greater than $4g+2$. Moreover, it has been proved in [73], that for any genus g there exist an action of $\mathbf{Z}_{4g+2}$ on $\mathcal{S}_g$, where the corresponding orbifold signature is $(0; \{2; 2g+1, 4g+2\})$. Signatures $(0; \{m_1\})$ where $m_1 \ne \infty$ and $(0; \{m_1, m_2\})$ with $m_1 \ne m_2$ are not admissible for any genus [37, Theorem IV.9.12].

The criterion 4) follows from solving a simple minimisation problem on values of branch indexes, see Lemma 10 in [63] for the argument. The last criterion is the Hurwitz bound, see Corollary 2.18. As the result we have the following algorithm.

Algorithm 4: Numerical solutions of Riemann-Hurwitz equation

Input: g: the genus of the surface
Output: N: the list of numerical solutions of (3.2)

```
N ← [];
for o ← 2 to 84(g − 1) do
    D ← [d | o div d = 0 and 1 < d and d < 4g + 3];
    M ← Vectors(D, 2g + 2);
    // sorted vectors over D up to length 2g + 2
    foreach m in M do                              // m = [m1, m2, ..., mr]
        l ← Length(m);
        for g′ ← 0 to g do
            if g′ = 0 then
                if l ≤ 1 then continue;
                if l = 2 and not m[1] = m[2] then continue;
            R ← 2g′ − 2 + o.l;
            for i ← 1 to l do
                R ← R − o div m[i]
            if 2g − 2 = R then Append (~N, ⟨g; g′; o; m⟩);
return N
```

Example 3.1. Even if the Riemann-Hurwitz formula is satisfied, the covering κ may not exist; for example, the signature $(0;\{2,3,7\})$ is not 2-admissible, $(1;\{2,2\})$ is not 3-admissible, etc. Table 3.1 displays a correspondence between numerical solutions of Riemann-Hurwitz equation and existing group actions for Riemann surfaces of genus 2.

$\|G\|$	Orbifold	Actions	$\|G\|$	Orbifold	Actions
1	$(2;\{\})$	Id	10	$(0;\{2,5,10\})$	$\mathbf{Z}_{10}$
2	$(1;\{2,2\})$	$\mathbf{Z}_2$	12	$(0;\{2,2,2,3\})$	D_{12}
2	$(0;\{2^6\})$	$\mathbf{Z}_2$	12	$(0;\{3,4,4\})$	$\mathbf{Z}_3:\mathbf{Z}_4$
3	$(1;\{3\})$	-	12	$(0;\{3,3,6\})$	-
3	$(0;\{3,3,3,3\})$	$\mathbf{Z}_3$	12	$(0;\{2,6,6\})$	$\mathbf{Z}_6\times\mathbf{Z}_2$
4	$(1;\{2\})$	-	15	$(0;\{3,3,5\})$	-
4	$(0;\{2^5\})$	$\mathbf{Z}_2\times\mathbf{Z}_2$	16	$(0;\{2,4,8\})$	QD_{16}
4	$(0;\{2,2,4,4\})$	$\mathbf{Z}_4$	20	$(0;\{2,5,5\})$	-
5	$(0;\{5,5,5\})$	$\mathbf{Z}_5$	24	$(0;\{3,3,4\})$	$SL(2,3)$
6	$(0;\{2,2,3,3\})$	$\mathbf{Z}_6$, Sym(3)	24	$(0;\{2,4,6\})$	$(\mathbf{Z}_6\times\mathbf{Z}_2):\mathbf{Z}_2$
6	$(0;\{2,2,2,6\})$	-	30	$(0;\{2,3,10\})$	-
6	$(0;\{3,6,6\})$	$\mathbf{Z}_6$	36	$(0;\{2,3,9\})$	-
8	$(0;\{2,2,2,4\})$	D_8	40	$(0;\{2,4,5\})$	-
8	$(0;\{4,4,4\})$	Q_8	48	$(0;\{2,3,8\})$	$GL(2,3)$
8	$(0;\{2,8,8\})$	$\mathbf{Z}_8$	84	$(0;\{2,3,7\})$	-
9	$(0;\{3,3,9\})$	-			

Table 3.1: Numerical solutions of genus 2 vs. actions of groups on genus 2 surface

3.2 Testing admissibility

In the previous step we obtained the list of numerical solutions as quadruples

$$\langle g; g'; o; \vec{m}\rangle,$$

where $o = |G|$ and $\vec{m} = (m_1, m_2, \ldots, m_r)$. Given a quadruple we determine the corresponding orbifold signature $\sigma = (g'; \{m_1, m_2, \ldots, m_r\})$ and we are able to construct the Fuchsian group Γ with the signature σ. It is a finitely-presented group with the presentation given in (3.3). We have to find all normal subgroups $K \trianglelefteq_o \Gamma$, such that every K is torsion-free. This is nothing but given K and $\kappa\colon \Gamma \to \Gamma/K = G$ we have to check

- for all elliptic elements in Γ $|\kappa(x_i)| = m_i$ and,
- no relation of Γ collapses.

To use computational algebra software packages is the first choice, possibly the only one for that check. Both, Magma [9] and GAP [38] provide facilities to solve this problem. In GAP we refer to the function `GQuotients`, which gives all epimorphisms $F \to H$, given a finitely-presented group F and a finite group H. Magma provides the function `LowIndexNormalSubgroups` [8], which can be used solving the problem.

Below is the listing of the Magma session, testing the numerical solution for the signature $\sigma = (0;\{2,4,6\})$, see Table 3.1. The group `G` is the Fuchsian group with signature σ. The group Γ contains two normal subgroups of index 24. The subgroup `K[1]` is torsion-free – it is checked in the lines 8–9 and 11–12, while the subgroup `K[2]` is not torsion-free – as it is displayed in the lines 24 and 25. We also show that the kernel of the epimorphism phi1 (defined on the line 7) is a copy of a surface group of genus 2 – see lines 14–21.

```
1  > G<x,y,z> := Group<x,y,z | x^2, y^4, z^6, x*y*z>;
2  > L:=LowIndexNormalSubgroups(G,24);
3  > K:=[elt‘Group : elt in L | elt‘Index eq 24];
4  > #K;
5  2
6
7  > phi1,_ := CosetAction(G,K[1]);
8  > [Order(phi1(e)) : e in Generators(G) ];
9  [ 2, 4, 6 ]
10
11 > Order(phi1(x)*phi1(y)*phi1(z));
12 1
13
14 > K1 := Kernel(phi1);
15 > D1,af1 := AbelianQuotient(K1);
16 > D1;
17 Abelian Group isomorphic to Z + Z + Z + Z
18 Defined on 4 generators (free)
19
20 > Index(K1,sub<K1 | {d @@ af1 : d in Generators(D1)}>);
21 1
22
23 > phi2,_ := CosetAction(G,K[2]);
24 > [Order(phi2(e)) : e in Generators(G) ];
25 [ 2, 4, 3 ]
```

Although both LowIndexNormalSubgroups and GQuotients are powerful and using contemporary computers we can manage many cases, both programs are essentially implementations of exponential algorithms. It may happen that computer memory is exceeded or the computation takes long time (weeks, months). There is sill a chance to obtain satisfactory results in reasonable time and not exceeding computer memory. We also managed solutions (lists of actions) by explicit constructions of order-preserving homomorphisms from corresponding Fuchsian groups to finite groups of given order (using library of small groups [4]). Using this approach we had to decide whether a pair of homomorphisms have the same kernel. In that case we say, that the pair of discrete actions of groups on surfaces is *equivalent*. Two discrete actions of groups on a Riemann surface, determined by epimorphisms $\varphi\colon \Gamma \to \mathrm{G}$ and $\psi\colon \Gamma \to \mathrm{G}$ are equivalent, if there exists an isomorphism $\alpha\colon \mathrm{G} \to \mathrm{G}$ such that for the generators x_i, a_j, b_j of the Fuchsian group Γ with signature $\sigma = (g'; \{m_1, m_2, \dots, m_r\})$ one has

$$\begin{aligned} \alpha(\varphi(x_i)) &= \psi(x_i), \quad i \in \{1, \dots, r\}, \\ \alpha(\varphi(a_j)) &= \psi(a_j), \quad j \in \{0, \dots, g'\}, \\ \alpha(\varphi(b_j)) &= \psi(b_j), \quad j \in \{0, \dots, g'\}. \end{aligned} \tag{3.4}$$

As an example we can take the action of the symmetric group Sym(3) on the Riemann surface of genus 3. We have two non-equivalent actions on genus 2 surface, described by vectors

$$(y^{-1}, y, x, x)$$
$$(y^{-1}, y^{-1}, xy, x)$$

where $\mathrm{Sym}(3) = \langle x, y \mid x^2, y^3, (y^{-1}x)^2 \rangle$ and the orbifold has the signature $(0; \{2, 2, 3, 3\})$.

4. Symmetric maps and hypermaps

4.1 Regular maps and hypermaps

Many authors contributed to the classification of Platonic solids (regular maps) since ancient times. One-skeletons of Platonic solids are precisely spherical regular maps which coincide with tessellations $\{3,3\}$, $\{3,4\}$, $\{4,3\}$, $\{3,5\}$, and $\{5,3\}$ of the sphere. The family of spherical regular maps also include the two infinite families $\{2,p\}$, $\{p,2\}$, where $p > 0$ is a prime, related with the embeddings of p-cycles and their duals.

The massive development of computers and methods of the computational algebra [88, 49] give us powerful tools to classify regular maps of higher genera. Regular maps on orientable surfaces up to genus $g \leq 301$ were classified completely in last decade [20, 21]. The reader can find several censa of regular hypermaps on Marston Conder's homepage [23]; those are created with respect to different conditions put on the classification. This result range to the genus $g = 101$ in the case of regular hypermaps on orientable surfaces.

All these results directly employs the theory described in Sections 2.4.3 and 2.4.4, namely Theorem 2.12 is of crucial importance. When we deal with regular hypermaps on a surface of fixed genus g, we are dealing with discrete actions of groups such that the orbifold $\mathcal{O} = \mathcal{S}/\mathrm{G}$ has signature $(0; \{k, m, n\})$. The case $n = 2$ covers the classification of regular maps. The quotient map, for both regular maps and regular hypermaps, is always the trivial dessin d'enfant on a spherical orbifold (cf. Section 2.4.5) and a voltage assignment (cf. Section 2.1.11) is given by a torsion-free epimorphism from the respective Fuchsian group to G. Let us just note that a Fuchsian group is nothing but the triangle group $\Delta^+(k, m, n)$, for some positive k, m, $n \in \mathbf{Z}$ such that $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} < 1$.

4.2 Cayley maps

Let us have a finite group G and a set X of generators of G such that $1_\mathrm{G} \notin X$ and if $x \in X$, then $x^{-1} \in X$. The *Cayley graph* $C = \mathrm{Cay}(\mathrm{G}, X)$ is then the finite graph with the vertex set $V(C) = \mathrm{G}$ and the set of darts $D(C) = \mathrm{G} \times X$. The incidence function $I((g, x)) = g$ is the projection to the first coordinate of the tuple. The dart-reversing involution is given by the mapping $L\colon (g, x) \mapsto (gx, x^{-1})$.

A *Cayley map* $\mathrm{CM}(\mathrm{G}, X, \xi, p)$ is a 2-cell embedding of the Cayley graph $\mathrm{Cay}(\mathrm{G}, X)$ on an orientable surface, with the same local cyclic permutation p of X at every vertex. The involution ξ denotes the inverse involution on X associating an element $x \in X$ with its inverse x^{-1}. Given a group G and its generating set X, the involution ξ is uniquely determined, hence it is usual to abbreviate $\mathrm{CM}(\mathrm{G}, X, \xi, p)$ to $\mathrm{CM}(\mathrm{G}, X, p)$. A comprehensive introduction† to the theory of Cayley maps can be found in [85].

Now observe that given a Cayley map $\mathbf{M} = \mathrm{CM}(\mathrm{G}, X, p)$, the group G acts semi-regularly on the set of darts and transitively on the vertex set of $\mathbf{M}$. By the left action, G acts as a group of automorphisms of $\mathbf{M}$ and can be also seen as a group of self-homeomorphisms of the underlying surface $\mathcal{S}_g$ of the map $\mathbf{M}$. Hence G is the group with discrete action on the underlying surface of $\mathbf{M}$ (see sections 2.4 and 2.5).

The map $\mathbf{M}$ is a vertex-transitive map and $\mathrm{Aut}^+(\mathbf{M}) = \mathrm{G}$ [85]. The quotient map $\bar{\mathbf{M}} = \mathbf{M}/\mathrm{G}$ is a one-vertex map embedded into an orbifold $\mathcal{O}$ with signature

† and more than the introduction

(g'; $\{m_1, m_2, \ldots, m_r\}$) (see also Sections 2.1.6 and 2.1.7). In Chapter 3 we described the method of classification of discrete group actions on Riemann surfaces with genus $g \geq 2$. The problem of construction of Cayley maps with a given group G on a surface $\mathcal{S}_g$ can be then reduced to the construction of all one-vertex maps on the respective orbifold $\mathcal{S}_g/\mathrm{G}$. Once one have all possible quotients, then the corresponding Cayley maps can be reconstructed employing lifts through voltage assignments (cf. Section 2.1.11).

However it looks like easy task, the problem is intractable in general. If the problem is stated such that one has to find all Cayley maps with a given group $\mathrm{G} = \langle X \rangle$ of a given genus g, then the complexity is in principle given by the order of X and grows very quickly. In fact it is hyper-exponential growth, depending on the number of involutions in $\mathrm{Sym}(X)$. We will meet with a particular instance of that problem in the following chapter.

Example 4.1. Cayley maps need not to be necessarily finite. A relaxation on the finiteness of the group G in the definition of Cayley map give us a possibility to represent e. g. tessellations of the upper half complex plane. In fact, this is the base step in the construction of (all) generating sets of Cayley maps of given genus with a given group.

First we show the essential example of construction of the standard presentation of a Fuchsian group determined by an orbifold with signature (g'; $\{m_1, m_2, \ldots, m_r\}$). We take the $2g'$-gonal canonical representation P of the orientable surface and arbitrarily we choose r distinct internal points of P creating the discrete set B. A point $b_i \in B$ obtain the branch index m_i. Further we choose the point $v \in P$, outside B and we draw r loops based on v such that a loop contains exactly one branch point $b_i \in B$. As a result we obtain a map on the surface of genus g', a bouquet of r loops, such that every loop contains a branch point and there is one long face containing no branch point.

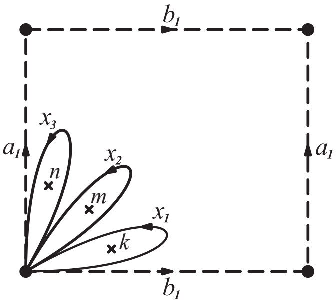


Figure 4.1: A Cayley map determining the presentation of a Fuchsian group

Every dart in the previously constructed map obtain an 'abstract voltage' x_i and the presentation of the corresponding Fuchsian group is derived such that every face-walk is written as a word w. The word w is raised to the power equal to the value corresponding to the branch index m_i. The word constructed out from the facial walk along the long face is raised to the power 1.

The particular example showed on Figure 4.1 use this construction to derive the presentation of the Fuchsian group with signature $(1; \{k, m, n\})$, which is

$$\Gamma = \langle x_1, x_2, x_3 \mid x_1^k = x_2^m = x_3^n = 1, [a_1, b_1]x_1x_2x_3 = 1 \rangle.$$

Let us remark that the infinite Cayley map on the upper half complex plane $\mathbb{H}$, lifted along the standard cover $\kappa\colon \mathbb{H} \to \mathbb{H}/\Gamma$, is the dual of the standard tessellation of $\mathbb{H}$ by fundamental regions of the Fuchsian group Γ (see Section 2.5).

4.3 Archimedean maps

There are 13 classical Archimedean solids (15 if the mirror images of two enantiomorphs are counted separately). Thus there are 13 (15) associated spherical Archimedean maps. Moreover, the five Platonic solids give rise to another five spherical Archimedean maps. To complete the list of spherical Archimedean maps we include the infinite families of maps associated with the n-prisms and n-anti-prisms, where $n \geq 3$ (including the 3-cube and the octahedron in this setting). For deeper insight we recommend e.g. the monograph by Conway et. al [27].

The classification of (classical) Archimedean solids is done in such fashion that every of them is fully determined by its local type. A *local type of a map* $\mathbf{M}$ *at a vertex* v is nothing but a cyclic sequence of lengths of faces incident to v following a given global orientation of the surface. In case of Archimedean solids local types do not depend on a choice of a vertex. Following [33], Archimedean solids are *uniform maps* – they have regular polygons as faces and they are vertex-transitive. Indeed, the family of uniform maps is broader than the family of classical Archimedean solids, determined already by Kepler. These solids comes as uniform 1-skeletons of *convex polyhedra* in 3-dimensional Euclidean space. A convex polyhedron is the convex hull of a finite set of points in the Euclidean space [45, 44]. By Steinitz's theorem [93] a graph forms the 1-skeleton of a convex polyhedron if and only if it is planar and 3-connected. Therefore a simple graph is called *polyhedral* if and only if it is planar and 3-connected. As follows, a 2-cell embedding of a polyhedral graph into the sphere – the *polyhedral map* – is uniquely determined by the graph, hence there is a correspondence between polyhedral graphs, polyhedral maps and polyhedra.

Archimedean solids classified by Kepler are vertex-transitive and polyhedral maps on the sphere. Both conditions are essential. Johnson solids are convex, they have 3-connected underlying graphs, but they are not vertex-transitive [51]. Planar embeddings of k-dipoles are vertex-transitive, but they are not 3-connected.

When we generalise the notion of Archimedean solid to orientable surfaces of higher genera, the definition of polyhedrality given by Steinitz becomes to be insufficient. Take the complete bipartite graph $K_{3,3}$. It is vertex-transitive and 3-connected. Any embedding of $K_{3,3}$ into the torus or into a surface of genus 2 is not polyhedral. Boundaries of faces of classical Archimedean solids are simple cycles and also they do not allow multiple touches of pairs of faces. Every toroidal embedding of $K_{3,3}$ has a pair of faces meeting in more than one edge. Genus 2 embedding of $K_{3,3}$ is even worse – it is a one-face embedding. The definition of polyhedrality must be therefore strengthened when we consider Archimedean maps on surfaces of higher genera. Following Theorem 5.5.12 in [80], the map is polyhedral if and only if its *face width*† is greater or equal to 3. Face width is defined as the minimum number of faces which a non-contractible curve on an orientable surface can intersect. By definition, the face width of planar embeddings is set to be infinity [80, Chapter 5].

Hereby it is said that polyhedrality is not a property of graph G itself, but it depends on the map $\mathbf{M}$, $\mathbf{M} = G \hookrightarrow \mathcal{S}$. Polyhedrality of a map cannot be expressed in pure group-theoretic terms. The invariants like valency of $\mathbf{M}$, co-valency of $\mathbf{M}$ and length of the shortest Petrie walk in $\mathbf{M}$ can be expressed as conditions on action of group of automorphism of $\mathbf{M}$‡, but also non-degenerate graphs may have non-polyhedral embeddings. The tetrahedron is the polyhedral embedding of K_4 into the sphere, while toroidal embedding of the same graph has face-width 1, thus it is not polyhedral although K_4 is a non-degenerate graph. On the other hand, given a combinatorial map $\mathbf{M} = (D; R, L)$ we have polynomial polyhedrality test.

† representativity, planar width, and face width are equivalent names ‡ particular words in the presentation of a group

Algorithm 5: Polyhedrality test

```
Input: (d; R, L): the map
Output: true, if map is polyhedral; false otherwise
vc ← Cycles(R); vcl ← Length(vc)
// 'Cycles' gives the array of sets, cycles of permutation
fc ← Cycles(R*L); fcl ← Length(fc)
for i ← 1 to fcl do
    // are boundaries of faces simple cycles?
    for j ← 1 to vcl do
        if Length(fc[i] ∩ vc[j]) > 1 then return false;
    // does meet a pair of faces in more than one edge?
    for k ← i + 1 to fcl do
        if Length(fc[i] ∩ fc[k]^L) > 1 then
            // 'L' is applied on points of the set 'fc[k]'
            return false
return true
```

Archimedean maps are vertex-transitive maps, i.e Aut($\mathbf{M}$) has one orbit on the vertices of the underlying graph of the map $\mathbf{M}$. In the case of spherical Archimedean maps† already group of orientation preserving automorphisms $\mathrm{Aut}^+(\mathbf{M})$ is vertex-transitive. The situation changes already on the torus. Archimedean maps arise by setting quadrangular fundamental region in one of 11 Archimedean tilings of Euclidean plane [46]. Let us note that the tiling with local type $(3^4.6)$ is a chiral tiling. The fundamental region has to be chosen such that the map coming from an identification of the sides of the quadrangular fundamental region has to be of face-width ≥ 3. In any case, the action of $\mathrm{Aut}^+(\mathbf{M})$ on toroidal Archimedean maps with local type (4.6.12) has two orbits on vertices of the corresponding map $\mathbf{M}$. Thanks to this phenomenon we have to be more careful with construction and classification of Archimedean maps of higher genera; we have to deal with two different‡ families of Archimedean maps. In [63] we recognise *Archimedean maps of type I* and *Archimedean maps of type II*. Types are defined such that an Archimedean map $\mathbf{M}$ is of type I if $\mathrm{Aut}^+(\mathbf{M})$ has one orbit on vertices of $\mathbf{M}$ or it is of type II otherwise.

The orientation-preserving automorphism group $\mathrm{G} = \mathrm{Aut}^+(\mathbf{M})$ of an Archimedean map $\mathbf{M}$ has discrete action on the respective surface. The quotient map $\bar{\mathbf{M}} = \mathbf{M}/\mathrm{G}$ is thus one- or two- vertex map on the orbifold. Given genus $g > 1$ we already determined all possible actions of finite groups determining the respective signatures of orbifolds. The Archimedean map $\mathbf{M}$ is reconstructed as a derived map from the T-reduced voltage assignment in G on darts of the quotient map $\bar{\mathbf{M}}$. The quotients of Archimedean map have to satisfy the following conditions

Proposition 4.2. *Let* $\mathbf{M}$ *be an Archimedean map on an orientable surface of genus* $g \geq 1$ *and let* $\bar{\mathbf{M}} = \mathbf{M}/\,\mathrm{Aut}^+(\mathbf{M})$ *be its k-vertex quotient* ($k = 1, 2$) *of valency* ℓ *on an orbifold* $\mathcal{O}(g'; m_0^k, m_1, \ldots, m_n)$. *Then*

1) $3 \leq \ell m_0 \leq 3 + \sqrt{12g - 3}$,
2) $m_i f_i > 2$ *for every face* $\bar{F}_i$, *where* $f_i = |\bar{F}_i|$ *is the face-valency,* $i \in \{1, \ldots, n\}$ *an* m_i *is an index of a branch-point associated with* $\bar{F}_i$,
3) $|\,\mathrm{Aut}^+(\mathbf{M})| > \frac{1}{k}\ell m_0^2$, *where* $k = [\mathrm{Aut}(\mathbf{M}) : \mathrm{Aut}^+(\mathbf{M})]$ *and the branch points with indexes with value* m_0 *(possibly* $m_0 = 1$*) are associated with vertices of* $\bar{\mathbf{M}}$.

† classical Archimedean solids ‡ but not necessarily distinct

Proposition 4.2 comes from [63], where it is written in a bit more broader form. It simply set the necessary conditions on the valency, the co-valency of the quotient map. The last item in Proposition 4.2 is the necessary condition on the size of the vertex stabiliser in the corresponding Archimedean map. Given genus g, we know all possible discrete actions of finite groups on $\mathcal{S}_g$. Hence it is easy to check if the embedding of a graph into the respective orbifold satisfies these conditions. What is not explicitly said here is that the end of any semi-edge must be a branch point of index two; we have a very convenient filter excluding most possibilities in the set of quotient maps. On the other hand, the number of quotient maps is terribly big though the valency of the quotient is bounded (see 1) in Proposition 4.2). The maximal valency of $\bar{\mathbf{M}}$ is $\ell = 7, 8, 9, 10, 11$ for genera $g = 2, 3, 4, 5, 6$ provided that $m_0 = 1$. We have to deal with potential quotient maps on $d_1 = 7, 8, 9 \ldots$ darts when Archimedean map is of type I and with maps on $d_2 = 14, 16, 18 \ldots$ darts in type II. The rotation $\bar{R}$ can be set as $\bar{R} = (1, 2, \ldots, d_1)$ or $\bar{R} = (1, 2, \ldots, \frac{d_2}{2})(\frac{d_2}{2}+1, \frac{d_2}{2}+2, \ldots, d_2)$, respectively. As concerns $\bar{L}$ we have almost no restriction. We know, that $\bar{L}^2 = id$ and that $\langle \bar{R}, \bar{L} \rangle$ must be transitive on $\bar{D} = \{1, 2, 3, \ldots, d_k\}$, $k = 1, 2$. Another restriction can be taken into account looking to signatures of orbifolds for given genus; involutions $\bar{L}$ with more fixed points than the maximum number of branch indexes two in signatures can be safely thrown away. In any case, we have to traverse through all involutions in symmetric groups Sym(1), Sym(2),..., Sym(d_k), where $k = 1, 2$. The biggest group we were able to provide this search is Sym(18).

Quotient maps we have just obtained, are not yet maps on orbifolds. Given a map $\bar{\mathbf{M}}$ on the surface $\mathcal{S}_{g'}$, we can set the discrete set of points in $\mathcal{S}_{g'}$ to be branch points according to a chosen signature. These points must be chosen with respect to the definition of map on orbifold. Given map a $\bar{\mathbf{M}}$ and a signature σ we have just finitely many choices to set the branch assignment b. In the case of two-vertex maps we have to take into account that branch-assignment must admit a reflection of the map $\bar{\mathbf{M}}$ on the orbifold, transposing the two vertices. We have the following proposition, essentially based on the results of papers [76, 77].

Proposition 4.3 ([63]). *Let* $\mathbf{M}$ *be an Archimedean map of type II. Then an orientation reversing automorphism* τ *of* $\mathbf{M}$ *transposing the two vertex orbits projects into orientation an reversing automorphism* $\bar{\tau}$ *of the map* $\bar{\mathbf{M}}$ *on the orbifold.*

As a result we have a condition on a T-reduced voltage assignment; certainly, on the corresponding quotient map $\bar{\mathbf{M}}$ on the orbifold $\mathcal{S}_g / \operatorname{Aut}^+(\mathbf{M})$ of signature σ and branch distribution b.

Theorem 4.4. *Let* $\varphi : \mathbf{M} \to \bar{\mathbf{M}} = \mathbf{M}/\operatorname{Aut}^+(\mathbf{M})$ *be a regular covering from an Archimedean map of genus* g *and let* b *be the induced distribution of branch-indices. Then there exists a* b*-compatible,* T*-reduced voltage-assignment* $\xi : D(\bar{\mathbf{M}}) \to \operatorname{Aut}^+(\mathbf{M})$ *such that the natural projection* $\pi_\xi : \mathbf{M}^\xi \to \bar{\mathbf{M}}$ *is equivalent to* φ*.*

Let us just note that Theorem 4.4 has been introduced in [63] in slightly different form. The changes are more-less cosmetic; at the time of writing that paper we did not have formulated 'walk-calculus' in maps on orbifold and also the notion of isomorphism of maps on orbifolds has not been explicitly stated. The term 'b-compatible' thus means that the two-vertex quotient map $\bar{\mathbf{M}}$ on the orbifold must admit a reflection mapping fundamental walks in the map into fundamental walks such that orders of these walks must be preserved. The condition on voltage assignment is technical, because the algorithm of map lifting works correctly if the voltage assignment provided as the argument is T-reduced. Every voltage assignment can be transformed to a T-reduced voltage assignment.

Example 4.5. Let us show the construction of voltage assignment on a particular map on orbifold with signature $\sigma = (0; \{2, 2, 2, 3\})$ arising from the action of the dihedral group D_6 on the Riemann surface of genus 2. The example is provided in MAGMA, using `LowIndexNormalSubgroups` to derive that voltage assignement.

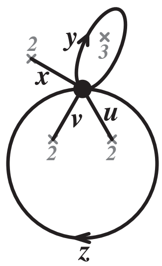


Figure 4.2: Quotient of an Archimedean map on orbifold with abstract voltages assigned

We begin with construction of the automorphism group U of the universal tessellation of respective Archimedean maps. The group

$$U = \langle x, y, z, u, v \mid x^2, y^3, u^2, v^2, zvu, z^{-1}y^{-}1x \rangle$$

is a finitely-presented group, isomorphic to the Fuchsian group with signature σ, see also Example 4.1. The relations in the presentation of U corresponds to the fundamental walks in the quotient map $\bar{\mathbf{M}}$ raised to the power corresponding with the respective branch index assigned to the face, end of semi-edge, or vertex. Walks here correspond to facial walks in $\bar{\mathbf{M}}$ respecting the orientation of 'abstract voltages'. Our task is just to find torsion-free normal subgroups of U of index 12 in this case. Epimorphisms we have determined at lines 15, 21, and 27 were constructed also in [63], so we are done just by checking whether the image of an epimorphism is isomorphic do D_6. All three derived maps, lifted from $\bar{\mathbf{M}}$ substituting the corresponding permutations are Archimedean maps; two of them (epimorphisms on lines 15 and 21) form a chiral pair, the last one is reflexible [63].

```
1  > U<x,y,z,u,v> := Group<x,y,z,u,v | x^2, y^3, u^2, v^2, z*v*u, z^-1*y^-1*x>;
2  > L:=LowIndexNormalSubgroups(U,12);
3  > L12 := [elt‘Group : elt in L | elt‘Index eq 12];
4
5  > #L12;
6  3
7
8  > for K in L12 do
9  for> epi, P := CosetAction(U,K);
10 for> if IdentifyGroup(P) eq <12,4> then
11 for|if> epi;
12 for|if> end if;
13 for> end for;
14 Homomorphism of GrpFP: U into GrpPerm: P, Degree 12 induced by
15     x |--> (1, 2)(3, 10)(4, 7)(5, 8)(6, 11)(9, 12)
16     y |--> (1, 3, 7)(2, 4, 10)(5, 9, 11)(6, 12, 8)
17     z |--> (1, 4)(2, 3)(5, 6)(7, 10)(8, 9)(11, 12)
18     u |--> (1, 5)(2, 8)(3, 9)(4, 6)(7, 11)(10, 12)
19     v |--> (1, 6)(2, 9)(3, 8)(4, 5)(7, 12)(10, 11)
20 Homomorphism of GrpFP: U into GrpPerm: P, Degree 12 induced by
21     x |--> (1, 2)(3, 10)(4, 7)(5, 11)(6, 9)(8, 12)
22     y |--> (1, 3, 7)(2, 4, 10)(5, 12, 9)(6, 8, 11)
23     z |--> (1, 4)(2, 3)(5, 6)(7, 10)(8, 9)(11, 12)
24     u |--> (1, 5)(2, 8)(3, 9)(4, 6)(7, 12)(10, 11)
```

```
    v |--> (1, 6)(2, 9)(3, 8)(4, 5)(7, 11)(10, 12)
Homomorphism of GrpFP: U into GrpPerm: P, Degree 12 induced by
    x |--> (1, 2)(3, 8)(4, 7)(5, 9)(6, 10)(11, 12)
    y |--> (1, 3, 7)(2, 8, 4)(5, 12, 10)(6, 9, 11)
    z |--> (1, 4, 3, 2, 7, 8)(5, 6, 12, 9, 10, 11)
    u |--> (1, 5)(2, 9)(3, 10)(4, 11)(6, 8)(7, 12)
    v |--> (1, 6)(2, 10)(3, 11)(4, 5)(7, 9)(8, 12)
```

In papers [62, 63] we established the definition of an Archimedean map. We described the method of classification of Archimedean maps up to map isomorphism. The classification of Archimedean maps of genera $2 \leq g \leq 4$ was completed [59]. The main result of [63] reads as follows.

Theorem 4.6 (Classification of Archimedean maps [63])**.** *There are* 17, 103 *and* 111 *isomorphism classes of Archimedean maps of genus* $2, 3, 4$, *respectively.*

4.4 Edge-transitive maps

Another important family of highly symmetrical maps are edge-transitive maps. A map is *edge-transitive* if the (full) automorphism group is transitive on the set of edges. Edge-transitive[†] maps were investigated by several authors including Graver and Watkins [42] and Orbanić et al. [84]. In the latter paper the characterisation in terms of 'symmetry diagrams' on surfaces with boundary induced by the action of the full automorphism group is given. It proves the results obtained by Graver and Watkins. However, the method used in [84] cannot be practically used for construction and classification of edge-transitive maps of higher genera. One have to look for normal subgroups of rather high index if certain finitely-presented groups (N.E.C. groups), which is due to to the complexity of involved algorithms almost intractable. The classification in the mentioned paper and A. Orbanić PhD. thesis [83] is complete up to genus $g = 4$ in case of orientable edge-transitive maps. Our approach in [64] is based on investigation of actions of groups of orientation-preserving automorphisms. This gives us the possibility to employ the results derived e.g. in [60].

As follows, the situation is much better. It is easy to see that the group of orientation-preserving automorphisms of an edge-transitive map $\mathbf{M}$ acts with at most two orbits on the edges of $\mathbf{M}$. It follows that the quotient map $\bar{\mathbf{M}} = \mathbf{M}/\operatorname{Aut}^+(M)$ of an edge-transitive map $\mathbf{M}$, is a map on an (quotient) orbifold with at most two edges. There are exactly 8 such quotient maps sitting on orbifolds with at most 4 singular points, seven are spherical and one is toroidal. The classification of discrete group actions admitting 'quadrangular' orbifolds was given by M. Conder [19] up to genus $g = 101$, a bit more problematic are maps yielding embedding of $\bar{\mathbf{M}}$ on the torus. We have classification of 'toroidal' orbifolds up to genus $g = 21$ [60] and this result can be extended further.

We have the following classification of all possible quotients $\bar{\mathbf{M}} = \mathbf{M}/\operatorname{Aut}^+(\mathbf{M})$ of edge-transitive maps.

Proposition 4.7 ([64])**.** *Let* $\mathbf{M}$ *be an edge-transitive map. Then* $\bar{\mathbf{M}} = \mathbf{M}/\operatorname{Aut}^+(\mathbf{M})$ *is one of the* 8 *maps,* $\mathbf{M}_1$, $\mathbf{M}_2$,…,$\mathbf{M}_8$, *on the orbifolds depicted in Table 4.1, where some of the branch indices may be trivial (taking value 1). We have* $\mathbf{M}_3^* = \mathbf{M}_4$ *and* $\mathbf{M}_5^* = \mathbf{M}_6$. *The remaining maps are self-dual.*

[†] but not necessarily regular

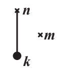
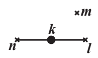
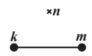
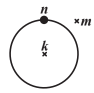
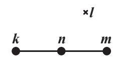
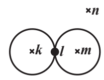
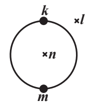
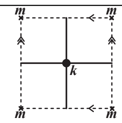
Fam.	Map	Conditions	$\bar{\mathbf{M}}$
E1	$\mathbf{M}_1$	$n \leq 2$	
E2	$\mathbf{M}_2$	$n \leq 2, n = l$	
E3	$\mathbf{M}_3$	none	
E3*	$\mathbf{M}_4$		
E4	$\mathbf{M}_5$	$k = m$	
E4*	$\mathbf{M}_6$		
E5	$\mathbf{M}_7$	none	
E6	$\mathbf{M}_8$	none	

Table 4.1

In [64] we have shown that for each of the 8 families the classification problem reduces to the problem of determining normal subgroups of bounded index in the associated N.E.C. group, all of them are displayed in Table 4.2. The abstract generators r and s in the displayed presentations give rise to external symmetries of the corresponding edge-transitive map. These external symmetries, in fact reflections, are lifts of the reflections of quotient maps (see Table 4.1), transposing the two edges of the quotient map. It happens that a quotient map admits two different reflections; we distinguished those reflections by marking 'subfamilies' by literals 'a' and 'b'. Certainly, it may happen, that a derived (lifted) edge-transitive map admit both reflections. If a quotient map $\mathbf{M}_i$, see e.g. family E4, has two edges, then the group of orientation-preserving automorphisms $\mathrm{Aut}^+(\mathbf{M})$ of the derived edge-transitive map $\mathbf{M}$ is of index two in the full automorphism group $\mathrm{Aut}(\mathbf{M})$, which is transitive on the edges of $\mathbf{M}$. Conversely, if the image of the subgroup of a universal group U (displayed in Table 4.2), generated by all generators but not r and s, is not of index two in image U, then the derived map $\mathbf{M}$ is discarded from the census. In that case $\mathbf{M}$ arises as an edge-transitive map of other family.

Similarly, as in Conder's list of regular maps, we managed the census of edge-transitive maps of genus g in terms of presentations of quotient groups of the corresponding N.E.C. group. For small genera, these quotients can be effectively constructed by using the low-index normal subgroup procedure in Magma [8].

In [64] we proved that any edge-transitive map on an orientable surface of genus $g > 1$ can be expressed as a derived map over one of the aforementioned quotients, taking voltages† from groups acting discretely on orientable surfaces of the corresponding genus. So we have the following classification theorem.

Theorem 4.8 (Projection Theorem [64])**.** *Let* $\mathbf{M}$ *be an edge-transitive map on an orientable surface of genus* $g > 1$*. Then, up to duality* $\mathbf{M}$ *is isomorphic to a derived map over a quotient map displayed in Table 4.1. In particular, there are 8 families of non-degenerate edge-transitive maps distinguished by the quotients* $\mathbf{M}/\mathrm{Aut}^+(\mathbf{M})$.

Example 4.9. We show the reconstruction of an edge-transitive map of genus 3. Let us have a quotient map $\mathbf{M}_2$ from Table 4.1. The dihedral group D_4 has an action on a surface of genus three with corresponding orbifold $(0; \{2, 2, 4, 4\})$. We take the

† voltages are images of generators of U, not including r and s

corresponding universal group from Table 4.2, namely

$$\mathrm{U} = \langle y_1, y_2, y_3, r \mid y_1^4 = y_2^2 = y_3^2 = (y_1^{-1}y_2y_3)^4 = 1, r^2 = 1, y_1^r = y_1^{-1}, y_2^r = y_3, y_3^r = y_2\rangle,$$

setting $k = m = 4$. The following record of Magma session present the core of the procedure obtaining edge transitive maps. Let us just note that the group `UP` is the corresponding Fuchsian group with signature $(0, \{2, 2, 4, 4\})$ (see Example 4.1), the preimage of D_4 acting on a surface of genus 3. We have to check all constraints: the index of the group `K` must be 16, since we are beginning with a N.E.C. group (`UP` extended by a reflection), the subgroup `K` must be torsion free and the image `autp` of `UP` in the epimorphism `phi` must be isomorphic to D_4†, of index 2 in the group `phi(U)`.

```
> U<Y1,Y2,Y3,R>:=Group<Y1,Y2,Y3,R | Y1^4, Y2^2, Y3^2, (Y1^-1*Y2*Y3)^4, \
> R^2, Y1^R=Y1^-1, Y2^R=Y3, Y3^R=Y2 >;
> UP := sub<U | Y1, Y2, Y3>;
> L := LowIndexNormalSubgroups(U,16);
> L16:=[i`Group : i in L | i`Index eq 16];
> #L16;
15

> TF := [];
> for K in L16 do
for> phi := CosetAction(U,K);
for> if Order(phi(Y1)) eq 4 and \
for|if>    Order(phi(Y2)) eq 2 and \
for|if>    Order(phi(Y3)) eq 2 and \
for|if>    Order(phi(Y1^-1*Y2*Y3)) eq 4 and \
for|if>    Order(phi(R)) eq 2 then
for|if>    Append(~TF,phi);
for|if> end if;
for> end for;
> #TF;
4

> ET := [];
> for phi in TF do
for> autp := phi(UP);
for> if Index(phi(U), autp) eq 2 and IdentifyGroup(autp) eq <8,3> then
for|if> Append(~ET,phi);
for|if> end if;
for> end for;
> #ET;
2
```

The two voltage assignments obtained at the end give rise to the map `E2.3.16` in the list [61].

In order to get the permutation representation of the maps, modified voltage assignments are used (cf. Section 2.1.11), similarly as in the classification of Archimedean maps. Compared to the method used by Orbanić et al. in [84], we control the genus g of the underlying surface by choosing a proper g-admissible orbifold. Moreover, the list of g-admissible groups of proper signatures is processed independently on the problem.

Theorem 4.10 (Isomorphism Theorem [64]). *Let* $\mathbf{M}$ *and* $\mathbf{M}'$ *be two edge-transitive maps defined by two T-reduced voltage assignments* ξ, ξ' *in* G, *defined on the quotient map* $\mathbf{M}_i$, $i \in \{1, 2, \ldots, 8\}$. *Then the isomorphism problem is equivalent to the problem whether the natural correspondence between the voltages on darts and vertices of* $\mathbf{M}_i$ *extends to an automorphism of* G.

† the code `<8,3>` in database of small groups [4]

E1	$\langle x_1, x_2 \mid x_1^k = x_2^2 = (x_1^{-1}x_2)^m = 1\rangle$
E2	$\langle y_1, y_2, y_3, r \mid y_1^k = y_2^2 = y_3^2 = (y_1^{-1}y_2y_3)^m = 1,$ $r^2 = 1, y_1^r = y_1^{-1}, y_2^r = y_3, y_3^r = y_2\rangle$
E3	$\langle x_1, x_2 \mid x_1^k = x_2^l = (x_2^{-1}x_1^{-1})^m = 1\rangle$
E4	$\langle y_1, y_2, y_3, r \mid y_1^k = y_2^l = y_3^m = (y_1^{-1}y_2^{-1}y_3^{-1})^n = 1, \rangle$ $r^2 = 1, y_1^r = y_2^{-1}, y_2^r = y_1^{-1}, y_3^r = y_3^{-1}\rangle$
E5a	$\langle y_1, y_2, y_3, r \mid y_1^k = y_2^l = (y_3y_2^{-1})^m = (y_3^{-1}y_1^{-1})^n = 1, \rangle$ $r^2 = 1, y_1^r = y_1^{-1}, y_2^r = y_2^{-1}, y_3^r = (y_2y_3y_2^{-1})^{-1}\rangle$
E5b	$\langle y_1, y_2, y_3, r \mid y_1^k = y_2^l = (y_3y_2^{-1})^m = (y_3^{-1}y_1^{-1})^n = 1, \rangle$ $r^2 = 1, y_1^r = y_2^{-1}, y_2^r = y_1^{-1}, y_3^r = (y_1y_3^{-1}y_1^{-1})^{-1}\rangle$
E6a	$\langle z, a, b, s \mid z^k = (z^{-1}ab^{-1}a^{-1}b)^m = 1,$ $s^2 = 1, z^s = z^{-1}, a^s = b^{-1}, b^s = a^{-1}\rangle$
E6b	$\langle z, a, b, s \mid z^k = (z^{-1}ab^{-1}a^{-1}b)^m = 1,$ $s^2 = 1, z^s = z^{-1}, a^s = b, b^s = a\rangle$

Table 4.2

The enumeration of edge-transitive maps with respect to genus is done in Table 4.3. Non-isomorphic edge-transitive maps are counted in the way that if a map admits two reflections, 'a' and 'b', then it is counted as a member of 'a'-family; thus, members of 'a-family' may admit also the reflection of type 'b'. The members of 'b-family' admit only the reflection of type 'b'.

g	Family / Subfamily								Maps
	E1	E2	E3	E4	E5a	E5b	E6a	E6b	
2	10	1	18	2	44	0	1	0	76
3	20	2	46	6	108	0	1	1	184
4	20	7	53	13	137	0	6	2	238
5	26	11	54	20	177	0	5	4	297
6	23	9	70	16	221	2	7	4	352
7	27	19	80	38	317	0	10	8	499
8	24	9	68	18	237	3	8	6	373
9	52	39	141	77	567	0	26	16	918
10	54	26	158	56	544	0	27	16	881

Table 4.3: Enumeration of edge-transitive maps by family and genus, see [61, 64]

Our classification employs different equivalence relation of maps, compared to classifications of edge-transitive maps published in other works [42, 84]. However, the general approach, as described in aforementioned works, leads to very expensive computations. The best result in [84] classifies edge-transitive maps up to orientable genus 4. On the other hand, with little effort we are able to adapt our method to solve the classification problem for non-orientable surfaces as well and obtain equivalent results as Orbanić at al. [84]. The main idea is based on constructions of 'half-quotients' through antipodal reflections, as introduced in [82]. It looks like that this way we will be able to reasonably decrease the difficulty of the computation.

5. Operations on hypermaps

Working on the classification of Archimedean maps [62], we recognised that many Archimedean maps can be obtained using local redrawings of regular maps. Local redrawings like truncation, creating medial, dual of the map, and others are common in geometry and map theory. In fact, it is most probable method that Kepler used to obtain all Archimedean solids from Platonic solids. As follows, the ideas based on redrawings can be interpreted in more deeper context. In Grothendieck theory of dessins d'enfants, where the underlying surfaces come endowed with a structure of a Riemann surface, deciding when two dessins share the same Riemann surface is a related and unsolved problem. Further, by Belyĭ Theorem every map uniquely determines a Riemann surface defined over a field of algebraic numbers. Local redrawings then give rise to transformations of Belyĭ functions related to regular maps and the family of Belyĭ functions related to maps of broader category. Hence, the geometric ideas of redrawings may have applications e.g. in number theory or cryptography.

Several authors considered distinct instances of problem of map transformations. Magot and Zvonkin determined in [75] explicit transformations of the Belyĭ functions that correspond to the classical operations taking Platonic solids to the Archimedean solids (i.e. Archimedean maps of genus 0). Singerman and Syddall in [92] studied operations on the uniform surface tilings of type $\{k, m, n\}$ preserving the underlying Riemann surface. Further development in this direction was done by Girondo in [41] where he described eight operations (called surgeries) on uniform dessins with the following property: If $\mathbf{H}_1$ and $\mathbf{H}_2$ are uniform dessins with the same underlying Riemann surface, then one can be obtained from the other by applying a sequence of Girondo's surgeries. Moreover, it turned out that a problem of inclusions of Fuchsian groups can be restated as a special instance of transformation problem; transformations of a particular kind appears implicitly in the proof of classification of inclusions of triangle groups in [90]. Moreover, the same 'trick' is used to complete the classification of inclusions of quadrangle groups in [15]. This work can be viewed as an extension of Singerman's classification of inclusions of triangle groups, although it follows slightly different direction.

5.1 Archimedean operations

The monodromy group of an oriented regular map $\mathbf{M}$ of type $\{p, q\}$ is an epimorphic image of $\Delta^+(p, q, 2)$ on a surface of genus $g \geq 2$. Let K be the kernel of this epimorphism. By Theorem 2.12, $\mathbb{H}/\mathrm{K}$ is a Riemann surface which we denote by $\mathcal{R}(\mathbf{M})$. In that setting, the group K can be considered as a group of isometries of $\mathbb{H}$. In [92] Singerman and Syddall consider the problem whether the assignment $\mathbf{M} \mapsto \mathcal{R}(\mathbf{M})$ is injective, or in other words, whether the same Riemann surface can underlie different regular maps. The late situation certainly happens for regular maps of genus 0, since the only Riemann surface of genus 0 is the Riemann sphere, with unique conformal structure. As concerns genus 1, up to duality, there are two infinite families of regular maps (maps of type $\{3, 6\}$ and of type $\{4, 4\}$) and there are two Riemann surfaces associated with these two families. To continue our discussion we need to define the following two functorial operations. By the *truncation* of $\mathbf{M}$ we mean the cubic map whose vertices are darts of $\mathbf{M}$ and two are joined by an edge if they form an angle of $\mathbf{M}$, or they underlie the same edge. By the *medial map* we mean the map whose vertices are the middle points of edges of $\mathbf{M}$, two being adjacent if the respective edges form an angle of $\mathbf{M}$. Finally, let $e(\mathbf{M})$ denotes the number of edges of $\mathbf{M}$. Using the above defined operations one can describe regular maps sharing the same Riemann surface in almost all cases.

Theorem 5.1 ([92]). *Let $\mathbf{M}$ and $\mathbf{N}$ be two orientably regular maps of genus > 1 with $\mathcal{R}(\mathbf{M}) = \mathcal{R}(\mathbf{N})$ and $e(\mathbf{M}) \leq e(\mathbf{N})$. Then one of the following statements holds:*

1) $\mathbf{N}$ is the medial map of $\mathbf{M}$,
2) $\mathbf{N}$ is the truncation of $\mathbf{M}$,
3) $\mathbf{N}$ has type $(7,3)$ and $\mathbf{M}$ has type $(7,7)$,
4) $\mathbf{N}$ has type $(8,3)$ and $\mathbf{M}$ has type $(8,8)$.
5) $\mathbf{N} \cong \mathbf{M}$ or to its dual map.

The paper [14] continues further in this direction. Well-known redrawings has been described in the rigorous manner such that the belong to the class of *Archimedean operations*. Further, the family of vertex-transitive maps arising from regular maps and hypermaps has been investigated there. We call this family of maps *maps of Archimedean class*. The defining criterion for this class is that the map $\mathbf{M}$ of Archimedean class comes from a regular hypermap $\mathbf{H}$ applying an Archimedean operation†. The automorphism group of the resulting vertex-transitive map $\mathbf{M}$ contains a copy of the automorphism group of regular hypermap $\mathbf{H}$ and $\mathbf{M}/\operatorname{Aut}^+(\mathbf{M})$ is the quotient map sitting on an orbifold with signature $(0;\{k,m,n\})$. The quotient map of the corresponding regular hypermap $\mathbf{H}/\operatorname{Aut}^+(\mathbf{H})$ is the trivial map sitting on the same orbifold. Thus, the Archimedean operation can be viewed as a (local) redrawing of the (vertex, edge, face) figure of the corresponding regular hypermap. Just note that face-width of map of Archimedean class does not play a rôle in [14], in contrast to the definition of Archimedean maps [62]. Not every Archimedean map is a map of Archimedean class, however, these categories have non-empty intersection. A representative of an Archimedean map which is not a map of Archimedean class has been treated in Example 4.5.

Main result of [14] reads as follows.

Theorem 5.2. *Every map $\mathbf{M}$ of Archimedean class of genus g is either a regular map, or comes from a regular hypermap $\mathbf{H}$ of genus g by applying one of the 10 Archimedean operations.*

With few exceptions, such as the (regular) maps and hypermaps operations introduced by Jones and Singerman [55] and James [50], many of non-regular maps operations are loosely defined. The definition of an operation on a dessin can be rigorously stated using Belyĭ's Theorem [1] (see also Section 2.4). As a result we have the following nice properties of operations [14], namely:

1) they form a monoid under composition;
2) they preserve the underlying Riemann surface and coverings between dessins;
3) they preserve the automorphism group.

More important yet, the Magot and Zvonkin's operations, as well as the Girondo's surgeries, can be then expressed in terms of our operations. Furthermore, inclusions between triangle groups determined by Singerman [90] give rise to particular instances of operations studied in [14]. Since regular maps and hypermaps are of Archimedean class, the result of [14] can be interpreted as an extension of Singerman's and Girondo's results describing regular dessins on the same Riemann surface [41, 91, 92].

† as follows, a series of Archimedean operations

Let $\mathbf{H}$ be a hypermap and $W(\mathbf{H})$ be its associated dessin[†] on a surface $\mathcal{S}_g$. Denote by $\omega_{\mathbf{H}} : W(\mathbf{H}) \to \mathbf{B}$ a Belyĭ function taking $W(\mathbf{H})$ onto the trivial dessin $\mathbf{B}$ on the Riemann sphere Σ. By definition, $\omega_{\mathbf{H}}$ takes $W(\mathbf{H})$ onto the 3-pointed sphere Σ^* with the singular points 0, 1, and ∞. In what follows, we use the convention that the fibres over 0, 1, and ∞ represent respectively (hyper)vertices, (hyper)edges, and centres of (hyper)faces of $\mathbf{H}$. Let $\mathbf{N}$ be a dessin on Σ^*. Then the lift $\omega_{\mathbf{H}}^{-1}(\mathbf{N})$ determines a dessin $W(\mathbf{H}')$ associated with a hypermap $\mathbf{H}'$. Note that the combinatorial structure of $W(\mathbf{H}')$ is uniquely determined and the Belyĭ function $\omega_{\mathbf{N}}$ satisfying $\omega_{\mathbf{N}}(\{0,1,\infty\}) \subseteq \{0,1,\infty\}$. Setting $\mathbf{H}' = T_{\mathbf{N}}(\mathbf{H}) = W^{-1}(\omega_{\mathbf{H}}^{-1}(\mathbf{N}))$ we get an *operation* $T_{\mathbf{N}} : \mathbf{H} \mapsto \mathbf{H}'$ from the set of hypermaps into the set of hypermaps. Clearly, $T_{\mathbf{N}}$ is universally defined on the category of oriented hypermaps and it depends just on the choice of $\mathbf{N}$. For technical reasons we consider taking mirror image of a hypermap $\mathbf{H} \mapsto \mathbf{H}^{-1}$ to be an operation. Moreover, any of the twelve dualities (see Section 2.2.3) is an operation $T_{\mathbf{N}}$ for some $\mathbf{N}$. The definition of an operation is then done in the similar fashion as it was introduced by Singerman and Syddall in [92, Section 5].

The following propositions gives a list of basic properties of operations.

Proposition 5.3 ([14]). *Let T, U be the operations determined by maps $\mathbf{N}_1$ and $\mathbf{N}_2$, respectively and let $\mathbf{H}$ be any hypermap. Then*

1) *$\mathbf{H}$ and $T(\mathbf{H})$ share the same Riemann surface, $\mathcal{R}(\mathbf{H}) = \mathcal{R}(T(\mathbf{H}))$;*
2) *$\mathrm{Aut}^+(\mathbf{H}) \leq \mathrm{Aut}^+(T(\mathbf{H}))$;*
3) *the composition $(U \circ T)(\mathbf{H}) = U(T(\mathbf{H}))$ is an operation,*
4) *the composition of operations is associative.*

Proposition 5.4 ([14]). *Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be two hypermaps, $\vartheta : \mathbf{H}_1 \to \mathbf{H}_2$ be a covering, and T be an operation. Then $\vartheta \circ T(\mathbf{H}_1) = T(\mathbf{H}_2) = T \circ \vartheta(\mathbf{H}_1)$.*

It follows that any operation can be seen as a functor of the category of orientable hypermaps. Further, a map $\mathbf{M}$ is of *Archimedean class* on an orientable surface S_g of genus g if its full group of automorphisms acts transitively on vertices and there exists an operation $T_{\mathbf{N}}$ such that $\mathbf{M} = T_{\mathbf{N}}(\mathbf{H})$ for some regular hypermap of genus g.

As it was already noted, $\mathrm{Aut}^+(\mathbf{H})$ of a regular hypermap is a quotient of a triangle group $\Delta^+(k,m,n)$ by a normal subgroup, for some $k,m,n > 1$. Vice-versa, any finite quotient G of a triangle group $\Delta^+(k,m,n)$ determines a regular hypermap defined as an algebraic hypermap $\mathbf{H} = (\mathrm{G}; x, y)$. In particular, if $\mathbf{H}$ is a regular hypermap, then $\omega_{\mathbf{H}}$ is a branched regular cover defined by the action of $\mathrm{Aut}^+(\mathbf{H})$ and $\omega_{\mathbf{H}} : \mathcal{S}_g \to \mathcal{S}_g/\mathrm{Aut}^+(\mathbf{H}) = \mathcal{O}$ with signature $(0; \{k,m,n\})$.

The operation $T_{\mathbf{N}}$ is called a *Platonic operation*, if there exist regular hypermaps $\mathbf{H}$ and $\mathbf{M}$, such that $T_{\mathbf{N}}(\mathbf{H}) = \mathbf{M}$. An operation $T_{\mathbf{N}}$ is an *Archimedean operation*, if there exist a map $\mathbf{M}$ of Archimedean class and a regular hypermap $\mathbf{H}$, such that $T_{\mathbf{N}}(\mathbf{H}) = \mathbf{M}$. Note that the mirror image of a hypermap and the dualities of all kinds are Platonic operations. Platonic operation on regular maps were investigated by Singerman and Syddall [92], while Girondo investigated operations between uniform dessins [41]. Note that all Girondo's operations (called surgeries in [41]) give rise to Platonic operations. Observe that many Archimedean operations can be obtained as compositions $T(P(\mathbf{H}))$, where P is a Platonic operation and T is an Archimedean operation.

Figure 5.1 shows 10 dessins on the 3-pointed sphere defining 10 Archimedean operations on hypermaps [14]. Most of the operations described by dessins in Figure 5.1 are well-known, and were intensively employed in distinguished contexts. In particular, all but T_2, T_3, and T_5 give always maps, the operations T_2 (truncation), T_3 and T_5 (snub) produce maps if they are applied to oriented maps. The operations $T_1, \ldots, T_5$, T_7, and

[†] better to said, a Walsh map, see Section 2.2.2

T_8 represent the classical operations transforming Platonic solids onto Archimedean ones.

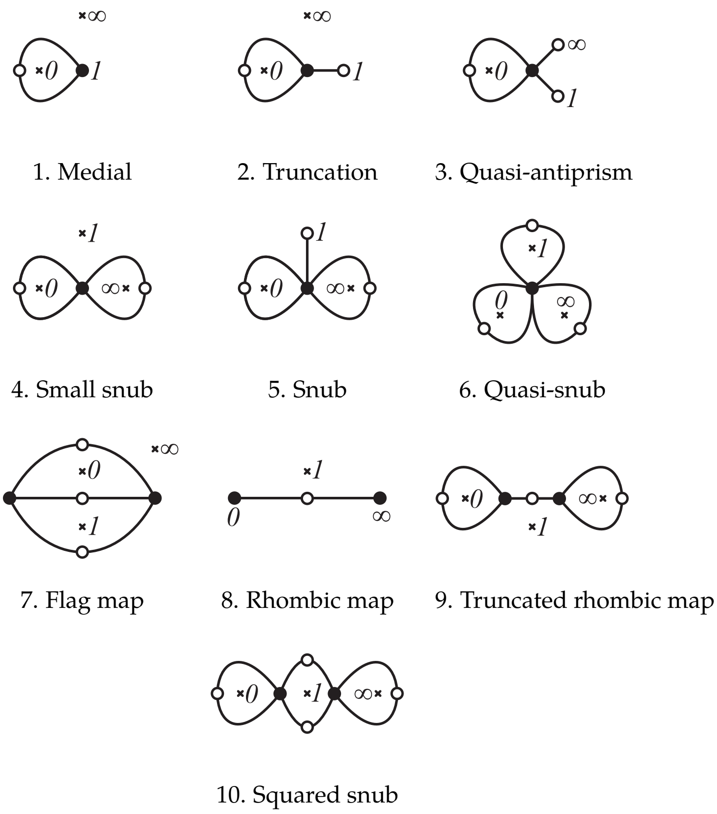


Figure 5.1: Dessins defining Archimedean operations

The main result of [14] involves dessins on Figure 5.1 and describes the set of Archimedean operations. It is straightforward, that it can be seen as a generalisation of Theorem 5.1. The following corollary is then the direct mimic of Singerman and Syddall's theorem.

Theorem 5.5 ([14]). *Let* $\mathbf{M}$ *be a map of Archimedean class of type* $\{k, m, n\}$ *and of genus* g. *Then there exist a regular hypermap* $\mathbf{H}$ *of genus* g *with* $\mathrm{Aut}^+(\mathbf{H}) = \mathrm{Aut}^+(\mathbf{M})$ *such that one of the following cases happen:*

1) $\mathbf{M} = \mathbf{H}$ *is a regular map;*
2) $\mathbf{M} \cong \mathrm{T}_j(\mathbf{H})$ *for some* $j \in \{1, 4, 6\}$, *i.e.* $\mathbf{M}$ *is medial, small snub, or quasi-snub of a regular hypermap* $\mathbf{H}$, *respectively;*
3) $n = 2$ *and* $\mathbf{M} \cong \mathrm{T}_j(\mathbf{H})$ *for some* $j \in \{2, 5\}$, *i.e.* $\mathbf{M}$ *is truncation, or snub of a regular hypermap* $\mathbf{H}$, *respectively;*
4) $k > 3$, $m = n = 2$, $\mathbf{H}$ *is a k-cycle in the sphere and* $\mathbf{M} \cong \mathrm{T}_3(\mathbf{H})$ *is a k-antiprism;*
5) $\mathbf{H}$ *is reflexible and* $\mathbf{M} \cong \mathrm{T}_7(\mathbf{H})$ *is the flag map of a reflexible regular hypermap;*
6) $k = m$, $\mathbf{H}$ *is* $(0, \infty)^-$*-self-dual and* $\mathbf{M} \cong \mathrm{T}_j(\mathbf{H})$, *for some* $j \in \{8, 9, 10\}$, *i.e.* $\mathbf{M}$ *is the rhombic, the truncated rhombic, or the squared snub map of* $\mathbf{H}$.

Corollary 5.6 ([14]). *Let* $\mathbf{M}_1 = T_i(\mathbf{H}_1)$ *and* $\mathbf{M}_2 = T_j(\mathbf{H}_2)$, $i, j \in \{1, \dots, 10\}$, *be two maps of Archimedean class, where* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *are regular hypermaps. Then* $\mathcal{R}(\mathbf{M}_1) = \mathcal{R}(\mathbf{M}_2)$ *if and only if* $\mathcal{R}(\mathbf{H}_1) = \mathcal{R}(\mathbf{H}_2)$.

Example 5.7. It is well-known that regular maps on the sphere are m-cycles, m-dipoles and the 2-skeletons of the five Platonic solids. In particular, we have

1) the infinite family of *m-dipoles*, $m \geq 1$ of local type (2^m);
2) the infinite family of *m-cycles*, $m \geq 1$ of local type (m^2);
3) *tetrahedron* of local type (3^3);
4) *cube* of local type (4^3);
5) *octahedron* of local type (3^4);
6) *dodecahedron* of local type (5^3);
7) *icosahedron* of local type (3^5).

The 2-skeletons of Archimedean solids are by definition polyhedral vertex-transitive maps. By Riemann-Hurwitz equation the following orbifold types of discrete actions of finite groups on the sphere are admissible: $O(0;\{n,n\})$, $n \geq 2$, $O(0;\{2,2,n\})$, $n \geq 2$, $O(0,\{2,3,3\})$, $O(0,\{2,3,4\})$ and $O(0,\{2,3,5\})$. Except type $O(0;\{n,n\})$ we may apply operations from Theorem 5.5 to construct all classical Archimedean solids (maps on sphere). The results are shown in Table 5.1

	(2^n), $n > 2$	(3^3)	(4^3)	(3^4)	(5^3)	(3^5)
T_1	$*$	(3^4)	(3.4.3.4)	(3.4.3.4)	(3.5.3.5)	(3.5.3.5)
T_2	$(4.4.n)$	(3.6.6)	(3.8.8)	(4.6.6)	(3.10.10)	(5.6.6)
T_3	$(3.3.3.n)$	$\times$	$\times$	$\times$	$\times$	$\times$
T_4	$*$	(3.4.3.4)	(3.4^3)	(3.4^3)	(3.4.5.4)	(3.4.5.4)
T_5	$*$	(3^5)	$(3^4.4)$	$(3^4.4)$	$(3^4.5)$	$(3^4.5)$
T_6	$*$	$*$	$*$	$*$	$*$	$*$
T_7	$*$	(4.6.6)	(4.6.8)	(4.6.8)	(4.6.10)	(4.6.10)
T_8	$\times$	(4^3)	$\times$	$\times$	$\times$	$\times$
T_9	$\times$	(3.8.8)	$\times$	$\times$	$\times$	$\times$
T_{10}	$\times$	(3.4^3)	$\times$	$\times$	$\times$	$\times$

Table 5.1: Archimedean solids arising from Platonic ones

Columns of Table 5.1 represent regular maps on the sphere indicated by the local type, while the rows correspond to the operations. The (i,j)-entry gives an information of the resulting map $T_i(\mathbf{M}_j)$, where $\mathbf{M}_j$ is the regular map represented by j-th column. All the maps are determined by their local types. Since the map $T_i(\mathbf{M}_j)$ may not be polyhedral, we mark this fact by '$*$' in the table. If an operation T_i applied on the map $\mathbf{M}_j$ does not give a map or the resulting map is not vertex-transitive (see Theorem 5.5), we mark this fact by the symbol '$\times$'. In Table 5.1 we also exclude the column representing the infinite family of cycles, since no operation applied on a cycle gives rise to a polyhedral, vertex-transitive map or does not give a map. Since Archimedean solids are uniquely determined by its local type (up to taking the mirror image), we can identify them in Table 5.1 by displaying their respective local type. Comparing the maps given in Table 5.1 with the classification of Archimedean solids we get complete census. List of Archimedean solids (see e.g. [27]) includes:

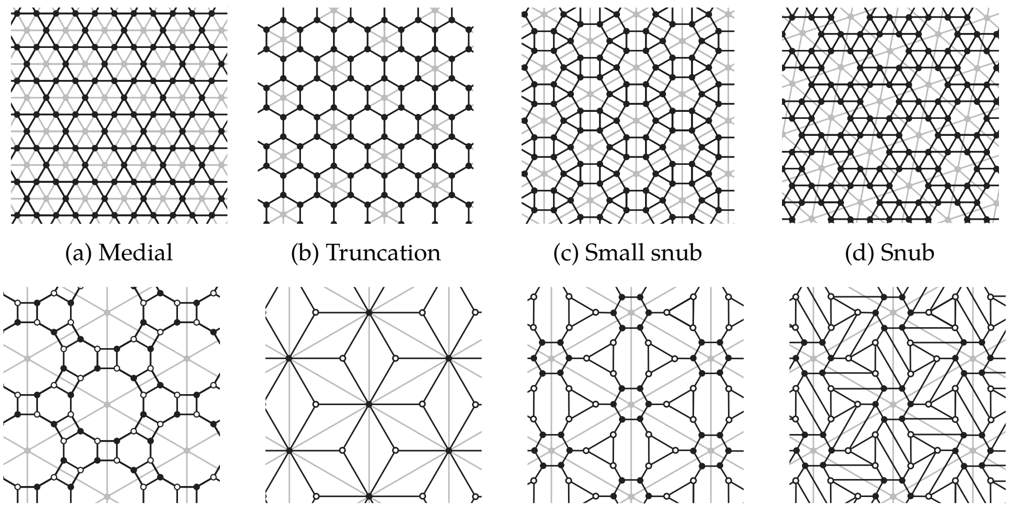
1) *truncated tetrahedron* of local type (3.6.6);
2) *truncated cube* of local type (3.8.8);
3) *truncated dodecahedron* of local type (3.10.10);
4) infinite series of *n-prisms*, $n > 2$ of local type $(4.4.n)$;
5) *truncated octahedron* of local type (4.6.6);
6) *truncated cuboctahedron* of local type (4.6.8);
7) *truncated icosidodecahedron* of local type (4.6.10);
8) *truncated icosahedron* ('soccer ball') of local type (5.6.6);
9) infinite series of *n-antiprisms*, $n > 2$ of local type $(3^3.n)$;

10) *cuboctahedron* of local type (3.4.3.4);
11) *rhombicuboctahedron* of local type (3.4^3);
12) *rhombicosidodecahedron* of local type (3.4.5.4);
13) *icosidodecahedron* of local type (3.5.3.5);
14) *snub cube* of local type $(3^4.4)^{\pm}$;
15) *snub dodecahedron* of local type $(3^4.5)^{\pm}$.

Note that the *snub cube* and the *snub dodecahedron* appears in two chiral forms.

Example 5.8. The Euclidean tiling $\{3,6\}$ is a 6-valent tessellation of the Euclidean plane by equilateral triangles. It is an example of the universal regular map with the automorphism group $\mathrm{G} = \Delta^{+}(2,3,6) = \langle x, y, z \mid x^2 = y^3 = z^6 = 1,\ xyz = 1\rangle$. The extended triangle group $\Delta(2,3,6)$ acts as a (full) group of automorphisms of $\{3,6\}$, therefore $\{3,6\}$ is a reflexible regular map.

Archimedean operations can be naturally used on infinite maps as it is shown here. The embeddings of the respective (infinite) regular maps on the Euclidean plane are displayed in grey on both figures (Figure 5.2 and Figure 5.3) to show relationship between map **M** and the transformed map $T_j(\mathbf{M})$.



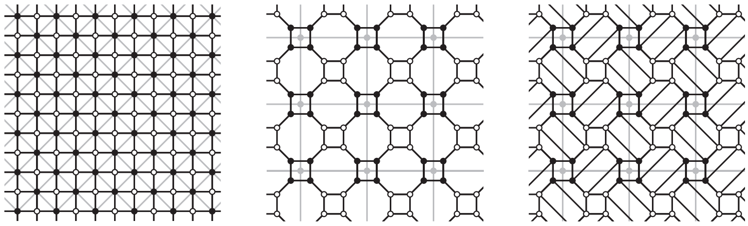
(a) Medial (b) Truncation (c) Small snub (d) Snub

(e) Flag map (f) Rhombic map (g) Trunc. rhombic map (h) Squared snub

Figure 5.2: Archimedean operations applied on Euclidean tiling $\{3,6\}$

Figure 5.2 shows the action of all the operations T_j, except T_3 and T_6, on the Euclidean tiling $\{3,6\}$. The operation T_3 gives a map only for branch assignment $\{0,1,\infty\} \to \{2,2,n\}$. It follows that $T_3(\{3,6\})$ is a hypermap. As concerns $T_6(\{3,6\})$ it can be easily seen that the resulting map is degenerate; it contains planar embeddings of a dipole, each vertex is incident with three such dipoles. The two orbits of the action of G on vertices of $T_j(\{3,6\})$, $j = 7,8,9,10$ are distinguished by black and white colours, respectively (Figure 5.2(e)-(h)). Using Theorem 5.5 we construct maps $T_j(\{3,6\})$, $j \in \{1,2,4,5,7\}$ and they are vertex-transitive. Since the Euclidean tiling $\{3,6\}$ is not self-dual, the operations T_8, T_9, and T_{10} do not give vertex-transitive maps.

On the other hand, operations T_8, T_9, and T_{10} applied to the infinite Euclidean tiling of type $\{4,4\}$ are vertex-transitive maps, as depicted on Figure 5.3. Again, the two orbits of an action of $\mathrm{H} = \langle x, y, z \mid x^2 = y^4 = z^4 = 1,\ xyz = 1\rangle$ are distinguished by different colours.



(a) Rhombic map (b) Trunc. rhombic map (c) Squared snub

Figure 5.3: Operations T_6, T_7, T_8 applied on Euclidean tiling $\{4,4\}$

Bibliography

[1] G. V. Belyĭ, *Galois extensions of a maximal cyclotomic field*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 2, 267–276, 479. MR 534593 (80f:12008)

[2] E. A. Bender and E. R. Canfield, *The asymptotic number of rooted maps on a surface*, J. Combin. Theory Ser. A **43** (1986), no. 2, 244–257. MR 867650 (88a:05080)

[3] P. Bergau and D. Garbe, *Nonorientable and orientable regular maps*, Groups—Korea 1988 (Pusan, 1988), Lecture Notes in Math., vol. 1398, Springer, Berlin, 1989, pp. 29–42. MR 1032808 (90k:57003)

[4] H. U. Besche, B. Eick, and E. A. O'Brien, *The Small Groups library*, `http://www.icm.tu-bs.de/ag_algebra/software/small/`, 2014.

[5] N. L. Biggs, *Automorphisms of imbedded graphs*, J, Combinatorial Theory Ser. B **11** (1971), 132–138.

[6] ______, *Homological coverings of graphs*, J. London Math. Soc. Second Series **30** (1984), no. 1, 1–14. MR 760867 (86e:05050)

[7] O. V. Bogopolski, *Classifying the actions of finite groups on orientable surfaces of genus* 4 *[translation of proceedings of the institute of mathematics, 30 (Russian), 48–69, Izdat. Ross. Akad. Nauk, Sibirsk. Otdel., Inst. Mat., Novosibirsk, 1996]*, Siberian Adv. Math. **7** (1997), no. 4, 9–38, Siberian Advances in Mathematics. MR 1604157 (2000e:30083)

[8] W. Bosma, J. Cannon, C. Fieker, and A. Steel, *Handbook of Magma functions*, Edition 2.16, 2010.

[9] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993).

[10] H. R. Brahana, *Systems of circuits on two-dimensional manifolds*, Ann. of Math. (2) **23** (1921), no. 2, 144–168. MR 1502608

[11] ______, *The Four-Color Problem*, Amer. Math. Monthly **30** (1923), no. 5, 234–243. MR 1520241

[12] ______, *Regular Maps and Their Groups*, Amer. J. Math. **49** (1927), no. 2, 268–284. MR 1506619

[13] A. J. Breda d'Azevedo, *The reflexible hypermaps of characteristic* -2, Math. Slovaca **47** (1997), no. 2, 131–153. MR 1476863 (98j:05056)

[14] A. J. Breda d'Azevedo, D. A. Catalano, J. Karabáš, and R. Nedela, *Maps of Archimedean class and operations on dessins*, accepted to Discrete Mathematics, preprint, 2013.

[15] ______, *Atlas of quadrangle group inclusions*, in progress, 2014.

[16] A. J. Breda d'Azevedo, R. Nedela, and J. Širáň, *Classification of regular maps of negative prime euler characteristic*, Trans. Amer. Math. Soc. **357** (2005), 4175–4190.

[17] S. A. Broughton, *Classifying finite group actions on surfaces of low genus*, J. Pure Appl. Algebra **69** (1991), no. 3, 233–270. MR 1090743 (92b:57021)

[18] W. Burnside, *Theory of groups of finite order*, Dover Publications, Inc., New York, 1955, 2d ed. MR 0069818 (16,1086c)

[19] M. D. E. Conder, *All large groups of automorphisms of compact Riemann surfaces of genus 2 to 101*, `http://www.math.auckland.ac.nz/~conder/BigSurfaceActions-Genus2to101-ByGenus.txt`, November 2011.

[20] ———, *All chiral (irreflexible) orientably-regular maps on surfaces of genus 2 to 301, up to isomorphism, duality and reflection, with defining relations for their automorphism groups*, `http://www.math.auckland.ac.nz/~conder/ChiralMaps301.txt`, 2012.

[21] ———, *All reflexible orientable regular maps on surfaces of genus 2 to 301,*, `http://www.math.auckland.ac.nz/~conder/OrientableRegularMaps101.txt`, 2012.

[22] ———, *All orientable regular maps on surfaces of genus 2 to 101, up to isomorphism and duality, with defining relations for their automorphism groups*, `http://www.math.auckland.ac.nz/~conder/OrientableRegularMaps101.txt`, 2014.

[23] ———, *Marston Conder's homepage*, `http://www.math.auckland.ac.nz/˜conder/`, 2014.

[24] M. D. E. Conder, R. Nedela, and J. Širáň, *Classification of regular maps of Euler characteristic $-3p$*, J. Combin. Theory Ser. B **102** (2012), no. 4, 967–981. MR 2927416

[25] M. D. E. Conder, P. Potočnik, and J. Širáň, *Regular maps with almost Sylow-cyclic automorphism groups, and classification of regular maps with Euler characteristic $-p^2$*, J. Algebra **324** (2010), no. 10, 2620–2635. MR 2725192 (2012b:20100)

[26] M. D. E. Conder, J. Širáň, and T. W. Tucker, *The genera, reflexibility and simplicity of regular maps*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 2, 343–364. MR 2608943 (2011i:05092)

[27] J. H. Conway, H. Burgiel, and C. Goodman-Strauss, *The symmetries of things*, A K Peters Ltd., Wellesley, MA, 2008. MR 2410150 (2009c:00002)

[28] J. H. Conway and D. H. Huson, *The orbifold notation for two-dimensional groups*, Structural Chemistry **13** (2002), no. 3-4, 247–257 (English).

[29] J.H. Conway, *The orbifold notation for surface groups*, Groups, Combinatorics & Geometry (Martin W. Liebeck and Jan Saxl, eds.), Cambridge University Press, 1992, Cambridge Books Online, pp. 438–447.

[30] R. Cori and A. Machì, *Maps, hypermaps and their automorphisms: a survey. I, II, III*, Exposition. Math. **10** (1992), no. 5, 403–427, 429–447, 449–467. MR 1190182 (94a:57006)

[31] T.H Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to algorithms*, 3rd edition ed., The MIT Press, 2009.

[32] D. Corn and D. Singerman, *Regular hypermaps*, European J. Combin. **9** (1988), 337–351.

[33] H. S. M. Coxeter, M. S. Longuet-Higgins, and J. C. P. Miller, *Uniform polyhedra*, Philos. Trans. Roy. Soc. London. Ser. A. **246** (1954), 401–450 (6 plates). MR 0062446 (15,980e)

[34] H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups*, fourth ed., Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 14, Springer-Verlag, Berlin-New York, 1980. MR 562913 (81a:20001)

[35] J. D. Dixon and B. Mortimer, *Permutation groups*, Graduate Texts in Mathematics, vol. 163, Springer-Verlag, New York, 1996. MR 1409812 (98m:20003)

[36] M. Drmota and R. Nedela, *Asymptotic enumeration of reversible maps regardless of genus*, Ars Math. Contemp. **5** (2012), no. 1, 77–97. MR 2869453 (2012m:05048)

[37] H. M. Farkas and I. Kra, *Riemann surfaces*, second ed., Graduate Texts in Mathematics, vol. 71, Springer-Verlag, New York, 1992. MR 1139765 (93a:30047)

[38] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.7.5*, 2014.

[39] D. Garbe, *Über die regulären Zerlegungen geschlossener orientierbarer Flächen*, J. Reine Angew. Math. **237** (1969), 39–55. MR 0246198 (39 #7502)

[40] A. Gardiner, R. Nedela, J. Širáň, and M. Škoviera, *Characterisation of graphs which underlie regular maps on closed surfaces*, J. London Math. Soc. (2) **59** (1999), no. 1, 100–108. MR 1688492 (2000a:05104)

[41] E. Girondo, *Multiply quasiplatonic Riemann surfaces*, Experiment. Math. **12** (2003), no. 4, 463–475. MR 2043996 (2005h:30078)

[42] J. E. Graver and M. E. Watkins, *Locally finite, planar, edge-transitive graphs*, Mem. Amer. Math. Soc. **126** (1997), no. 601, vi+75. MR 1361828 (97i:05001)

[43] J. L. Gross and T. W. Tucker, *Topological graph theory*, Dover Publications Inc., Mineola, NY, 2001, Reprint of the 1987 original [Wiley, New York; MR0898434 (88h:05034)] with a new preface and supplementary bibliography. MR 1855951

[44] B. Grünbaum and G. C. Shepard, *Edge-transitive planar graphs*, J. Graph Theory **11** (1987), 141–155.

[45] B. Grünbaum and G. C. Shephard, *Convex polytopes*, Bull. London Math. Soc. **1** (1969), 257–300. MR 0250188 (40 #3428)

[46] ______, *Tilings and patterns*, W. H. Freeman and Company, New York, 1987. MR 857454 (88k:52018)

[47] W. J. Harvey, *Cyclic groups of automorphisms of a compact Riemann surface*, Quart. J. Math. Oxford Ser. (2) **17** (1966), 86–97. MR 0201629 (34 #1511)

[48] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354 (2002k:55001)

[49] D. F. Holt, B. Eick, and E. A. O'Brien, *Handbook of computational group theory*, Discrete Mathematics and its Applications (Boca Raton), Chapman & Hall/CRC, Boca Raton, FL, 2005. MR 2129747 (2006f:20001)

[50] L. D. James, *Operations on hypermaps and outer automorphism*, European J. Combin. **9** (1988), 551–560.

[51] N. W. Johnson, *Convex solids with regular faces*, Canadian Journal of Mathematics **18** (1966), 169–200.

[52] G. A. Jones, *Maps on surfaces and Galois groups*, Math. Slovaca **47** (1997), no. 1, 1–33, Graph theory (Donovaly, 1994). MR 1476746 (98i:05055)

[53] ______, *Riemann surfaces and their automorphism groups*, GAC 2010 workshop lecture notes, Harish-Chandra Research Insitute Allahabad, India, 2010.

[54] G. A. Jones and D. Singerman, *Theory of maps on orientable surfaces*, Proc. of the London Math. Soc. Third Series **37** (1978), no. 2, 273–307. MR 0505721 (58 #21744)

[55] ______, *Complex functions*, Cambridge University Press, Cambridge, 1987, An algebraic and geometric viewpoint. MR 890746 (89b:30001)

[56] ______, *Belyĭ functions, hypermaps and Galois groups*, Bull. London Math. Soc. **28** (1996), no. 6, 561–590. MR 1405488 (97g:11067)

[57] G. A. Jones and D. B. Surowski, *Regular cyclic coverings of the Platonic maps*, European J. Combin. **21** (2000), no. 3, 333–345. MR 1750888 (2001a:05076)

[58] G. A. Jones and J. S. Thornton, *Operations on maps, and outer automorphisms*, J. Combin. Theory Ser. B **35** (1983), no. 2, 93–103. MR 733017 (85m:05036)

[59] J. Karabáš, *Archimedean maps of higher genera*, `http://www.savbb.sk/~karabas/science.html#arch`, 2012.

[60] ______, *Actions of finite groups on Riemann surfaces of higher genera*, `http://www.savbb.sk/~karabas/finacts.html`, 2013.

[61] ______, *Edge transitive maps on orientable surfaces*, `http://www.savbb.sk/~karabas/science.html#etran`, 2014.

[62] J. Karabáš and R. Nedela, *Archimedean solids of genus two*, 6th Czech-Slovak International Symposium on Combinatorics, Graph Theory, Algorithms and Applications, Electron. Notes Discrete Math., vol. 28, Elsevier, Amsterdam, 2007, pp. 331–339. MR 2324036

[63] ______, *Archimedean maps of higher genera*, Math. Comp. **81** (2012), no. 277, 569–583. MR 2833509

[64] ______, *Discrete group actions and edge-transitive maps on a given surface*, in progress, 2014.

[65] S. Katok, *Fuchsian groups*, The University of Chicago Press, 1992.

[66] F. Klein, *Ueber die Transformation siebenter Ordnung der elliptischen Functionen*, Math. Ann. **14** (1878), no. 3, 428–471. MR 1509988

[67] A. Kuribayashi and H. Kimura, *Automorphism groups of compact Riemann surfaces of genus five*, J. Algebra **134** (1990), no. 1, 80–103. MR 1068416 (91j:30033)

[68] I. Kuribayashi and A. Kuribayashi, *Automorphism groups of compact Riemann surfaces of genera three and four*, J. Pure Appl. Algebra **65** (1990), no. 3, 277–292. MR 1072285 (92a:30041)

[69] S. K. Lando and A. K. Zvonkin, *Graphs on surfaces and their applications*, Encyclopaedia of Mathematical Sciences, vol. 141, Springer-Verlag, Berlin, 2004, With an appendix by Don B. Zagier, Low-Dimensional Topology, II. MR 2036721 (2005b:14068)

[70] J. Lehner and M. Newman, *On Riemann surfaces with maximal automorphism groups*, Glasgow Math. J. **8** (1967), 102–112. MR 0220924 (36 #3976)

[71] V. A. Liskovets, *A census of nonisomorphic planar maps*, Algebraic methods in graph theory, Vol. I, II (Szeged, 1978), Colloq. Math. Soc. János Bolyai, vol. 25, North-Holland, Amsterdam-New York, 1981, pp. 479–494. MR 642058 (83a:05073)

[72] ______, *Enumeration of nonisomorphic planar maps.*, Selecta Math. Sovietica **4** (1985), 303–323.

[73] ______, *A multivariate arithmetic function of combinatorial and topological significance*, INTEGERS **10** (2010), 155–177.

[74] A. Machì, *On the complexity of a hypermap*, Discrete Math. **42** (1982), no. 2-3, 221–226. MR 677055 (84a:05024)

[75] N. Magot and A. K. Zvonkin, *Belyi functions for Archimedean solids*, Discrete Mathematics **217** (2000), no. 13, 249–271.

[76] A. Malnič, R. Nedela, and M. Škoviera, *Lifting graph automorphisms by voltage assignments*, European J. Combin. **21** (2000), no. 7, 927–947.

[77] ______, *Regular homomorphisms and regular maps*, European J. Combin. **23** (2002), no. 4, 449–461. MR 1914482 (2003g:05045)

[78] W. S. Massey, *A basic course in algebraic topology*, Graduate Texts in Mathematics, vol. 127, Springer-Verlag, New York, 1991. MR 1095046 (92c:55001)

[79] A. D. Mednykh and R. Nedela, *Enumeration of unrooted maps of a given genus*, J. Combin. Theory Ser. B **96** (2006), no. 5, 706–729. MR 2236507 (2007g:05088)

[80] B. Mohar and C. Thomassen, *Graphs on surfaces*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, MD, 2001. MR 1844449 (2002e:05050)

[81] R. Nedela, *Regular maps – combinatorial objects relating different fields of mathematics*, J. Korean Math. Soc. **38** (2001), no. 5, 1069–1105, Mathematics in the new millennium (Seoul, 2000). MR 1849340 (2002k:05071)

[82] R. Nedela and M. Škoviera, *Exponents of orientable maps*, Proc. London Math. Soc. (3) **75** (1997), no. 1, 1–31. MR 1444311 (98i:05059)

[83] A. Orbanić, *Edge-transitive maps*, Ph.D. thesis, University of Ljubljana, `http://www.ijp.si/RegularMaps/`, 2006.

[84] A. Orbanić, D. Pellicer, T. Pisanski, and T. W. Tucker, *Edge-transitive maps of low genus*, Ars Mathematica Contemporanea **4** (2011), 385–402.

[85] B. Richter, J. Širáň, R. Jajcay, T. W. Tucker, and M. E. Watkins, *Cayley maps*, J. Combin. Theory Ser. B **95** (2005), no. 2, 189–245. MR 2171363 (2006g:05063)

[86] G. Ringel and J. W. T. Youngs, *Solution of the Heawood map-coloring problem*, Proc. Nat. Acad. Sci. U.S.A. **60** (1968), 438–445. MR 0228378 (37 #3959)

[87] J. J. Rotman, *An introduction to the theory of groups*, fourth ed., Graduate Texts in Mathematics, vol. 148, Springer-Verlag, New York, 1995. MR 1307623 (95m:20001)

[88] C. C. Sims, *Computation with finitely presented groups*, Encyclopedia of Mathematics and its Applications, vol. 48, Cambridge University Press, Cambridge, 1994. MR 1267733 (95f:20053)

[89] D. Singerman, *Subgroups of Fuchsian groups and finite permutation groups*, Bull. London Math. Soc. **2** (1970), 319–323. MR 0281805 (43 #7519)

[90] ______, *Finitely maximal Fuchsian groups*, J. London Math. Soc. (2) **6** (1972), 29–38. MR 0322165 (48 #529)

[91] ______, *Riemann surfaces, Belyi functions and hypermaps*, Topics on Riemann surfaces and Fuchsian groups (Madrid, 1998), London Math. Soc. Lecture Note Ser., vol. 287, Cambridge Univ. Press, Cambridge, 2001, pp. 43–68. MR 1842766 (2002g:14047)

[92] D. Singerman and R. I. Syddall, *The Riemann surface of a uniform dessin*, Beiträge Algebra Geom. **44** (2003), no. 2, 413–430. MR 2017042 (2004k:14053)

[93] E. Steinitz, *Polyeder und raumeinteilungen*, Teubner, 1910.

[94] W.P. Thurston and S. Levy, *Three-dimensional geometry and topology*, Luis A.Caffarelli, no. zv. 1, Princeton University Press, 1997.

[95] W. T. Tutte, *A census of planar maps*, Canad. J. Math. **15** (1963), 249–271. MR 0146823 (26 #4343)

[96] A. Vince, *Combinatorial maps*, J. Combin. Theory Ser. B **34** (1983), no. 1, 1–21. MR 701167 (84i:05048)

[97] ______, *Flag transitive maps*, Proceedings of the fifteenth Southeastern conference on combinatorics, graph theory and computing (Baton Rouge, La., 1984), vol. 45, 1984, pp. 235–250. MR 777723 (86e:05037)

[98] J. Širáň, *The "walk calculus" of regular lifts of graph and map automorphisms*, Proceedings of the 10th Workshop on Topological Graph Theory (Yokohama, 1998), vol. 47, 1999, pp. 113–128.

[99] T. R. S. Walsh, *Hypermaps versus bipartite maps*, J. Combinatorial Theory Ser. B **18** (1975), 155–163. MR 0360328 (50 #12778)

[100] A. T. White, *Graphs, groups and surfaces*, second ed., North-Holland Mathematics Studies, vol. 8, North-Holland Publishing Co., Amsterdam, 1984. MR 780555 (86d:05047)

[101] S. E. Wilson, *Cantankerous maps and rotary embeddings of K_n*, J. Combin. Theory Ser. B **47** (1989), no. 3, 262–273. MR 1026064 (90j:05115)

[102] C. K. Wong, *A uniformization theorem for arbitrary Riemann surfaces with signature*, Proc. Amer. Math. Soc. **28** (1971), 489–495. MR 0279303 (43 #5026)

6. Reprints of papers

Available online at www.sciencedirect.com
ScienceDirect

Electronic Notes in Discrete Mathematics 28 (2007) 331–339

Electronic Notes in DISCRETE MATHEMATICS

www.elsevier.com/locate/endm

Archimedean solids of genus two

Ján Karabáš [1,3]

[a] *Science and Research Institute, Mathei Bel University, Cesta k amfiteátru 1, 974 01 Banská Bystrica, Slovakia*

Roman Nedela [2,3]

[b] *Science and Research Institute, Mathei Bel University, Cesta k amfiteátru 1, 974 01 Banská Bystrica, Slovakia*

Abstract

The problem of classifying orientable vertex-transitive maps on a surface with genus two is considered. We construct and classify all simple orientable vertex-transitive maps, with face width at least 3 which can be viewed as generalisations of classical Archimedean solids. The proof is computer-aided. The developed method applies to higher genera as well.

Keywords: Archimedean solid, map, surface, group, graph embedding

[1] Email: `karabas@savbb.sk`

[2] Email: `nedela@savbb.sk`

[3] The research of author was partially supported by the Ministry of Education of Slovak Republic, grant APVV-51-009605

doi:10.1016/j.endm.2007.01.047

1 Archimedean solids of higher genera

By a *map* $\mathcal{M}$ we mean a 2-cell decomposition of a compact connected orientable surface $\mathcal{S}_g$ of genus g. In other words a map $\mathcal{M}$ can be described as a 2-cell embedding $i : \Gamma \hookrightarrow \mathcal{S}_g$ of the *underlying graph* Γ into a surface $\mathcal{S}_g$. The connectivity components of $\mathcal{S}_g \setminus i(\Gamma)$ are called *faces*. A map $\mathcal{M}$ is *vertex-transitive* if its orientation-preserving automorphism group $\mathrm{Aut}^+(\mathcal{M})$ acts transitively on vertices of Γ. A spherical *Archimedean solid* [3] is a three-dimensional convex polyhedron – a solid which consists of a collection of polygons (faces) such that a local permutation of faces in a vertex v (a local type) does not depend on the choice of v. By Steinitz's theorem [11], Γ is *polyhedral* if an only if it is planar, simple and 3-connected. Hence, Archimedean solids can be viewed as particular maps sharing a combinatorial symmetry. The classification of spherical Archimedean solids is a classical result due to Plato, Archimedes, Kepler etc. The following question arises:

How to define non-spherical Archimedean solids?

As mentioned above, the classification of spherical Archimedean solids is done by their *local types*. A local type of map $\mathcal{M}$ at a vertex v is nothing but a local permutation of lengths of faces incident to v. If the local types of $\mathcal{M}$ do not depend on a choice of a vertex, we say that $\mathcal{M}$ is of that local type. We write the type of a map $\mathcal{M}$ in a multiplicative form (e.g. 3.4.6.4 or $3.3.3.3 = 3^4$) also known as Cundy and Rollett symbol.

In the spherical case it happens that for each Archimedean solid $\mathcal{M}$ the *group of orientation preserving automorphisms* $\mathrm{Aut}^+(\mathcal{M})$ acts transitively on vertices of the underlying graph Γ. Hence the requirement to have a local type can be replaced by a stronger condition of vertex-transitivity of $\mathcal{M}$. Simplicity and 3-connectivity of Γ are another characteristic features of classical Archimedean solids. Recall that the 3-connectivity forces a boundary cycle of any face to be a simple cycle. However, one can easily find 2-cell embeddings of simple 3-connected graphs with self-touching faces. Hence this condition is not enough to guarantee a polyhedrality of maps of higher genera.

The following invariant will be useful.

Definition 1.1 Let $\mathcal{M}$ be a map of genus $g > 0$. We say that $\mathcal{M}$ has a face-width $r(\mathcal{M}) \geq 3$ if the closure of a face or the closure of two faces sharing a vertex in common is contractible.

Definition 1.2 A map $\mathcal{M}$ of genus $g > 0$ with a simple underlying graph Γ will be called Archimedean solid if it $\mathrm{Aut}^+(\mathcal{M})$ acts transitively on vertices and the face-width $r(\mathcal{M})$ is at least three.

Proposition 1.3 *The underlying graph of an Archimedean solid of genus $g > 0$ is 3-connected.*

Proof. Mader [10] proved that a simple vertex-transitive graph Γ of valency at least 3 is 3-connected. □

Note that, by Proposition 1.3, Archimedean solids are necesarilly polyhedral (see [5,7] for definition). Toroidal Archimedean solids are quotients of the uniform (vertex-transitive) tilings of the Euclidean plane [6, p. 63]. There are 11 such tilings of 10 types, two of them forming a chiral pair. Each tiling gives rise to an infinite family of toroidal Archimedean solids. As we shall see later, this does not happen for genera ≥ 2 (see Proposition 2.1). The reader may be confused by the fact that there are 12 uniform tilings of E^2 depicted in [6, p. 63]. The reason is that the tiling of type 4.6.12 is not vertex-transitive by our definition.

2 Classification method

In what follows, we explain an outline of our approach. It is well-known [8] that a map on an orientable surface can be described by means of a triple $\mathcal{M} = (D; R, L)$, where D is a set of *darts of* $\mathcal{M}$, $R, L \in \mathrm{Sym}(D)$ are permutations of darts of $\mathcal{M}$ such that $L^2 = 1$ and $\langle R, L\rangle$ is transitive on D. The permutation R is called a *rotation* of $\mathcal{M}$ and L is called a *dart reversing involution*. The orbits of R are vertices of $\mathcal{M}$, the orbits of L are edges of $\mathcal{M}$. Edges that correspond to orbits of length one are called *semiedges*. In this description, map automorphisms are permutations of D commuting with both R and L. Every vertex-transitive map $\mathcal{M}$ with $\mathrm{Aut}^+(\mathcal{M}) = \mathcal{A}$ has a one-vertex quotient $\mathcal{N} = \mathcal{M}/\mathcal{A} = (\bar{D}; \bar{R}, \bar{L})$ of genus γ, where $\bar{D} = \{[x]_{\mathcal{A}} \mid x \in D\}$, $\bar{R}[x] = [Rx]$ and $\bar{L}[x] = [Lx]$. The genera g and γ of $\mathcal{M}$ and $\mathcal{N}$, respectively, are related by the Riemann-Hurwitz equation

$$2 - 2g = |\mathrm{Aut}^+(\mathcal{M})|(2 - 2\gamma - \sum_{i=1}^{r}(1 - \frac{1}{m_i})),\ m_i \geq 2,$$

where m_i, $i = 1 \dots r$, are branch-indices of a discrete set of brach-points on $\mathcal{S}_\gamma$. Surface of genus γ with a distinguished discrete set of points endowed with branch-indices m_i, $i = 1 \dots r$ will be called an *orbifold* $\mathcal{O}(\gamma; m_1, m_2 \dots, m_r)$. Hence $\mathcal{N}$ is not just a map on $\mathcal{S}_\gamma$ but on the orbifold $\mathcal{O}(\gamma; m_1, m_2 \dots, m_r)$. It follows that a distribution of branch-indices into the set of faces, free ends of semiedges and into the unique vertex of $\mathcal{N}$ is prescribed. An orbifold

$\mathcal{O} = \mathcal{O}(\gamma; m_1, m_2 \ldots, m_r)$ is called *g-admissible* if there exist a group G acting on $\mathcal{S}_g$ such that $\mathcal{O} \cong \mathcal{S}_g/G$.

A consequence of Riemann-Hurwitz equation reads as follows.

Proposition 2.1 *Given a vertex-transitive map of genus $g > 1$ the order of $\mathcal{A}$ is bounded by $|\mathcal{A}| \leq 84(g-1)$. In particular, there exist only finitely many Archimedean solids of genus g.*

The following problem arises.

Problem. Classify all Archimedean solids of given genus $g > 1$.

The aim of our paper is to derive a solution for genus two.

For technical reasons we slightly modify a notation of an orbifold, where a one-vertex map is embedded into. Let $\mathcal{N}$ be a one-vertex map on an orbifold $\mathcal{O}$. Let n be the sum of the numbers of its faces and semiedges. For each face and for each semiedge let $m_i \geq 1$, $i = 1, \ldots, n$, denotes the respective branch-index, and let $m_0 \geq 1$ be a branch-index associated with the unique vertex of $\mathcal{N}$. Moreover, we assume that m_1 is associated with a face of $\mathcal{N}$. With this notation in mind we write $\mathcal{O} = \mathcal{O}(\gamma; m_0, m_1, m_2 \ldots, m_n)$, $m_i \geq 1$ for $i = 0, 1, \ldots, n$.

The following proposition lists some properties of the one-vertex quotient maps.

Proposition 2.2 *Let $\mathcal{M}$ be an Archimedean solid of genus g and let $\mathcal{N}$ be its one-vertex quotient of valency k on an orbifold $\mathcal{O}(m_0, m_1, \ldots, m_n)$. Then*

(i) *a branch-point is either an internal point of a face or the free end of a semiedge or the vertex of $\mathcal{N}$,*

(ii) *each face contains at most one branch-point,*

(iii) *a branch-point at a free end of a semiedge is of index 2,*

(iv) $3 \leq m_0 k \leq 3 + \sqrt{12g-3}$,

(v) $m_i|f_i| > 2$ *for every face f_i, where $|f_i|$ denotes the length of a boundary walk of f_i,*

(vi) $|\mathrm{Aut}^+(\mathcal{M})| > m_0^2 k$.

By Proposition 2.2 one can derive a list of all the possible one-vertex maps arising as quotients of Archimedian solids of genus g. For example, for $g = 2$ we have found 45 spherical one-vertex maps satisfying condition (iv). Using the branch-data taken from [2] we have derived 94 non-isomorphic one-vertex maps on 2-admissible orbifolds.

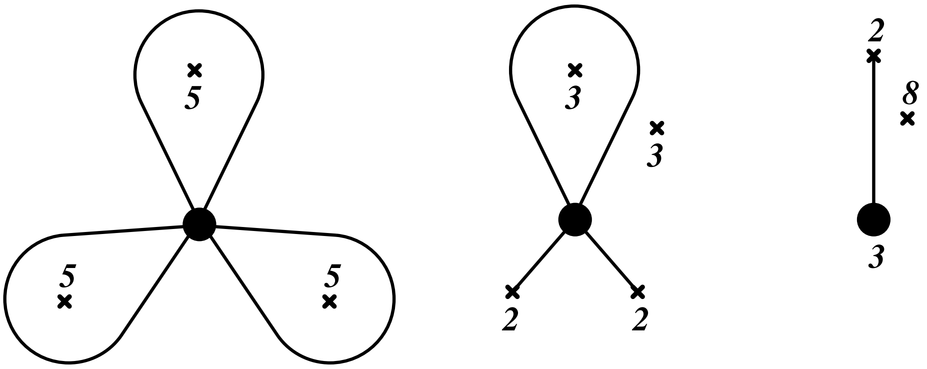


Fig. 1. Three examples of maps on spherical orbifolds

Given one-vertex map $\mathcal{N}$ on an orbifold $\mathcal{O}(\gamma; m_0, m_1, m_2 \ldots, m_n)$ one can derive a presentation of a universal group $\mathcal{U}$ of a universal covering map $\widetilde{\mathcal{M}} \to \mathcal{N}$. The group $\mathcal{U}$ acts transitively on vertices of the universal map $\widetilde{\mathcal{M}}$ and every Archimedean solid $\mathcal{M}$ which projects onto $\mathcal{N}$ on $\mathcal{O}(\gamma; m_0, m_1, m_2 \ldots, m_n)$ is covered by $\widetilde{\mathcal{M}}$. Moreover, $\widetilde{\mathcal{M}}$ is the least map with the above universal property. To determine $\mathcal{U}$ from the one-vertex map $\mathcal{N}$ let us assign an abstract generator x_d to every dart d of $\mathcal{N}$ such that each pair of opposite darts d, $L(d)$ gain a generator x_d and its inverse x_d^{-1} respectively. Note that if $d = L(d)$ then the order of the associated generator is two. If the vertex v of $\mathcal{N}$ is a branch-point, v is assigned by a generator z. The generator z generates the stabilizer of a vertex in $\mathcal{M}$. For each semiedge $\{d\} = \{L\,d\}$ let $m_j = 2$ be the respective branch-index associated with the free end of $\{d\}$. We set $\mathcal{R}_j = x_d^{m_j} = x_d^2$. For each face f there is a corresponding boundary cycle in RL. We derive a word w substituting an appearance of a dart d by the respective generator x_d in the boundary cycle. We set $\widetilde{\mathcal{R}}_i = w^{m_i}$, $i \geq 2$ if the branch-index of f is m_i. To summarize, the presentation of $\mathcal{U}$ reads as follows

$$\mathcal{U} = \langle x_1, x_2, \ldots x_t, z \mid z^{m_0}, (z^{-1}w_1)^{m_1}, \widetilde{\mathcal{R}}_2, \widetilde{\mathcal{R}}_3, \ldots, \widetilde{\mathcal{R}}_n \rangle,$$

with a convention that if $z = 1$ the presentation reduces to

$$\mathcal{U} = \langle x_1, x_2, \ldots x_t \mid w_1^{m_1}, \widetilde{\mathcal{R}}_2, \widetilde{\mathcal{R}}_3, \ldots, \widetilde{\mathcal{R}}_n \rangle.$$

In Figure 2 there are two one-vertex maps on spherical orbifolds. In Case a) we have three generators and walking around the faces we get four relators, so

$$\mathcal{U}_a = \langle x_1, x_2, x_3 \mid x_1^5 = x_2^5 = x_3^5 = (x_1^{-1} x_2^{-1} x_3^{-1})^5 = 1 \rangle$$

is desired presentation. In Case b) the vertex is a branchpoint with branchin-

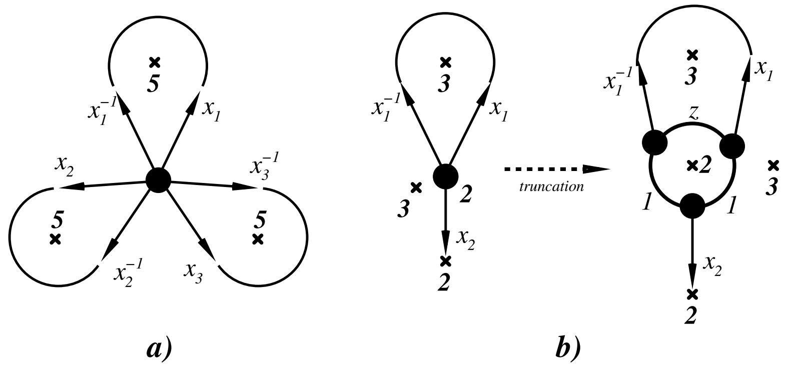


Fig. 2. Two examples to construct the presentation of $\mathcal{U}$

dex 2 hence

$$\mathcal{U}_b = \langle x_1, x_2, z \mid (z^{-1}x_1)^3 = x_2^3 = z^3 = (x_1^{-1}x_2)^3 = 1 \rangle.$$

Note that $\mathcal{U}$ is known as a Fuchsian group associated with $\mathcal{O}(\gamma; m_0, m_1, m_2 \ldots, m_n)$. However, the derived presentation is not standard in general, since it depends on an embedded one-vertex map $\mathcal{N}$.

The following theorem is of crucial importance.

Theorem 2.3 *Let $\mathcal{M}$ be an Archimedean solid of genus g and automorphism group $\mathcal{A}$. Let $\mathcal{N} = \mathcal{M}/\mathcal{A}$ be its one-vertex quotient on an orbifold $\mathcal{O}(\gamma; m_0, m_1, m_2 \ldots, m_n)$. Then there exist a torsion-free normal subgroup $K \lhd \mathcal{U}$ of genus g such that $\mathcal{A} \cong \mathcal{U}/K$ and $\mathcal{M} \cong \widetilde{\mathcal{M}}/K$. In particular, the index $[\mathcal{U} : K]$ is given by Riemann-Hurwitz equation.*

In fact g-admissible orbifolds as well as groups acting on S_g for $g \leq 5$ are known (see [1,2,9]). Hence given $\mathcal{U}$ we may set $\mathcal{A} \cong \mathcal{U}/K$ to be a concrete finite group G from the list. For each voltage assignment $\xi : (z, x_1, \ldots, x_t) \to G^{t+1}$ one can check whether or not the universal relations given by the presentation of $\mathcal{U}$ are satisfied, such a voltage-assignment ξ will be called *$\mathcal{U}$-compatible*. It follows that a systematical check through all $\mathcal{U}$-compatible voltage assignments ξ yields a list of presentations of $\mathcal{U}/K$ The goal is that given presentation

$$\mathcal{A} \cong \mathcal{U}/K = \langle x_1, x_2, \ldots x_t, z \mid z^{m_0}, (z^{-1}w_1)^{m_1}, \widetilde{\mathcal{R}}_2, \widetilde{\mathcal{R}}_3, \ldots, \widetilde{\mathcal{R}}_n, \ldots \rangle$$

the respective vertex-transitive map covering $\mathcal{N}$ can be reconstructed as follows. For introduction for theory of voltage assignemts see [4, Chapters 2 – 4] Let $\mathcal{N} = (\bar{D}; \bar{R}, \bar{L})$ with a voltage assignment $\xi : \bar{D} \to G$. Since $\mathcal{N}$ is an one-vertex map, we may assume that $\bar{D} = \{1, 2, \ldots, k\}$, $\bar{R} = (1, 2, \ldots, k)$ and $\bar{L} \in \mathrm{Sym}(k), L^2 = 1$.

Then the derived map $\mathcal{M} = (D; R, L)$ is defined as follows

$$
\begin{aligned}
D &= \bar{D} \times G, \\
L(i,g) &= (\bar{L}i,\, \xi_i g),
\end{aligned}
$$

(1) and

$$
R(i,g) = \begin{cases} (i+1, g);\ i \neq k \\ \\ (1, zg); \quad i = k \end{cases}
$$

Some of the vertex-transitive covers of $\mathcal{N}$, however, may not be Archimedean. In general, either the underlying graph may not be simple or the face-width of $\mathcal{M}$ may be ≤ 2. Given voltage assignment ξ the simplicity of $\mathcal{M}$ can be tested directly. As concerns the face width of the lifted map this can be done by investigating self-touchings of faces and multiple adjacences of pairs of faces.

The following algorithm is used to determine Archimedean solids of genus two:

(i) Determine all spherical and toroidal one-vertex maps $\mathcal{N}$ satisfying Proposition 2.2 (iv),

(ii) find all embeddings of the maps $\mathcal{N}$ determined in Step (i) into 2-admissible orbifolds,

(iii) for every embedding determined in Step (ii), derive the universal group $\mathcal{U}$,

(iv) find all $\mathcal{U}$-compatible voltage assignments in G, where G ranges through the list of groups in [2] (omitting voltage assignments giving loops and multiple adjacences),

(v) derive the lifted maps in terms $(D; R, L)$ following (1),

(vi) determine the isomorphism classes of lifted maps,

(vii) for each representative of an isomorphism class test whether the face-width $r(\mathcal{M}) \geq 3$.

Table 1 gives a list of Archimedean solids of genus two in a condensed form. By Table 1 there are 19 Archimedean solids of genus 2, eight of them appear in chiral pairs, the remaining five are reflexible. Recall that $\mathcal{M}_1 = (D_1; R_1, L_1)$ and $\mathcal{M}_2 = (D_2; R_2, L_2)$ form a chiral pair if $\mathcal{M}_1 \ncong \mathcal{M}_2$ but

$\mathcal{M}_1 \cong (D_2; R_2^{-1}, L_2)$. Some of the listed Archimedean solids can be obtained from regular maps and hypermaps of genus two taking their truncations and medials.

Ref.	Type	$\lvert\mathcal{M}\rvert$	Val(v)	Stab(v)	$\mathrm{Aut}^+(\mathcal{M})$	Comment	#
$\mathcal{M}_1$	$3^3.4.3.4$	12	6	1	D_{12}	chiral	2
$\mathcal{M}_2$	$3^4.4^2$	12	6	1	D_{12}	chiral	2
$\mathcal{M}_3$	3^7	12	7	1	D_{12}	chiral	2
$\mathcal{M}_4$	3^7	12	7	1	D_{12}	chiral	2
$\mathcal{M}_5$	3^7	12	7	1	D_{12}	chiral	2
$\mathcal{M}_6$	$4^3.6$	24	4	1	$(C_6 \times C_2) \rtimes C_2$	reflexible	1
$\mathcal{M}_7$	$3^2.4.3.6$	24	5	1	$(C_6 \times C_2) \rtimes C_2$	chiral	2
$\mathcal{M}_8$	$3.6.4.6$	24	4	1	$SL_2(3)$	reflexible	1
$\mathcal{M}_9$	$3^5.4$	24	6	1	$SL_2(3)$	reflexible	1
$\mathcal{M}_{10}$	$6^2.8$	48	3	1	$GL_2(3)$	reflexible	1
$\mathcal{M}_{11}$	$3.4.8.4$	48	4	1	$GL_2(3)$	reflexible	1
$\mathcal{M}_{12}$	$3^4.8$	48	5	1	$GL_2(3)$	chiral	2

Table 1
Census of Archimedean solids of genus two

References

[1] Bogopolski, O. V., *Classifying the actions of finite groups on orientable surfaces of genus* 4, Siberian Adv. Math. **7** (1997), pp. 9–38.

[2] Broughton, S. A., *Classifying finite group actions on surfaces of low genus*, J. Pure Appl. Algebra **69** (1991), pp. 233–270.

[3] Cromwell, P. R., "Polyhedra," Cambridge University Press, New York, 1997.

[4] Gross, J. L. and T. W. Tucker, "Topological graph theory," Dover Publications Inc., Mineola, NY, 2001, second edition.

[5] Grünbaum, B., "Convex polytopes," Graduate Texts in Mathematics **221**, Springer-Verlag, New York, 2003, second edition.

[6] Grünbaum, B. and G. C. Shephard, "Tilings and Patterns," W. H. Freeman and Co., New York, 1986.

[7] Jendroľ, S. and H.-J. Voss, *A local property of polyhedral maps on compact two-dimensional manifolds*, Discrete Math. **212** (2000), pp. 111–120.

[8] Jones, G. A. and D. Singerman, *Theory of maps on orientable surfaces*, Proc. London Math. Soc. (3) **37** (1978), pp. 273–307.

[9] Kuribayashi, A. and H. Kimura, *Automorphism groups of compact Riemann surfaces of genus five*, J. Algebra **134** (1990), pp. 80–103.

[10] Mader, W., *Über den Zusammenhang symmetrischer Graphen*, Arch. Math. (Basel) **21** (1970), pp. 331–336.

[11] Steinitz, E., *Polyeder und raumeinteilungen*, Enzyklopedie Math. Wiss. **3 (Geometrie)**, Leipzig, 1922 pp. 1 – 139.

MATHEMATICS OF COMPUTATION
Volume 81, Number 277, January 2012, Pages 569–583
S 0025-5718(2011)02502-0
Article electronically published on May 13, 2011

ARCHIMEDEAN MAPS OF HIGHER GENERA

JÁN KARABÁŠ AND ROMAN NEDELA

Abstract. The paper focuses on the classification of vertex-transitive polyhedral maps of genus from 2 to 4. These maps naturally generalise the spherical maps associated with the classical Archimedean solids. Our analysis is based on the fact that each Archimedean map on an orientable surface projects onto a one- or a two-vertex quotient map. For a given genus $g \geq 2$ the number of quotients to consider is bounded by a function of g. All Archimedean maps of genus g can be reconstructed from these quotients as regular covers with covering transformation group isomorphic to a group G from a set of g-admissible groups. Since the lists of groups acting on surfaces of genus $2, 3$ and 4 are known, the problem can be solved by a computer-aided case-to-case analysis.

1. Introduction

By a *map* $\mathbf{M}$, we mean a 2-cell decomposition of a compact connected orientable surface $\mathcal{S}_g$ of genus g. In other words, a map $\mathbf{M}$ can be described as a 2-cell embedding $\varepsilon : \Gamma \hookrightarrow \mathcal{S}_g$ of the *underlying graph* Γ into a surface $\mathcal{S}_g$. The connected components of $\mathcal{S}_g \setminus \varepsilon(\Gamma)$ are called *faces*. Given a map $\mathbf{M}$, an automorphism of the underlying graph Γ which extends to a self-homeomorphism of $\mathcal{S}_g$ is called an *automorphism* of $\mathbf{M}$. A map $\mathbf{M}$ is *vertex-transitive* if its automorphism group $\mathrm{Aut}(\mathbf{M})$ acts transitively on vertices of Γ. In a vertex-transitive map on an orientable surface, the group of orientation-preserving automorphisms $\mathrm{Aut}^+(\mathbf{M})$ acts on the vertex-set either with one or with two orbits.

Graphs considered in this paper may have loops, multiple edges and semiedges. More precisely, a graph is a quadruple $\Gamma = (D, V; I, L)$, where $D = D(\Gamma)$ and $V = V(\Gamma)$ are disjoint nonempty finite sets, $I : D \to V$ is a surjective mapping and L is an involutory permutation on D. The elements of D and V are *darts* and *vertices*, respectively, I is the *incidence function* assigning to every dart its *initial vertex* and L is the *dart-reversing involution*. The orbits of the group $\langle L \rangle$ on D are *edges* of Γ. It may happen that $x.L = x$ for some dart $x \in \Gamma$, and in this case the corresponding edge is called a *semiedge*. If $I(x.L) = I(x)$ but $x.L \neq x$, then the corresponding edge is called a *loop*. The remaining edges are called *links*. Two links $\{x, x.L\}$ and $\{y, y.L\}$ are *parallel* if $I(x) = I(y)$ and $I(x.L) = I(y.L)$, or $I(x) = I(y.L)$ and $I(x.L) = I(y)$. A graph without semiedges, loops and parallel

Received by the editor September 14, 2007 and, in revised form November 4, 2010.
2010 *Mathematics Subject Classification.* Primary 05C30; Secondary 05C10, 05C25.
Key words and phrases. Polyhedron, Archimedean solid, map, surface, group, graph embedding.
Both authors were partially supported by the grants APVV-51-009605 and VEGA 1/0722/08, grants of Slovak Ministry of Education.

links is called a *simple* graph. The above definition of a graph follows the approach of Jones and Singermann [17] and is the same as in [25].

A (convex) polyhedron is defined in [15, 16] as the convex hull of a finite set of points in the Euclidean space. By Steinitz's theorem [30] a graph forms a 1-skeleton of a convex polyhedron if and only if it is planar and 3-connected. Therefore a simple graph is called *polyhedral* if and only if it is planar and 3-connected. A 2-cell decomposition of the sphere given by a 2-cell embedding of a polyhedral graph into the sphere will be called a *spherical polyhedral map*. Note that a 2-cell embedding of a polyhedral graph into the sphere is uniquely determined by the graph, hence there is a correspondence between polyhedral graphs, polyhedral maps and polyhedra. To generalise the concept of a polyhedral map to higher genera we shall relax the condition on polyhedrality as follows.

We say that two faces of a map are *adjacent* if they are incident to the same vertex. In a spherical polyhedral map a boundary of a face is a simple cycle and the boundary cycles of two adjacent faces intersect either in a single vertex, or in a single edge together with the two incident vertices. Following Mohar and Thomassen [24, Proposition 5.5.12] and Brehm and Schulte [4], we say that a map on a surface of genus g is *polyhedral of genus g* if the boundary of every face is a simple cycle and the boundary cycles of any two faces are either disjoint or intersect either in a single vertex or in a single edge (together with the two incident vertices). By Mohar and Thomassen [24, Proposition 5.5.12] the underlying graph of a polyhedral map is simple and 3-connected. An *Archimedean map* is a polyhedral map such that its automorphism group is transitive on its vertices. Note that in [26] a weaker definition of polyhedrality is used.

The classification of classical Archimedean solids (maps) is done by their local types. A *local type of a map* $\mathbf{M}$ *at a vertex* v is nothing but a cyclic sequence of lengths of faces incident to v following a given global orientation of the surface. In the case of Archimedean maps the local types do not depend on a choice of a vertex. Therefore for an Archimedean map $\mathbf{M}$ we can talk about a *type of* $\mathbf{M}$. We write the type of a map $\mathbf{M}$ in a multiplicative form (e.g., $(3.4.4.6.4) = (3.4^2.6.4)$ or $(3.3.3.3) = (3^4)$) also known as Cundy and Rollett symbol [9]. Since we shall consider a map and its mirror image to be isomorphic, a local type and its mirror image will be considered to be the same. As a rule we shall use the lexically minimal representative of local type of a map. For each local type there exists a universal Archimedean map (tiling) of the sphere, Euclidean or hyperbolic plane covering every Archimedean map of that type; see [8].

There are 13 classical Archimedean solids (15 if the mirror images of two enantiomorphs are counted separately). Thus there are 13 (15) associated spherical Archimedean maps. Moreover, the five Platonic solids give rise to another five spherical Archimedean maps. To complete the list of spherical Archimedean maps we include the infinite families of maps associated with the prisms and anti-prisms. For more information see e.g. the Wikipedia [32].

Toroidal Archimedean maps are quotients of the uniform (vertex-transitive) tilings of the Euclidean plane [16, p. 63]. There are 11 distinct *Archimedean tilings* of the Euclidean plane E_2 (12 if the mirror images of two enantiomorphs of type $(3^4.6)$ are counted separately). Each toroidal Archimedean map can be constructed as a quotient of one of the universal tilings by a group of translations. Each of the universal tilings give rise to an infinite family of toroidal Archimedean maps. A more detailed classification can be found in [31]. For further information see [1, 26, 27, 31] and [14, pp. 291–295].

From the above notes it is clear that there are infinitely many Archimedean maps of both genus 0 and genus 1. In contrast, by the well-known Riemann-Hurwitz bound combined with Proposition 5, there are finitely many Archimedean maps for each genus g, $g \geq 2$. In particular, we have the following proposition.

Proposition 1 (Riemann-Hurwitz bound, [13])**.** *Given a vertex-transitive map* $\mathbf{M}$ *of genus* $g > 1$ *the order of* $\mathrm{Aut}(\mathbf{M})$ *is bounded by* $|\mathrm{Aut}(\mathbf{M})| \leq 168(g-1)$ *and the order of the group of orientation-preserving automorphisms is bounded by* $|\mathrm{Aut}^+(\mathbf{M})| \leq 84(g-1)$*. In particular, the number of vertices of* $\mathbf{M}$ *is at most* $168(g-1)$*.*

In the present paper the following classification problem will be considered.

Problem. Classify isomorphism classes of Archimedean maps of given genus $g > 1$. A similar problem, the classification of regular maps of genus $2 \leq g \leq 101$, has already been solved by Conder [7]. The main result of our paper written in the enumerative form follows. Complete lists of Archimedean maps of genus $2, 3, 4$ can be found at the website [18].

Theorem 2 (Main Theorem)**.** *There are* 17*,* 103 *and* 111 *isomorphism classes of Archimedean maps of genus* $2, 3, 4$*, respectively.*

2. Two-dimensional orbifolds and maps on orbifolds

The idea of an orbifold comes from geometry of manifolds, where they are defined in a more general setting. Generally, orbifolds can be viewed as homomorphic images of manifolds, in particular, a quotient orbifold is induced by an action of a discrete group of automorphisms of a manifold.

For the purpose of this paper we define an (orientable) *orbifold* to be an orientable surface of genus $\gamma \geq 0$ together with a distinguished finite set of points $\mathcal{B} = \{b_1, b_2, \ldots, b_r\}$, to each point b_i in $\mathcal{B}$ there is associated an integer $m_i \geq 2$, $i = 1, 2, \ldots r$. The numbers $m_1, m_2, \ldots, m_r$ are called *branch-indices*. Each orbifold is determined by its *signature* $(\gamma; m_1, m_2, \ldots, m_r)$, where the branch-indices are ordered in a nondecreasing sequence $m_1 \leq m_2 \leq \cdots \leq m_r$.

Every vertex-transitive map $\mathbf{M}$ on a surface of genus g covers regularly a quotient map $\bar{\mathbf{M}} = \mathbf{M}/\mathrm{Aut}^+(\mathbf{M})$. The quotient map $\bar{\mathbf{M}}$ is of genus $\gamma \leq g$. The underlying surface S_γ can be viewed as a *two-dimensional quotient orbifold* $\mathcal{O}(\gamma; m_1, m_2, \ldots, m_r)$ $= S_g/\mathrm{Aut}^+(\mathbf{M})$, where the parameters γ, g, $m_1, \ldots, m_r$ are related by the Riemann-Hurwitz equation:

$$2 - 2g = |\mathrm{Aut}^+(\mathbf{M})|(2 - 2\gamma - \textstyle\sum_{i=1}^{r}(1 - \frac{1}{m_i})),\ m_i \geq 2, \qquad (2.1)$$
$$|\mathrm{Aut}^+(\mathbf{M})| \equiv 0 \bmod m_i.$$

Note that given a surface, its admissible quotient orbifolds were classified by Broughton [5] for genus 2 and 3, and by Bogopolski [2] for genus four. Lists of admissible orbifolds for surfaces of higher genera can be found in [19].

Our approach is based on the fact that given a quotient orbifold $\mathcal{O}$ we can identify all the potential quotient maps $\bar{\mathbf{M}} = \mathbf{M}/\mathrm{Aut}^+(\mathbf{M})$ on $\mathcal{O}$ which have by definition one or two vertices. Moreover, given a map $\bar{\mathbf{M}}$ on $\mathcal{O}$, we can reconstruct the covering map $\mathbf{M}$ over $\bar{\mathbf{M}}$ and verify whether it is polyhedral or not. To do this effectively we first need a combinatorial description of a map on an orbifold and to derive more facts on the quotients of Archimedean maps. The reconstruction is explained in the following section.

It is well known [17, 25] that a map on an orientable surface can be described by means of a triple $\mathbf{M} = (D; R, L)$, where D is a set of *darts of* $\mathbf{M}$, $R, L \in \mathrm{Sym}(D)$ are permutations of darts of $\mathbf{M}$ such that $L^2 = 1$ and $\langle R, L\rangle$ is transitive on D. The permutation R is called the *rotation* of $\mathbf{M}$ and L is called the *dart reversing involution*. The orbits of R are identified with the set V of vertices of $\mathbf{M}$, the orbits of L are identified with the set E of edges of $\mathbf{M}$ and the cycles of RL form boundary walks of the set F of faces of $\mathbf{M}$. The edges that correspond to orbits of length one are called *semiedges*. A valency $\mathrm{val}(x)$ of a vertex, edge or face x is the size of the corresponding orbit, i.e., $\mathrm{val}(x) = |x|$. It follows that the valency of an edge e is either one or two depending on whether e is a semiedge or not.

Let $\mathbf{M}_1 = (D_1; R_1, L_1)$ and $\mathbf{M}_2 = (D_2; R_2, L_2)$. Then a map homomorphism $\varphi : \mathbf{M}_1 \to \mathbf{M}_2$ is a mapping $\varphi : D_1 \to D_2$ such that $\varphi R_1 = R_2\varphi$ and $\varphi L_1 = L_2\varphi$. By transitivity of $\langle R_2, L_2\rangle$ every map homomorphism is surjective, therefore map homomorphisms are called *coverings*. The orientation preserving map automorphisms are permutations of D commuting with both R and L. By transitivity of $\langle R, L\rangle$ the action of $\mathrm{Aut}^+(\mathbf{M})$ on D is semiregular (see Dixon and Mortimer [10, Theorem 4.2a]). A permutation τ of $\mathbf{M}$ is an *orientation reversing automorphism* of $\mathbf{M}$ if $\tau R\tau^{-1} = R^{-1}$ and $\tau L\tau = L$. The automorphism group $\mathrm{Aut}(\mathbf{M})$ is formed by orientation preserving and orientation reversing automorphisms of $\mathbf{M}$. Since a composition of two orientation reversing automorphisms belongs to $\mathrm{Aut}^+(\mathbf{M})$, we have $[\mathrm{Aut}(\mathbf{M}) : \mathrm{Aut}^+(\mathbf{M})] \le 2$. If there are no orientation reversing automorphisms $\mathbf{M}$ is called *chiral*, otherwise it is called *reflexible*. An orientation reversing automorphism of order two fixing a vertex, an edge or a face is called a *reflection*.

Every vertex-transitive map $\mathbf{M}$ with a group of orientation-preserving automorphisms $\mathrm{G} \le \mathrm{Aut}^+(\mathbf{M})$ covers a quotient $\bar{\mathbf{M}} = \mathbf{M}/\mathrm{G} = (\bar{D}; \bar{R}, \bar{L})$ of genus γ, where $\bar{D} = \{[x]_{\mathrm{G}};\ x \in D\}$, $[x]\bar{R} = [xR]$ and $[x]\bar{L} = [xL]$. The mapping $x \mapsto [x]_{\mathrm{G}}$ is a covering $\mathbf{M} \to \bar{\mathbf{M}}$ called a *standard covering*.

Proposition 3. *Let* $\mathbf{M}$ *be an Archimedean map on an orientable surface. Then the quotient map* $\bar{\mathbf{M}} = \mathbf{M}/\mathrm{Aut}^+(\mathbf{M})$ *is either a one-vertex map, or a two-vertex map.*

Proof. Since $[\mathrm{Aut}(\mathbf{M}) : \mathrm{Aut}^+(\mathbf{M})] \le 2$, the group of orientation preserving automorphisms acts on the vertex set either with one, or with two orbits. It follows that the quotient map has at most two vertices. □

It follows that Archimedean maps are of two kinds, which will be treated separately. If $\mathrm{Aut}^+(\mathbf{M})$ acts on the vertices of $\mathbf{M}$ inducing one orbit, then $\mathbf{M}$ will be called an *Archimedean map of type I*. In the other case if $\mathrm{Aut}^+(\mathbf{M})$ acts on the vertices of $\mathbf{M}$ inducing two orbits, $\mathbf{M}$ will be called an *Archimedean map of type II*.

The quotient map $\bar{\mathbf{M}} = \mathbf{M}/\mathrm{Aut}^+(\mathbf{M})$ lies on a surface of genus $\gamma \le g$. The genus γ can be counted by the Euler-Poincaré formula $2 - 2\gamma = \bar{V} - \bar{E} + \bar{F}$. Moreover, the covering $f : \mathbf{M} \to \bar{\mathbf{M}}$ associates with each vertex, edge, or face $\bar{x} = f(x)$ an integer $b(\bar{x}) = \frac{\mathrm{val}(x)}{\mathrm{val}(f(x))}$. It follows that f determines an orbifold $\mathcal{O}$ with signature $(\gamma; m_1, m_2, \ldots, m_r)$, where the set of branch-indices is given by $\{b(\bar{x}) \mid b(\bar{x}) \ge 2 \text{ and } x \in V \cup E \cup F\}$. In general, a map $\mathbf{M} = (D; R, L)$ is on an orbifold $\mathcal{O}(\gamma; m_1, m_2, \ldots, m_r)$ if it is of genus γ and a function $b : V \cup E \cup F \to \{1, m_1, m_2, \ldots, m_r\}$ satisfying the following conditions is given:

(M1) Let x be an edge. Then $b(x) \leq 2$, and $b(x) = 2$ if and only if x is a semiedge.

(M2) For each $i \in \{1, 2, \ldots, r\}$ there is exactly one $x \in V \cup E \cup F$ such that $b(x) = m_i$.

If b is defined by setting $b(\bar{x}) = \frac{\mathrm{val}(x)}{\mathrm{val}(f(x))}$, for some regular covering $f : \mathbf{M} \to \bar{\mathbf{M}}$, we say that b *is induced by* f. We will say that a covering $f : \mathbf{M} \to \bar{\mathbf{M}}$ is *regular* if the group of covering transformations acts transitively (and hence regularly) on each fibre over a dart. For more details on regular coverings between maps see [22].

The above definition of a map on an orbifold comes from [23]. The following proposition is a useful extension of Proposition 3.

Proposition 4. *Let* $\mathbf{M}$ *be an Archimedean map of type II. Then an orientation reversing automorphism* τ *of* $\mathbf{M}$ *transposing the two vertex orbits projects into orientation reversing automorphism* $\bar{\tau}$ *of* $\bar{\mathbf{M}}$.

In particular, the branch-indices of a face $\bar{F}$ *and its image* $\bar{\tau}(\bar{F})$ *coincide and the branch-indices of the two vertices are the same.*

Proof. The following diagram has to commute:

$$\begin{tikzcd} x \arrow[r, "\tau"] \arrow[d, "f"'] & \tau x \arrow[d, "f"] \\ \bar{x} \arrow[r, "\bar{\tau}"] & \bar{\tau}\bar{x} \end{tikzcd}$$

Let $G = \mathrm{Aut}^+(\mathbf{M})$. Set $f : \mathbf{M} \to \bar{\mathbf{M}}$, $f : x \mapsto [x]_G$, where $[x]_G$ denotes the orbit of G containing a dart x of $\mathbf{M}$. Since $G \trianglelefteq \mathrm{Aut}(\mathbf{M})$,

$$[g\tau x]_G = [\tau g' x]_G = [\tau x]_G.$$

It follows that $\bar{\tau} : [x]_G \mapsto [\tau x]_G$ is well defined. Moreover, valencies of $\bar{F}$ and $\bar{\tau}(\bar{F})$, as well as the valencies of their preimages in $\mathbf{M}$, are the same. The result follows. $\square$

For technical reasons we slightly modify a notation of an orbifold, where a one- or two-vertex map is embedded. Let $\bar{\mathbf{M}}$ be a one-vertex map on an orbifold $\mathcal{O}$. Let n be the sum of the numbers of its faces and semiedges. For each face and for each semiedge let $m_i \geq 1$, $i = 1, \ldots, n$, denote the respective branch-index, and let $m_0 \geq 1$ be a branch-index associated with the unique vertex of $\bar{\mathbf{M}}$. In other words, we admit that some branch-indices are equal to one. With this notation in mind we write $\mathcal{O} = \mathcal{O}(\gamma; m_0, m_1, m_2, \ldots, m_n)$, $m_i \geq 1$ for $i = 0, 1, \ldots, n$, with the convention that $m_0 \geq 1$ is a branch-index of the unique vertex. Alternatively, if $\bar{\mathbf{M}}$ is a 2-vertex quotient map, the respective orbifold will be described as $\mathcal{O} = (\gamma; m_0^2, m_1, \ldots, m_n)$ with the convention that the two branch-points of index $m_0 \geq 1$ are associated with the two vertices. By Proposition 4 the two vertices of the quotient have the same branch-index.

The following proposition lists some properties of quotient maps.

Proposition 5. *Let* $\mathbf{M}$ *be an Archimedean map on an orientable surface of genus* $g \geq 1$ *and let* $\bar{\mathbf{M}} = \mathbf{M}/\mathrm{Aut}^+(\mathbf{M})$ *be its quotient of valency* ℓ *on an orbifold* $\mathcal{O}(\gamma; m_0, m_1, \ldots, m_n)$ *or* $\mathcal{O}(\gamma; m_0^2, m_1, \ldots, m_n)$, *respectively. Then*

(1) $3 \leq \ell m_0 \leq 3 + \sqrt{12g - 3}$,

(2) $m_i f_i > 2$ *for every face* $\bar{F}_i$, *where* $f_i = |\bar{F}_i|$ *is the face-valency,* $i \in \{1, \ldots, n\}$ *and* m_i *is an index of a branch-point associated with* $\bar{F}_i$,
(3) $|\mathrm{Aut}^+(\mathbf{M})| > \ell m_0^2$ *if* $\mathbf{M}$ *is of type I,*
(4) $|\mathrm{Aut}^+(\mathbf{M})| > \frac{1}{2}\ell m_0^2$ *if* $\mathbf{M}$ *is of type II.*

Proof. Observe that m_0 is the order of a vertex stabiliser in $\mathbf{M}$ and $k = \ell m_0$ is the valency of $\mathbf{M}$. In the case $\ell m_0 \leq 2$ the lifted map cannot be polyhedral, since its valency is $k \leq 2$.

Now we shall prove the upper bound $k \leq 3 + \sqrt{12g - 3}$. Let v, e and f denote the number of vertices, edges and faces of $\mathbf{M}$, respectively. By the Euler-Poincaré equation it holds that $v - e + f = \chi = 2 - 2g$. By vertex-transitivity, the valency k of a lifted map $\mathbf{M}$ is equal to $k = \ell m_0$ and $e = kv/2$. Since $\mathbf{M}$ has to be simple, $f \leq kv/3$. If $g = 1$, then direct computation yields $k \leq 6$. If $g \geq 2$, then for $k \leq 6$ the upper bound is satisfied. Hence we may assume $k \geq 7$. Inserting the previous terms in the Euler-Poincaré equation and using $v > k$ we get

$$-\chi \geq kv\left(\frac{1}{6} - \frac{1}{k}\right) > k^2\left(\frac{1}{6} - \frac{1}{k}\right).$$

Hence $0 > k^2 - 6k + 6(-\chi)$ and statement (1) holds.

To see (2) observe that $m_i f_i$ for $i \geq 1$ is the valency of a face F_i covering $\bar{F}_i$. Since the face-valency in a polyhedral map is at least 3 we are done.

Denote $\mathrm{Aut}^+(\mathbf{M}) = \mathrm{G}$. To prove (3) recall that $k = \ell m_0$ is the valency of $\mathbf{M}$ while $\frac{|\mathrm{G}|}{|\mathrm{G}_v|} = \frac{|\mathrm{G}|}{m_0}$, where G_v is the vertex stabilizer in G. Since $\mathbf{M}$ is a simple map, the number of vertices of $\mathbf{M}$ is $\frac{|\mathrm{Aut}^+(\mathbf{M})|}{m_0} > \ell m_0$.

To see (4) observe that if $\mathbf{M}$ is of type II we have $\frac{v}{2} = \frac{|\mathrm{G}|}{|\mathrm{G}_v|} = \frac{|\mathrm{G}|}{m_0}$. Since $\mathbf{M}$ is simple, $v > k = \ell m_0$ and consequently $\frac{2|\mathrm{G}|}{m_0} > \ell m_0$. □

3. Reconstruction of Archimedean maps

An explicit construction of the lift over $\bar{\mathbf{M}}$ requires a practical method of description of the covering. Such a method is provided by voltage assignments [14]. Usually, voltages are assigned to darts. However, in order to encompass all regular covers (including those which are not valency preserving), we need to employ voltage assignments on angles. We shall follow the approach developed in [22].

Let $\mathbf{M} = (D; R, L)$ be a map. An (oriented) *angle* of $\mathbf{M}$ is an ordered pair $a = (x, y)$, where x and y are darts of $\mathbf{M}$ such that $y \in \{xR, xR^{-1}, xL\}$. We always consider the angles (x, xR) and (x, xR^{-1}) to be distinct. The darts x and y are called the *initial* and the *terminal* dart of a, respectively. The angle $a^{-1} = (y, x)$ is the *inverse* of $a = (x, y)$. We denote by $A(\mathbf{M})$ the set of all angles of $\mathbf{M}$; obviously, $|A(\mathbf{M})| = 3|D|$.

An *angle-walk* (or briefly a *walk*) is a sequence $W = a_1 a_2 \ldots a_n$ of angles of $\mathbf{M}$ such that the terminal dart of a_i coincides with the initial dart of a_{i+1}, for each index $i = 1, 2, \ldots, n-1$. The initial dart of a_1 and the terminal dart of a_n are called the *initial* and the *terminal* dart of W, respectively. The term *closed angle-walk* has the obvious meaning. If $W = a_1 a_2 \ldots a_n$ is a walk originating at x and terminating at y, then $W^{-1} = a_n^{-1} a_{n-1}^{-1} \ldots a_1^{-1}$ is a walk originating at y and terminating at x, called the *inverse* of W.

Let $\mathbf{M}$ be a map and let G be a finite group. A *voltage assignment* on $\mathbf{M}$ valued in G is a function $\xi : A(\mathbf{M}) \to \mathrm{G}$ such that for any angle a one has $\xi(a^{-1}) =$

$\xi(a)^{-1}$. Note that if x is the unique dart of a semiedge, then the voltage $\xi(x, xL)$ is necessarily an involution. In what follows we shall denote $\xi(a) = \xi_a$. The group G is called the *voltage group.*

A voltage assignment can be extended to walks in the obvious way. Let $W = a_1 a_2 \dots a_n$ be an angle-walk on $\mathbf{M}$. The *voltage* of W is defined to be the product $\xi_W = \xi_{a_1}\xi_{a_2}\dots\xi_{a_n}$. The subgroup of G generated by voltages of all closed walks based at a fixed dart x will be called the *local voltage group* G^x at x.

Given a voltage assignment ξ on $\mathbf{M} = (D; R, L)$ valued in G, set $D^\xi = D \times \mathrm{G}$, and define the permutations of D^ξ by

$$
\begin{array}{rcl}
(x,h)R^\xi & = & (xR, h \cdot \xi(x, xR)),\\
(x,h)L^\xi & = & (xL, h \cdot \xi(x, xL)).
\end{array}
\tag{3.1}
$$

If the group $\langle R^\xi, L^\xi\rangle$ is transitive, then $\mathbf{M}^\xi = (D^\xi; R^\xi, L^\xi)$ is the *derived map* determined by $\mathbf{M}$ and ξ. The group $\langle R^\xi, L^\xi\rangle$ is transitive if $G^x = G$. In what follows we assume this property to be satisfied automatically.

It is easy to see that the *natural projection* $\pi_\xi : \mathbf{M}^\xi \to \mathbf{M}$ erasing the second coordinate is a map covering. Observe that for each element $a \in \mathrm{G}$ the mapping $\psi_a : (x, g) \mapsto (x, ag)$ is a fibre-preserving automorphism of $\mathbf{M}^\xi$ and that the group $\tilde{\mathrm{G}} = \{\psi_a;\ a \in \mathrm{G}\}$ is isomorphic to G. Moreover, the projection $\pi_{\tilde{G}} : M^\xi \to \mathbf{M}^\xi/\tilde{\mathrm{G}}$ is clearly equivalent to π_ξ. Therefore, π_ξ is a regular map homomorphism. The converse holds as well; see [22, Theorem 5.1].

Theorem 6 ([22])**.** *Let $\varphi : \tilde{\mathbf{M}} \to \mathbf{M} = \tilde{\mathbf{M}}/\mathrm{G}$ be a standard map homomorphism. Then there exists a voltage assignment ξ on $\mathbf{M}$ valued in* G *such that the natural projection $\pi_\xi : \mathbf{M}^\xi \to \mathbf{M}$ is equivalent to φ.*

Let v be a vertex of valency k. A *boundary angle walk for a vertex v* is a closed walk $a_1 a_2 \dots a_k$ such that $a_i = (xR^{i-1}, xR^i)$ for some dart x emanating from v, $i = 1, \dots, k$. A *boundary angle walk for a semiedge $s = \{x\}$* is the closed walk (x, xL). A *boundary angle walk for an edge $e = \{x, xL\}$*, $x \neq xL$ is the closed walk $(x, xL)(xL, x)$. A *boundary angle walk for a face F* of valency k is a closed walk $a_1 a_2 \dots a_{2k}$ where $a_{2i} = (x(RL)^{i-1}R, x(RL)^{2i})$ and $a_{2i-1} = (x(RL)^{i-1}, x(RL)^{i-1}R)$, $i = 1, 2, \dots, k$. For $y \in V \cup E \cup F$ we set $\xi_y = \xi_W$, where W is the boundary angle walk for y. Let ψ be a map automorphism on $\mathbf{M}$. We say that a voltage-assignment ξ on $\mathbf{M}$ is *locally-ψ-invariant* if for every closed angle walk W we have $\xi_W = 1$ implies $\xi_{\psi W} = 1$ [22, p. 455].

Let $\mathbf{M}$ be a map on an orbifold $\mathcal{O}(\gamma; m_1, m_2, \dots, m_r)$ with a distribution function $b : V \cup E \cup F \to \{1, m_1, m_2, \dots, m_r\}$. A voltage-assignment ξ on $\mathbf{M}$ will be called *b-compatible* if for every $y \in V \cup E \cup F$ we have $|\xi_y| = b(y)$.

The next theorem follows from Theorem 6 and Propositions 3 and 4.

Theorem 7. *Let $\varphi : \mathbf{M} \to \bar{\mathbf{M}} = \mathbf{M}/\mathrm{Aut}^+(\mathbf{M})$ be a regular covering from an Archimedean map of genus g and let b be the induced distribution of branch-indices. Then there exists a b-compatible voltage-assignment $\xi : A(\bar{\mathbf{M}}) \to \mathrm{Aut}^+(\mathbf{M})$ such that the natural projection $\pi_\xi : \mathbf{M}^\xi \to \bar{\mathbf{M}}$ is equivalent to φ.*

Furthermore, if $\mathbf{M}$ is of type II, then ξ is locally-$\bar{\tau}$-invariant for some orientation-reversing map automorphism $\bar{\tau}$ transposing the two vertices of $\bar{\mathbf{M}}$.

Remark 8. Let $\mathbf{N}$ be a map. Using some standard arguments in topological graph theory [14, Theorem 2.5.3, Theorem 2.5.4] one can prove that a voltage assignment $\xi : A(\mathbf{N}) \to \mathrm{G}$ can be reduced with respect to a rooted spanning tree T of $\mathbf{N}$ as follows. For each vertex v there is a unique dart d_v (based at v) on the unique path in T joining v to the root. We form an equivalent voltage assignment satisfying the following conditions:

(1) all angles of the form (x, xR), (x, xR^{-1}) except (d_v, d_vR), (d_vR, d_v), $v \in V(\mathbf{N})$, receive trivial voltages,
(2) for angles of the form $a = (x, xL)$, $\xi_a = 1$ if x is contained in the spanning tree T.

The assignment ξ satisfying (1) and (2) will be called *T-reduced*. In a T-reduced voltage assignment ξ, the voltage ξ_a on an angle $a = (x, xL)$ can be interpreted as a voltage on the dart x while for every vertex $v \in V$ the voltage on the angle (d_v, d_vR) can be interpreted as ξ_v. It follows that a T-reduced voltage assignment can be viewed as a voltage assignment $\xi : V \cup D \to \mathrm{G}$.

Remark 9. A b-compatible voltage assignment $\xi : A(\bar{\mathbf{M}}) \to \mathrm{G}$ on a one-vertex map $\bar{\mathbf{M}}$ on an orbifold $\mathcal{O} = \mathcal{O}(\gamma; m_1, m_2, \ldots, m_r)$ determines G as a quotient of the fundamental group $\pi_1(\mathcal{O})$ of the orbifold $\mathcal{O}$ which is an F-group with presentation:

$$\pi_1(\mathcal{O}) = \langle a_1, \ldots, a_\gamma, b_1, \ldots, b_\gamma, x_1, \ldots, x_r \mid x_1^{m_1}, \ldots, x_r^{m_r}, \prod_{j=1}^{r} x_j \prod_{k=1}^{\gamma} [a_k, b_k] \rangle.$$

Given an orbifold $\mathcal{O}$ the *canonical one-vertex map* is the one whose faces induce the universal relators in the above presentation [12, Section 1.2].

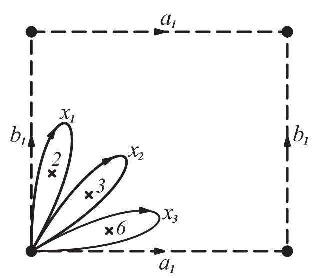


FIGURE 1. The canonical map on the orbifold $\mathcal{O}(1; \{2, 3, 6\})$ defines F-group $\pi(\mathcal{O}) = \langle a_1, b_1, x_1, x_2, x_3 \mid x_1^2, x_2^3, x_3^6, x_1x_2x_3[a_1, b_1] \rangle$.

4. Description of computation

At this moment we are ready to determine Archimedean maps of a given genus. In what follows we provide steps of an algorithm reconstructing all Archimedean maps of genera $g = 2, 3, 4$.

Step 1 (*Solving Riemann-Hurwitz equation*). The lists of admissible orbifolds were derived in [5] for genus 2 and 3 and in [2] for surface of genus 4. For our purpose we have derived the catalogues independently. It turned out that there is a misprint in the catalogue in [2] (compare with [19]). The processing is divided into two substeps:

a) solving the Riemann-Hurwitz equation numerically,

b) for a given set of parameters $|\mathrm{G}|, g, \gamma, m_1, m_2, \ldots, m_r$ we determine all groups G acting on $\mathcal{S}_g$ giving rise to the g-admissible orbifold $\mathcal{O} = \mathcal{O}(\gamma; m_1, m_2, \ldots, m_r)$. This can be done by constructing all b-compatible voltage assignments defined on a canonical one-vertex map on $\mathcal{O}$.

In the first step the following bound turned out to be useful.

Lemma 10. *Given $g > 1$, the number of branch-indices r in the Riemann-Hurwitz equation* (2.1) *is bounded by $2g + 2 \geq r$.*

Proof. Assume the covering is branched. Hence $|\mathrm{G}| \geq 2$. Setting $m_i = 2$ for each $i = 1, 2, \ldots, r$, minimises the summands and maximises r. Now

$$2 - 2\gamma + \frac{2g-2}{|\mathrm{G}|} = \frac{r}{2}.$$

To maximise r we set $\gamma = 0$ and $|\mathrm{G}| = 2$. □

The computation iterates through $\gamma \in \{0 \ldots g-1\}$ and $|\mathrm{G}| \in \{2, \ldots, 84(g-1)\}$ testing the values of m_i such that $m_i \leq |\mathrm{G}|$ divides $|\mathrm{G}|$ for each $i = 1, 2, \ldots, r$ including cases with $r = 0$ (smooth covers $\mathcal{S}_g \to \mathcal{S}_\gamma$).

Note that all the groups of orders given by the numerical solution are contained (at least up to genus 15, see [19]) in Small Groups Library [29], which is contained in Magma [3]. Thus employing the software Magma, the candidates for the voltage groups can be identified inside these lists.

Step 2 (*Determination of quotient maps*). Given a group $\mathrm{G} \leq \mathrm{Aut}^+(\mathbf{M})$ acting with one or two orbits on the set of vertices of $\mathbf{M}$, we use Proposition 5 to obtain the maximal valency ℓ of the quotient map $\bar{\mathbf{M}} = (\bar{D}; \bar{R}, \bar{L})$. In the one-vertex case we assume $\bar{R} = (1, 2, \ldots, \ell)$ and in the two-vertex case we assume $\bar{R} = (1, 2, \ldots, \ell)(\ell + 1, \ell + 2, \ldots, 2\ell)$. For sake of clarity the two cycles of $\bar{R}$ in two-vertex quotient map are denoted as the "black" and "white" vertex. Every quotient map is determined by setting $\bar{L} \in \mathrm{Sym}(\ell)$ or $\bar{L} \in \mathrm{Sym}(2\ell)$, respectively, where $\bar{L}^2 = 1$ and $\langle \bar{R}, \bar{L} \rangle$ is transitive on $\bar{D}$. In the case of a two-vertex quotient we first determine all $\bar{\tau}$ transposing the two vertices such that $\bar{R}^{\bar{\tau}} = \bar{R}^{-1}$ (see Proposition 4) and then we determine all dart-reversing involutions $\bar{L}$ such that $\bar{L}^{\bar{\tau}} = \bar{L}$. This approach significantly decreases the computation time. In the case of a one-vertex quotient map $\bar{\tau}$ is always set to be the identity of $\mathrm{Sym}(\ell)$. By reasons of symmetry it is sufficient to consider a unique representative of each conjugacy class $\bar{L}^{\langle \bar{R} \rangle}$.

We fix one dart d of $\bar{\mathbf{M}}$ to be a root and reorder the branch-indices such that the face whose boundary walk contains d has branch-index m_1. In the one-vertex case we simply choose the dart $d = 1$. In the two-vertex case we choose a dart emanating from the "white" vertex and terminating in the "black" vertex.

Given a quotient map $\bar{\mathbf{M}}$ of genus γ and a g-admissible orbifold $\mathcal{O}(\gamma; m_1, m_2, \ldots, m_r)$, we derive all possible distributions of branch-points among the faces, semiedges and vertices of $\bar{\mathbf{M}}$. The distribution b of branch-indices is done such that the conditions of Proposition 5 are satisfied. The distribution b is checked for consistency with $\bar{\tau}$, i.e., the images of vertices, semiedges and faces should have the same branch-assignment as the respective preimages. The search of possible quotient embeddings is done through all g-admissible signatures (from Step 1) and all quotient maps.

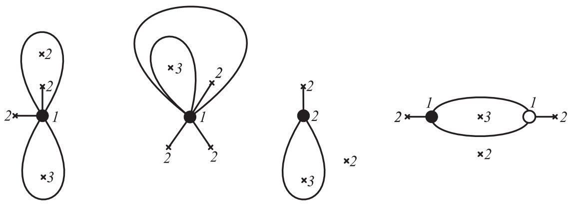


FIGURE 2. Different quotients on the orbifold $\mathcal{O}(0; \{2, 2, 2, 3\})$.

Step 3 (*Determining voltage assignments*). By an embedding $\bar{\mathbf{M}} \hookrightarrow \mathcal{O}$ a partial presentation of a voltage group is determined. Namely, for each angle $\bar{a}$ we have a generator $\xi_{\bar{a}}$ and each $x \in \bar{V} \cup \bar{E} \cup \bar{F}$ gives rise to the relator $\xi_x^{b(x)}$, where ξ_x is written as a product of generators. This way a universal voltage group U is defined. Given $\bar{\mathbf{M}}$ on the orbifold $\mathcal{S}_g/\mathrm{G}$ we try to construct an order-preserving epimorphism $\xi : \mathrm{U} \to \mathrm{G}$. Actually, we use Magma `HomomorphismsProcess` procedure, as described in The Magma Handbook of Functions [6, Part IX].

In fact, following Remark 8 we are going to construct a T-reduced b-compatible voltage assignment $\xi : \bar{V} \cup \bar{D} \to \mathrm{G}$, which allows us to reduce the number of generators. Keeping in mind that the derived map has to be simple we get additional constrains:

- for every dart x in $\bar{D}$ not belonging to T, $\xi_x \neq 1$,
- two darts, x, y emanating from the same vertex and terminating in the same vertex are endowed with different voltages, i.e., $\xi_x \neq \xi_y$,

Step 4 (*Lifting the maps*). The construction described in (3.1) was used to determine Archimedean maps. The lifted maps were tested whether they satisfy the required criteria (genus, number of darts, vertices, etc.).

In the case of two-vertex quotient maps we have tested whether the automorphism $\bar{\tau}$ lifts. For this purpose we have recorded the darts in the fibre over a dart emanating from the "black" or from the "white" vertex of the quotient, respectively. We choose a dart y from the "black fibre" covering a fixed dart d in the quotient and determine the set of darts $\{x_1, x_2, \ldots, x_{|\mathrm{G}|}\}$ from the "white fibre" covering the dart $d \cdot \bar{\tau}$. We check whether at least one mapping $\tau : y \mapsto x_i$ extends into a map homomorphism $\tau : \mathbf{M} \to \mathbf{M}^{-1}$.

Step 5 (*Polyhedrality test*). After completing Step 4 we have obtained a list of simple maps of a given genus in terms of $\mathbf{M} = (D; R, L)$. Each $\mathbf{M}$ in the list is vertex-transitive. The polyhedrality is checked directly for each map in the list following the definition.

Step 6 (*Recognition of isomorphism classes*). The list of polyhedral maps may contain (and actually it contains) isomorphic maps represented differently. In fact, in the previous computations we have constructed all the pairs of the form $(\mathbf{M}, \mathrm{G})$, where $\mathbf{M}$ is an Archimedean map on $\mathcal{S}_g$ and $\mathrm{G} \leq \mathrm{Aut}^+(\mathbf{M})$ acts either with one, or with two orbits on vertices of $\mathbf{M}$. To solve the isomorphism problem for a pair of maps $\mathbf{M}_1$ and $\mathbf{M}_2$ the following observation is crucial. Given a mapping ψ of a dart $x \mapsto y$, $x \in D_1$ and $y \in D_2$, employing the commuting rules $R_1^{\psi} = R_2$ and

$L_1^\psi = L_2$ the mapping ψ either extends to a unique isomorphism $\psi : \mathbf{M}_1 \to \mathbf{M}_2$, or there is no such isomorphism. The algorithms determining $\mathrm{Aut}(\mathbf{M})$, checking whether $\mathbf{M}$ is reflexible or chiral and so on, reduce to the same procedure.

5. Census of Archimedean maps of genera 2, 3 and 4

Each map in the list in Appendix A is described by a three-row record. The first row contains the following description of an Archimedean map

$$A_{g.i} = \mathbf{M}(\mathrm{G}_{g.i}, \mathrm{G}_v, (x_1, x_2, \dots, x_\ell)^{m_0}), \tag{5.1}$$

where g is genus and i is a unique integer identifier. The group $\mathrm{G} = \mathrm{G}_{g.i} \leq \mathrm{Aut}^+(\mathbf{M})$ is either vertex-transitive or it acts on the vertex set with two orbits. The group $\mathrm{G}_v = \langle z \rangle \leq \mathrm{G}$ is a vertex stabilizer ($\mathrm{G}_v \cong C_{m_0}$, where $m_0 \mid \mathrm{val}(\mathbf{M})$). In general, the same map admits more than one vertex-transitive action of a group of automorphisms. Our program chooses the representative with the minimum order. The (local) type of $\mathbf{M}$ and information about the type (Type I or II) and reflexibility is displayed in this row.

In the case of maps of Type I the vector $(x_1, x_2, \dots, x_\ell)$ gives a fixed cyclic order of the set of generators $\{x_1, x_2, \dots, x_\ell\}$ induced by the rotation of $\mathbf{M}$. The set $\{x_1, x_2, \dots, x_\ell\}$ is closed under taking inverses. The above description generalises the standard description of a Cayley map [28]. A map $\mathbf{M}$ is a *Cayley map* if there exist $\mathrm{G} \leq \mathrm{Aut}^+(\mathbf{M})$ acting regularly on the vertices of $\mathbf{M}$.

In case of maps of Type II the vector $(x_1, x_2, \dots, x_\ell)$ gives a fixed cyclic order of the set of generators $\{x_1, x_2, \dots, x_\ell\}$ at the "black" vertices induced by the rotation of $\mathbf{M}$. The rotation at the "white" vertices is compatible with the cyclic permutation $(\bar{\tau}x_1, \bar{\tau}x_2, \dots, \bar{\tau}x_\ell)$. In the case $m_0 = 1$ the action of the considered automorphism group is regular on vertices and the corresponding Archimedean map is an unoriented Cayley map (see [20] for more details).

The second row of the record shows a presentation of $\mathrm{G} = \mathrm{G}_{g.i}$ and structural information about G [3, 29].

The last row gives the quotient map $\bar{\mathbf{M}} = (\bar{D}; \bar{R}, \bar{L})$; we use the abbreviation $\bar{\mathbf{M}} = [\bar{R}; \bar{L}]$. If $\mathbf{M}$ is of Type I, then $\bar{R}$ is by definition $\bar{R} = (1, 2, \dots, \ell)$ (see Theorem 11), and $\bar{L} \in \mathrm{Sym}(\ell)$ is written as a product of transpositions. If $\mathbf{M}$ is of Type II, then $\bar{R}$ is by definition $\bar{R} = (1, 2, \dots, \ell)(\ell{+}1, \ell{+}2, \dots, 2\ell)$ (see Theorem 11), and $\bar{L} \in \mathrm{Sym}(2\ell)$ is written as a product of transpositions. The quotient $\bar{\mathbf{M}}$ is embedded into the orbifold of the signature $\mathcal{O}(\gamma; m_1, m_2, \dots, m_r) = \mathcal{S}_g/\mathrm{G}_{g.i}$; see the extended catalogue [18].

The records in the list are sorted lexicographically according to the local type. Full records of Archimedean maps of genera $2, 3, 4$ can be found at the website [18].

The output of our computation can be found in the following theorems.

Theorem 11. *The following table enumerates Archimedean maps of genera 2, 3 and 4.*

Genus	All	Type I	Type II	Reflexible	Chiral pairs
2	17	9	8	13	4
3	103	78	25	63	40
4	111	76	35	77	34

Since our algorithm constructs all pairs $(\mathbf{M}, \mathrm{G})$ *where* $\mathbf{M}$ *is Archimedean of genus* $2 \leq g \leq 4$ *and* $\mathrm{G} \leq \mathrm{Aut}^+(\mathbf{M})$ *is either transitive on vertices of* $\mathbf{M}$ *or it acts with two orbits, we get as a by-product a list of non-Cayley Archimedean maps of genera* $2, 3, 4$. *Most of the Archimedean maps of type I of small genera are Cayley maps.*

Theorem 12. *Every Archimedean map of type I and of genus two is a Cayley map. There are* **3** *non-Cayley Archimedean maps of genus three of type I. There are* **2** *non-Cayley Archimedean maps of genus four and of type I.*

The description of non-Cayley Archimedean maps and genus $2 \leq g \leq 4$ follows.

$\mathrm{A}_{3.191}$ $\mathbf{M}(\mathrm{G}_{3.191},\ C_3,\ (x)^3)$ is of local type (7^3), type I, reflexible;
$\mathrm{G}_{3.191} = \langle x, y \mid x^2, y^3, (xy)^7, (xy^{-1}xy)^4 \rangle \cong PSL_3(2)$;
$\bar{\mathbf{M}} = [(1); ()]$ on $\mathcal{O}(0; \{2, 3, 7\})$;

$\mathrm{A}_{3.382}$ $\mathbf{M}(\mathrm{G}_{3.382},\ C_2,\ (x, x^{-1})^2)$ is of local type $(3.7.3.7)$, type I, reflexible;
$\mathrm{G}_{3.382} = \langle x, y \mid y^2, (xy)^3, x^7, (yx^{-3})^4 \rangle \cong PSL_3(2)$;
$\bar{\mathbf{M}} = [(1, 2); (1, 2)]$ on $\mathcal{O}(0; \{2, 3, 7\})$.

$\mathrm{A}_{3.1383}$ $\mathbf{M}(\mathrm{G}_{3.1383}, C_4, (x, x^{-1})^4)$ is of local type (3^8), type I, reflexible;
$\mathrm{G}_{3.1383} = \langle x, y \mid x^3, y^4, (x^{-1}y^{-1})^3, (yx^{-1})^3 \rangle \cong (C_4 \times C_4) : C_3$;
$\bar{\mathbf{M}} = [(1, 2); (1, 2)]$ on $\mathcal{O}(0; \{3, 3, 4\})$.

$\mathrm{A}_{4.656}$ $\mathbf{M}(\mathrm{G}_{4.656}, C_2, (x, x^{-1})^2)$ is of local type (5^4), type I, reflexible;
$\mathrm{G}_{4.656} = \langle x, y \mid y^2, x^5, (x^2y)^3, (yxyx^{-2})^2, (xy)^5 \rangle \cong A_5$;
$\bar{\mathbf{M}} = [(1, 2); (1, 2)]$ on $\mathcal{O}(0; \{2, 4, 5\})$.

$\mathrm{A}_{4.1874}$ $\mathbf{M}(\mathrm{G}_{4.1874}, C_2, (z, z^{-1}, x, y)^2)$ is of local type $(3^3.4.3^3.4)$, type I, reflexible;
$\mathrm{G}_{4.1874} = \langle x, y, z, u \mid u^2, x^2, y^2, z^3, z^{-1}xy, zxz^{-1}y, (zuy)^2, (uy)^3, (ux)^3, (yxux)^2 \rangle$
$\cong S_4$;
$\bar{\mathbf{M}} = [(1, 2, 3, 4); (1, 2)]$ on $\mathcal{O}(0; \{2, 2, 2, 4\})$.

Appendix A. Archimedean maps of genus 2

In what follows, we display the record containing basic data for Archimedean maps of genus two only. For genus 3 and 4 the list with other additional information can be found at the website [18].

$\mathrm{A}_{2.1}$ $\mathbf{M}(\mathrm{G}_{2.1},\ 1,\ (x, y, z))$ is of local type $(4.6.16)$, type II, reflexible;
$\mathrm{G}_{2.1} = \langle x, y, z \mid x, (yz^{-1})^2, (zx^{-1})^3, (y^{-3}z^{-1})^2, (xy^{-1})^8 \rangle \cong GL_2(3)$;
$\bar{\mathbf{M}} = [(1, 2, 3)(4, 5, 6); (1, 4)(2, 6)(3, 5)]$ on $\mathcal{O}(0; \{2, 3, 8\})$;

$\mathrm{A}_{2.7}$ $\mathbf{M}(\mathrm{G}_{2.7},\ 1,\ (x, y, z))$ is of local type $(4.8.12)$, type II, reflexible;
$\mathrm{G}_{2.7} = \langle x, y, z \mid x, (xy^{-1})^2, z^2yz^2y^{-1}, (yz^{-1})^4, (zx^{-1})^6 \rangle \cong (C_6 \times C_2) : C_2$;
$\bar{\mathbf{M}} = [(1, 2, 3)(4, 5, 6); (1, 4)(2, 6)(3, 5)]$ on $\mathcal{O}(0; \{2, 4, 6\})$;

$\mathrm{A}_{2.27}$ $\mathbf{M}(\mathrm{G}_{2.27},\ 1,\ (x, y, y^{-1}))$ is of local type $(6.6.8)$, type I, reflexible;
$\mathrm{G}_{2.27} = \langle x, y \mid x^2, (xy^{-1})^3, y^8, (y^3xy)^2, (y^{-2}xyx)^2 \rangle \cong GL_2(3)$;
$\bar{\mathbf{M}} = [(1, 2, 3); (2, 3)]$ on $\mathcal{O}(0; \{2, 3, 8\})$;

$\mathrm{A}_{2.33}$ $\mathbf{M}(\mathrm{G}_{2.33},\ 1,\ (x, x^{-1}, y, y^{-1}))$ is of local type $(3.4.8.4)$, type I, reflexible;
$\mathrm{G}_{2.33} = \langle x, y \mid y^3, (x^{-1}y^{-1})^2, x^8, (x^{-1}yx^{-2})^2 \rangle \cong GL_2(3)$;
$\bar{\mathbf{M}} = [(1, 2, 3, 4); (1, 2)(3, 4)]$ on $\mathcal{O}(0; \{2, 3, 8\})$;

$\mathrm{A}_{2.39}$ $\mathbf{M}(\mathrm{G}_{2.39},\ 1,\ (x, x^{-1}, y, y^{-1}))$ is of local type $(3.6.4.6)$, type I, reflexible;
$\mathrm{G}_{2.39} = \langle x, y \mid y^3, x^4, yx^{-1}yxyx^{-1}, yx^{-1}yxyx^{-1}, x^{-2}yx^{-2}y^{-1} \rangle \cong SL_2(3)$;
$\bar{\mathbf{M}} = [(1, 2, 3, 4); (1, 2)(3, 4)]$ on $\mathcal{O}(0; \{3, 3, 4\})$;

$\mathrm{A}_{2.43}$ $\mathbf{M}(\mathrm{G}_{2.43},\ 1,\ (x, x^{-1}, y, y^{-1}))$ is of local type $(4.4.4.6)$, type I, reflexible;
$\mathrm{G}_{2.43} = \langle x, y \mid y^4, (x^{-1}y^{-1})^2, (y^{-1}x)^2, x^6 \rangle \cong (C_6 \times C_2) : C_2$;
$\bar{\mathbf{M}} = [(1, 2, 3, 4); (1, 2)(3, 4)]$ on $\mathcal{O}(0; \{2, 4, 6\})$;

$A_{2.46}$ $\mathbf{M}(G_{2.46},\ 1,\ (z,r,x,s))$ is of local type (4.4.4.6), type II, reflexible;
$G_{2.46} = \langle x,y,z,r,s \mid z,x^2,y^2,s^2,(sx)^2,rsxy,r^{-1}srs,(ys)^2,(sz^{-1})^2,(zr^{-1})^3\rangle \cong D_{12}$;
$\bar{\mathbf{M}} = [(1,2,3,4)(5,6,7,8);(1,5)(2,8)(4,6)]$ on $\mathcal{O}(0;\{2,2,2,3\})$;

$A_{2.51}$ $\mathbf{M}(G_{2.51},\ 1,\ (x,y,z,r))$ is of local type (4.4.4.6), type II, reflexible;
$G_{2.51} = \langle x,y,z,r \mid x,z^2,(yz)^2,(rx^{-1})^2,(zr^{-1})^2,(r^{-1},y^{-1}),(xy^{-1})^3\rangle \cong D_{12}$;
$\bar{\mathbf{M}} = [(1,2,3,4)(5,6,7,8);(1,5)(2,8)(3,7)(4,6)]$ on $\mathcal{O}(0;\{2,2,2,3\})$;

$A_{2.89}$ $\mathbf{M}(G_{2.89},\ 1,\ (x,y,y^{-1},z,z^{-1}))$ is of local type $(3^4.8)$, type I, chiral;
$G_{2.89} = \langle x,y,z \mid x^2,xy^{-1}z^{-1},z^3,y^8,zy^{-2}xz^{-1}y^{-3}\rangle \cong GL_2(3)$;
$\bar{\mathbf{M}} = [(1,2,3,4,5);(2,3)(4,5)]$ on $\mathcal{O}(0;\{2,3,8\})$;

$A_{2.93}$ $\mathbf{M}(G_{2.93},\ 1,\ (x,y,y^{-1},z,z^{-1}))$ is of local type (3.3.4.3.6), type I, chiral;
$G_{2.93} = \langle x,y,z \mid x^2,xy^{-1}z^{-1},y^4,(z^{-1}y)^2,z^6\rangle \cong (C_6\times C_2):C_2$;
$\bar{\mathbf{M}} = [(1,2,3,4,5);(2,3)(4,5)]$ on $\mathcal{O}(0;\{2,4,6\})$;

$A_{2.97}$ $\mathbf{M}(G_{2.97},\ 1,\ (z,r,s,s^{-1},x))$ is of local type (3.4^4), type II, reflexible;
$G_{2.97} = \langle x,y,z,r,s,t \mid z,x^2,y^2,yx,s^{-1}t^{-1},s^3,t^3,rxs^{-1},rs^{-1}xz^{-1},zr^{-1}yt^{-1}\rangle \cong C_6$;
$\bar{\mathbf{M}} = [(1,2,3,4,5)(6,7,8,9,10);(1,10)(2,6)(3,4)(8,9)]$ on $\mathcal{O}(0;\{2,2,3,3\})$;

$A_{2.99}$ $\mathbf{M}(G_{2.99},\ 1,\ (x,x^{-1},y,y^{-1},z,z^{-1}))$ is of local type $(3^5.4)$, type I, reflexible;
$G_{2.99} = \langle x,y,z \mid x^3,z^3,x^{-1}y^{-1}z^{-1},y^4,xy^{-2}zy^{-1}\rangle \cong SL_2(3)$;
$\bar{\mathbf{M}} = [(1,2,3,4,5,6);(1,2)(3,4)(5,6)]$ on $\mathcal{O}(0;\{3,3,4\})$;

$A_{2.105}$ $\mathbf{M}(G_{2.105},\ 1,\ (x,z,z^{-1},r,y,r^{-1}))$ is of local type $(3^4.4^2)$, type I, chiral;
$G_{2.105} = \langle x,y,z,r \mid x^2,y^2,xz^{-1}r^{-1},z^3,zr^{-2},xzr,(ry)^2,(yx)^2,(z^{-1}y)^2\rangle \cong D_{12}$;
$\bar{\mathbf{M}} = [(1,2,3,4,5,6);(2,3)(4,6)]$ on $\mathcal{O}(0;\{2,2,2,3\})$;

$A_{2.109}$ $\mathbf{M}(G_{2.109},\ 1,\ (z,z^{-1},r,s,r^{-1},x))$ is of local type$(3^4.4^2)$, type II, reflexible;
$G_{2.109} = \langle x,y,z,r,s,t,u \mid s,x^2,y^2,yx,tz^{-1},zur^{-1},$
$t^3,z^3,z^{-1}r^{-1}x,xzu^{-1},tyu^{-1},rs^{-1}us\rangle \cong C_6$;
$\bar{\mathbf{M}} = [(1,2,3,4,5,6)(7,8,9,10,11,12);(1,2)(3,5)(4,10)(7,12)(9,11)]$ on $\mathcal{O}(0;\{2,2,3,3\})$;

$A_{2.111}$ $\mathbf{M}(G_{2.111},\ 1,\ (r,r^{-1},s,x,y,s^{-1},z))$ is of local type (3^7), type I, chiral;
$G_{2.111} = \langle x,y,z,r,s \mid x^2,y^2,z^2,rs^{-2},r^3,r^{-1}s^{-1}z,sxy,zrs,(zx)^2\rangle \cong D_{12}$;
$\bar{\mathbf{M}} = [(1,2,3,4,5,6,7);(1,2)(3,6)]$ on $\mathcal{O}(0;\{2,2,2,3\})$;

$A_{2.114}$ $\mathbf{M}(G_{2.114},\ 1,\ (z,z^{-1},r,s,t,r^{-1},x))$ is of local type (3^7), type II, reflexible;
$G_{2.114} = \langle x,y,z,v,r,s,t,u \mid s,x^2,y^2,yx,uz^{-1},rs^{-1}t,$
$st^{-1}v,z^3,z^{-1}r^{-1}x,uyv^{-1},u^3,ztr^{-1},xzt^{-1}\rangle \cong C_6$;
$\bar{\mathbf{M}} = [(1,2,3,4,5,6,7)(8,9,10,11,12,13,14);(1,2)(3,6)(4,11)(5,12)(8,14)(10,13)]$ on $\mathcal{O}(0;\{2,2,3,3\})$;

$A_{2.116}$ $\mathbf{M}(G_{2.116},\ 1,\ (z,r,x,s,t,t^{-1},u))$ is of local type (3^7), type II, reflexible;
$G_{2.116} = \langle x,y,z,v,r,s,t,u \mid z,x^2,y^2,xy,tr^{-1},t^{-1}v^{-1},$
$uz^{-1}y,t^3,st^{-1}u^{-1},v^3,rys^{-1},zr^{-1}v^{-1}\rangle \cong C_6$;
$\bar{\mathbf{M}} = [(1,2,3,4,5,6,7)(8,9,10,11,12,13,14);(1,12)(2,9)(4,8)(5,6)(7,14)(10,11)]$ on $\mathcal{O}(0;\{2,2,3,3\})$;

Acknowledgements

The authors would like to thank Jin-Ho Kwak, Marston Conder, and Daniel Pellicer for fruitful discussions about the method and algorithms used here.

References

[1] L. Babai, *Vertex-transitive graphs and vertex-transitive maps*, J. Graph Theory **15** (1991), no. 6, pp. 587–627. MR1133814 (92m:05064)

[2] O. V. Bogopolski, *Classifying the actions of finite groups on orientable surfaces of genus* 4, Siberian Adv. Math. **7** (1997), pp. 9–38. MR1604157 (2000e:30083)

[3] W. Bosma, J. Cannon and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput., **24**(3-4):235-265, 1997. MR1484478

[4] U. Brehm and E. Schulte, *Polyhedral maps* in "Handbook of discrete and computational geometry", J. E. Goodman and J. O'Rourke (Eds.), Chapman & Hall/CRC, Boca Raton, FL, 2004, pp. 477–491. MR1730174
[5] S. A. Broughton, *Classifying finite group actions on surfaces of low genus*, J. Pure Appl. Algebra **69** (1991), pp. 233–270. MR1090743 (92b:57021)
[6] J. Cannon, W. Bosma, C. Fieker and A. Steel (Eds.), Handbook of Magma Functions, Edition 2.16 (2010), Sydney, 5017 p.
[7] M. Conder and P. Dobcsányi, *Determination of all regular maps of small genus*, J. Combin. Theory Ser. B **81** (2001), pp. 224–242. MR1814906 (2002f:05088)
[8] J. H. Conway, H. Burgiel and Ch. Goodman-Strass, "Symmetries of things", A. K. Peters Ltd. 2008. MR2410150 (2009c:00002)
[9] H. M. Cundy and A. P. Rollett, "Mathematical models", 2nd Ed. Oxford University Press, 1961. MR0124167 (23:A1484)
[10] J. D. Dixon and B. Mortimer, "Permutation groups", Springer-Verlag, New York, 1996. MR1409812 (98m:20003)
[11] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*; 2006, (`http://www.gap-system.org`).
[12] G. Gromadzki, "Groups of Automorphisms of Compact Riemann and Klein Surfaces", Wydawnictwo U. W. Sz. Pedag., Bydgoszc, 1993.
[13] A. Hurwitz, *Über algebraische Gebilde mit Eindeutigen Transformationen in sich*, Mathematische Annalen **41** (3), (1893), pp. 403–442 MR1510753
[14] J. L. Gross and T. W. Tucker, "Topological graph theory", Dover Publications Inc., Mineola, NY, 2001, second edition. MR1855951
[15] B. Grünbaum, "Convex polytopes", Graduate Texts in Mathematics **221**, Springer-Verlag, New York, 2003, second edition. MR1976856 (2004b:52001)
[16] B. Grünbaum and G. C. Shephard, "Tilings and Patterns", W. H. Freeman and Co., New York, 1986. MR992195 (90a:52027)
[17] G. A. Jones and D. Singerman, *Theory of maps on orientable surfaces*, Proc. London Math. Soc. (3) **37** (1978), pp. 273–307. MR0505721 (58:21744)
[18] J. Karabáš, *Archimedean solids*, `http://www.savbb.sk/~karabas/science.html#arch`.
[19] J. Karabáš, *Actions of finite groups on Riemann surfaces of higher genera*, `http://www.savbb.sk/~karabas/science.html#rhsu`.
[20] J. H. Kwak and Y. S. Kwon, *Unoriented Cayley Maps*, Studia Scientiarum Mathematicarum Hungarica **43**(2) (2006), pp. 137–157 MR2229619 (2007g:05049)
[21] A. Malnič, R. Nedela and M. Škoviera, *Lifting graph automorphisms by voltage assignments*, European J. Combin. **21** (2000), no. 7, pp. 927–947. MR1787907 (2001i:05086)
[22] A. Malnič, R. Nedela and M. Škoviera, *Regular homomorphisms and regular maps*, European J. Combin. **23** (2002), no. 4, pp. 449–461. MR1914482 (2003g:05045)
[23] A. Mednykh and R. Nedela, *Enumeration of unrooted maps of a given genus*, J. Combin. Theory Ser. B **96** (2006), pp. 706–729. MR2236507 (2007g:05088)
[24] B. Mohar and C. Thomassen, "Graphs on Surfaces", John Hopkins University Press, Baltimore, 2001. MR1844449 (2002e:05050)
[25] R. Nedela and M. Škoviera, *Exponents of orientable maps*, Proc. London Math. Soc. (3), **75** (1997), no. 1, pp. 1–31. MR1444311 (98i:05059)
[26] D. Pellicer and A. Ivić Weiss, *Uniform maps on surfaces of non-negative Euler characteristic*, preprint.
[27] V. Proulx, "Classification of the Toroidal Groups", Ph.D. thesis, Columbia University, 1977. MR2627033
[28] R. B. Richter, J. Širáň, R. Jajcay, T. W. Tucker and M. E. Watkins, *Cayley maps*, J. Combin. Theory Ser. B **95** (2005), pp. 189–245. MR2171363 (2006g:05063)
[29] "The Small Groups Library", `http://www-public.tu-bs.de:8080/~hubesche/small.html`.
[30] E. Steinitz, *Polyeder und raumeinteilungen*, "Enzyklopedie Math. Wiss". 3 (Geometrie), Leipzig, 1922, pp. 1 – 139.
[31] O. Šuch, *Vertex-transitive maps on torus*, preprint.
[32] "Archimedean solid", Wikipedia, `http://en.wikipedia.org/wiki/Archimedean_solid`.

(J. Karabáš) Science and Research Institute, Matej Bel University, Ďumbierska 1, 974 11 Banská Bystrica, Slovakia

E-mail address: karabas@savbb.sk

(R. Nedela) Faculty of Natural Sciences, Matej Bel University, Tajovského 40, 974 01 Banská Bystrica, Slovakia

(R.Nedela) Mathematical Institute, Slovak Academy of Sciences, Ďumbierska 1, 974 11 Banská Bystrica, Slovakia

E-mail address: nedela@savbb.sk

MAPS OF ARCHIMEDEAN CLASS AND OPERATIONS ON DESSINS

ANTONIO BREDA D'AZEVEDO, DOMENICO CATALANO, JÁN KARABÁŠ, AND ROMAN NEDELA

ABSTRACT. In the present paper we introduce a family of functors (called operations) of the category of hypermaps (dessins) preserving the underlying Riemann surface. The considered family of functors include as particular instances the operations considered by Magot and Zvonkin (2000), Singerman and Syddall (2003), and Girondo (2003). We identify a set of 10 operations in the above infinite family which produce vertex-transitive dessins out of regular ones. This set is complete in the following sense: If a vertex-transitive map arises from a regular dessin **H** applying an operation, then it can be obtained from a regular dessin on the same surface (possibly different from **H**) applying one of the 10 operations. The statement includes the classical case when the underlying surface is the sphere.

1. INTRODUCTION

In classical crystallography the Archimedean solids can be constructed from the Platonic solids by applying few operations, generally determined by local changing of their 2-skeletons. This is the case of the operations defined by taking a dual, truncation, medial, or dual of barycentric subdivision of a map, to name but a few. While some operations like the barycentric subdivision destroy vertex transitivity, the operations that take the Platonic solids to the Archimedean solids preserve it. However, all the classical operations considered for maps share the following common properties:

- they preserve the underlying surface;
- they preserve the automorphism group of the original Platonic map, i.e. the automorphism group of the resulting map contains the automorphism group of the original Platonic map.

Map operations based on local redrawings are common in geometry and map theory to construct new maps from a given one. In the Grothendieck theory of dessins d'enfants, where the underlying surfaces come endowed with a structure of a Riemann surface, deciding when two dessins share the same Riemann surface is a related and unsolved problem. By Belyĭ's Theorem, every map (dessin) uniquely determines a Riemann surface defined over a field of algebraic numbers. Several authors have considered distinct instances of this problem. Magot and Zvonkin, for example, have determined in [11] explicit transformations of the Belyĭ functions that correspond to the classical operations taking Platonic solids to the Archimedean solids. Singerman and Syddall in [15] have looked into operations on the uniform surface tilings of type $\{k, m, n\}$ preserving the underlying Riemann surface structure. Further development in this direction was done by Girondo in [8]

2010 *Mathematics Subject Classification.* 05C25; 05C10, 57M60, 57M12.
Key words and phrases. action of a group on a surface, Belyĭ function, dessin, hypermap, map, map covering, orbifold.
Printed October 15, 2014.

where he described eight operations (called surgeries) on uniform dessins with the following property: If $\mathbf{H}_1$ and $\mathbf{H}_2$ are uniform dessins with the same underlying Riemann surface, then one can be obtained from the other by applying a sequence of Girondo's surgeries.

Recall that a hypermap is a bipartite map on an orientable surface with a fixed colouring of the two classes of vertices. A hypermap $\mathbf{H}$ is regular if the automorphism group of $\mathbf{H}$ is transitive on the edges of $\mathbf{H}$ [5]. In this paper we generalise the classical Archimedean operations, transforming the Platonic solids into the Archimedean ones, to higher genera. In particular, we investigate the family of vertex-transitive maps arising from regular maps and hypermaps applying these operations. We will call this family of maps, more precisely defined in Section 4, *maps of Archimedean class*. A short version of our main result follows.

Theorem 1. *Every map $\mathbf{M}$ of Archimedean class of genus $g \geq 0$ is either a regular map, or comes from a regular hypermap $\mathbf{H}$ of genus g by applying one of the 10 operations defined further.*

Note that applying an operation to a regular hypermap might not give a map of Archimedean class (see Section 5). With a few exceptions, such as the regular maps and hypermaps operations introduced by Jones and Singerman [10] and Lynne James [9], many of non-regular map operations are loosely defined. In contrast, we use Belyĭ's Theorem [1] to precise our definition of operation on dessins (see Section 3 for details). As a result we prove that our operations possess the following nice properties (cf. Lemma 2, Lemma 3):

- they form a monoid under composition;
- they preserve the underlying Riemann surface and coverings between dessins;
- they preserve the automorphism group.

More important yet, the Magot and Zvonkin's operations, as well as the Girondo's surgeries, can be explained in terms of our operations. Furthermore, inclusions between triangle groups determined by Singerman [13] give rise to particular instances of our operations.

Since regular maps and hypermaps are of Archimedean class, our main result can be interpreted as an extension of Singerman's and Girondo's results describing regular dessins on the same Riemann surface [8, 14, 15], see Corollary 17.

2. Maps and hypermaps

It is well-known that maps on orientable surfaces can be described by means of two permutations. An *oriented map* $\mathbf{M}$ is a triple $\mathbf{M} = (D; R, L)$, where D is the set of *darts*, the *rotation* R and the *dart-reversing involution* L are permutations in $\mathrm{Sym}(D)$ such that $L^2 = id$. Moreover, the *orientable monodromy group of the map* $\mathrm{Mon}^+(\mathbf{M}) = \langle R, L \rangle$ is transitive on D. Throughout the paper we adopt a convention that $\mathrm{Mon}^+(\mathbf{M})$ acts on D from right. The orbits of $\langle R \rangle$ will be called *vertices* of $\mathbf{M}$. The orbits of $\langle L \rangle$ determine the *edges* of the map $\mathbf{M}$. Our maps may have loops, multiple edges and even *semi-edges*. The semi-edges of $\mathbf{M}$ correspond to the fixed points of L. The orbits of $\langle RL \rangle$ define the *faces* of the map $\mathbf{M}$. The sets of vertices, edges, and faces of $\mathbf{M}$ will be denoted by $V(\mathbf{M})$, $E(\mathbf{M})$, and $F(\mathbf{M})$, respectively.

The incidence relation between vertices and edges of $\mathbf{M}$, given by non-empty intersections of the respective orbits defines the *underlying graph X of* $\mathbf{M}$. Transitivity of $\mathrm{Mon}^+(\mathbf{M})$ implies that X (and hence the map $\mathbf{M}$ itself) is connected. Following Jones and Singerman [10], the map $\mathbf{M}$ can be viewed as an embedding $\varepsilon\colon X \hookrightarrow \mathcal{S}_g$, where $\mathcal{S}_g$ is a closed orientable surface, such that the connected components of $\mathcal{S}_g \setminus \varepsilon(X)$ are homeomorphic to open discs and they are in one-to-one correspondence with the faces of $\mathbf{M}$. Thus $\mathbf{M}$ can be regarded as a *topological map*, that is a 2-cell decomposition of $\mathcal{S}_g$. Moreover, each cycle of R permutes the darts incident to a vertex of $\mathbf{M}$ following a fixed global orientation in $\mathcal{S}_g$. For further details we refer to paper by Jones and Singerman [10]. An oriented map is *non-degenerate* if it has no semi-edges, every vertex has valency at least 3, and every face has valency at least 3.

Relaxing the condition $L^2 = id$ in the above definition of oriented map we get the definition of *oriented hypermap*. Maps and hypermaps in this paper are finite, unless stated otherwise.

A *hypermap homomorphism* (or *covering* of hypermaps) $\vartheta : \mathbf{H}_1 \to \mathbf{H}_2$ is defined as a mapping $\vartheta : D_1 \to D_2$ satisfying $R_1\vartheta = \vartheta R_2$ and $L_1\vartheta = \vartheta L_2$. If ϑ is a bijection, the hypermaps $\mathbf{H}_1$ and $\mathbf{H}_2$ are isomorphic, $\mathbf{H}_1 \cong \mathbf{H}_2$. A *hypermap automorphism* $\vartheta : \mathbf{H} \to \mathbf{H}$ is a hypermap homomorphism where $\vartheta \in \mathrm{Sym}(D)$. The automorphisms of a hypermap $\mathbf{H}$ form the *orientation-preserving automorphism group* $\mathrm{Aut}^+(\mathbf{H})$. Let $\mathbf{H} = (D; R, L)$ be a hypermap. Any subgroup $\mathrm{G} \leq \mathrm{Aut}^+(\mathbf{H})$ determines a *quotient hypermap* $\mathbf{N} = (\bar{D}; \bar{R}, \bar{L}) = \mathbf{H}/\mathrm{G}$ by setting $\bar{D} = \{[x]_\mathrm{G} \mid x \in D\}$, $[x]_\mathrm{G}.\bar{R} = [xR]_\mathrm{G}$ and $[x]_\mathrm{G}.\bar{L} = [xL]_\mathrm{G}$, where $[x]_\mathrm{G}$ denotes the orbit of the action of G on the dart x. The quotient of a map is again a map. A hypermap covering $\mathbf{H} \to \mathbf{N}$ is called *regular* if $\mathbf{N} \cong \mathbf{H}/\mathrm{G}$ for some $\mathrm{G} \leq \mathrm{Aut}^+(\mathbf{H})$. Oriented hypermaps together with (orientation-preserving) homomorphisms form a category OHM. In this paper we shall exclusively consider hypermaps as members of this category.

Since the monodromy group $\mathrm{Mon}^+(\mathbf{H}) = \langle R, L \rangle$ is transitive on the set D of darts, the action of $\mathrm{Aut}^+(\mathbf{H})$ is semi-regular on D. The oriented hypermap $\mathbf{H}$ is *regular* if $\mathrm{Aut}^+(\mathbf{H})$ acts transitively on the set of darts, hence $\mathrm{Aut}^+(\mathbf{H}) \cong \mathrm{Mon}^+(\mathbf{H}) = \mathrm{G}$ and $|D| = |\mathrm{G}|$, compare with Dixon and Mortimer [6, Theorem 4.2A]. It follows that a regular hypermap can be algebraically described by the triple $(\mathrm{G}; R, L)$, where the monodromy elements R and L act on the set of darts $g \in \mathrm{G}$ by right multiplication. Regular hypermaps are the "most symmetric hypermaps". Besides regular hypermaps, highly symmetrical hypermaps are vertex-transitive hypermaps. A hypermap $\mathbf{H}$ is *vertex-transitive* if its automorphism group $\mathrm{Aut}(\mathbf{H})$ acts transitively on the vertices of $\mathbf{H}$.

Given a hypermap $\mathbf{H} = (D; R, L)$, the hypermap $\mathbf{H}^{-1} = (D; R^{-1}, L^{-1})$ is called the *mirror image* of $\mathbf{H}$. The hypermap $\mathbf{H}^{-1}$ may, or may not, be isomorphic to $\mathbf{H}$. When $\mathbf{H} \cong \mathbf{H}^{-1}$ we say that $\mathbf{H}$ is *reflexible*, otherwise it is *chiral*. An isomorphism $\varrho \in \mathrm{Sym}(D)$ sending $\mathbf{H}$ onto $\mathbf{H}^{-1}$ will be called a *reflection*. The group generated by (orientation-preserving) automorphisms and reflections of a hypermap $\mathbf{H}$ is called the *full automorphism group* $\mathrm{Aut}(\mathbf{H})$. A reflection fixing a dart of D is uniquely determined. Thus $\mathrm{Aut}^+(\mathbf{H}) \leq \mathrm{Aut}(\mathbf{H})$ is a subgroup of index at most 2. If $\mathbf{H}$ is chiral, then $\mathrm{Aut}(\mathbf{H}) = \mathrm{Aut}^+(\mathbf{H})$.

Together with reflections, cell-operations will play an important rôle in this paper. The dual of a map $\mathbf{M}$ is another map which may, or may not, be isomorphic to $\mathbf{M}$. Dual is thus a duality operation on the category of maps. If a map $\mathbf{M}$ is self-dual (i.e., isomorphic to its dual), the dual operation can be seen as an external symmetry of $\mathbf{M}$ swapping faces with vertices. In the hypermap category we may have six kinds of cell-operations distinguished by the respective permutations of their *cells*, namely the sets of hypervertices (marked by 0), hyperedges (marked by 1), and hyperfaces (marked by ∞). A *cell-operation* is an operation on the category of hypermaps, preserving the underlying surface, which is induced by a particular outer-automorphism of the free group $\langle r, \ell \rangle$ of rank 2. Cell-operations preserve sets of darts and act on monodromy groups as group automorphisms. As a consequence, each cell-operation can be orientation-preserving or orientation-reversing. Thus, we have 12 kinds of cell-operations on hypermaps. Let $\sigma^-_{(0,1)}$ be the involutory cell-operation (duality operation) $r \mapsto \ell^{-1}$, $\ell \mapsto r^{-1}$ that transforms the hypermap $\mathbf{H} = (D; R, L)$ to the hypermap $\mathbf{H}^{-}_{(0,1)} = (D; L^{-1}, R^{-1})$, and let $\sigma^{+}_{(0,\infty)}$ be the involutory cell-operation sending $\mathbf{H}$ to $\mathbf{H}^{+}_{(0,\infty)} = (D; L^{-1}R^{-1}, L)$ corresponding to the outer automorphism determined by $r \mapsto \ell^{-1}r^{-1}$ and $\ell \mapsto \ell$. It is easily seen that the group of cell-operations generated by the dualities $\sigma^{-}_{(0,1)}$ and $\sigma^{+}_{(0,\infty)}$ is the dihedral group of order 12; the central involution is the mirror-image operation $\left(\sigma^{-}_{(0,1)}\sigma^{+}_{(0,\infty)}\right)^3$. A permutation of cells (vertices, edges, and faces) and a sign (where '+' means orientation-preserving) uniquely determines one of the twelve cell-operations. For instance, $\sigma^{-}_{(0,1)}$ is the cell-operation that fixes faces and transposes vertices with edges while the global orientation is reversed. Let $\alpha \in \mathrm{Sym}(\{0, 1, \infty\})$ and $e \in \{+, -\}$. We say that a hypermap is *α^e-self-dual* if $\mathbf{H} \cong \sigma^e_\alpha(\mathbf{H}) = (D; \sigma^e_\alpha(R), \sigma^e_\alpha(L))$. The following table describes all possible cell-operations σ^e_α acting on a hypermap $\mathbf{H} = (D; R, L)$.

α	e	σ^e_α	e	σ^e_α
id		$R \mapsto R,\ L \mapsto L$		$R \mapsto R^{-1},\ L \mapsto L^{-1}$
$(0,1)$		$R \mapsto L,\ L \mapsto R$		$R \mapsto L^{-1},\ L \mapsto R^{-1}$
$(0,\infty)$	$+$	$R \mapsto L^{-1}R^{-1},\ L \mapsto L$	$-$	$R \mapsto LR,\ L \mapsto L^{-1}$
$(1,\infty)$		$R \mapsto R,\ L \mapsto L^{-1}R^{-1}$		$R \mapsto R^{-1},\ L \mapsto LR$
$(0,1,\infty)$		$R \mapsto L,\ L \mapsto R^{-1}L^{-1}$		$R \mapsto L^{-1},\ L \mapsto RL$
$(0,\infty,1)$		$R \mapsto R^{-1}L^{-1},\ L \mapsto R$		$R \mapsto RL,\ L \mapsto R^{-1}$

TABLE 1. The 12 possible cell-operations.

Let $\mathbf{H} = (D; R, L)$ be a regular hypermap with $\mathrm{G} = \mathrm{Aut}^+(\mathbf{H}) \cong \langle R, L\rangle$. Clearly, every α^e-self-duality of $\mathbf{H}$ induces an automorphism of G. Therefore we can form an *extended group of automorphisms* $\mathrm{Aut}^*(\mathbf{H})$ of the hypermap $\mathbf{H}$, generated by G and all reflections and self-dualities of $\mathbf{H}$. There is a natural homomorphism from $\mathrm{Aut}^*(\mathbf{H})$ into the dihedral group of order 12 with kernel G. For more details on maps and hypermaps we refer the reader to [5, 10].

3. Dessins

For each hypermap $\mathbf{H} = (D; R, L)$, there is an associated bipartite map $w(\mathbf{H}) = (D_w; R_w, L_w)$ defined as follows:

$$\begin{aligned} D_w &= D \times \{\pm 1\}, \\ (x,1)R_w &= (xR, 1), \\ (x,-1)R_w &= (xL, -1), \\ (x,i)L_w &= (x, -i). \end{aligned}$$

A *topological dessin* (or shortly dessin) $W(\mathbf{H})$ defined by the hypermap $\mathbf{H}$ is the topological map associated with $w(\mathbf{H})$ together with a fixed bi-colouring of the vertices. We use convention that black vertices of $W(\mathbf{H})$ represent vertices of $\mathbf{H}$ and white vertices of $W(\mathbf{H})$ represent the edges of $\mathbf{H}$. By definition, dessins do not admit semi-edges. A homomorphism between dessins $W(\mathbf{H}_1) \to W(\mathbf{H}_2)$ is a colour-preserving covering of the maps. Dessins together with their homomorphisms form the category DES. It is well-known that $W : \mathbf{H} \mapsto W(\mathbf{H})$ is an invertible functor OHM$\to$DES.

Dessins are defined as topological objects. However, dessins and hypermaps can be viewed as geometric objects as well. Let $\mathbf{B} = W(\mathbf{I})$ be the trivial dessin (with two vertices, black and white, one edge and one face) embedded into the sphere $\mathcal{S}_0$. Then $\mathbf{B}$ can be viewed as a map embedded into the Riemann sphere Σ with the black vertex located at 0, the white vertex located at 1 and the unique edge being the unit interval $(0,1)$. The point ∞ is the centre of the face of $\mathbf{B}$. Let $\mathbf{H}$ be a hypermap of genus g. By Belyĭ's theorem it is possible to introduce a structure of Riemann surface $\mathcal{R}(\mathbf{H})$ on $\mathcal{S}_g$ such that the canonical covering $\omega_{\mathbf{H}} : W(\mathbf{H}) \to \mathbf{B}$ extends to a meromorphic function $\omega^*_{\mathbf{H}}$ with exactly three singular values 0, 1, and ∞. The function $\omega^*_{\mathbf{H}}$ is called the Belyĭ function associated with $\mathbf{H}$. Conversely, any Belyĭ function f determines a dessin $\mathbf{H} = f^{-1}(\mathbf{B})$ on a Riemann surface. It is well known that the Belyĭ function associated with a hypermap $\mathbf{H}$ is determined up to the action of the group of Möbius transformations of Σ.

Since map homomorphisms can be viewed as continuous mappings defined on the underlying surface $\mathcal{S}_g$ of genus g, the covering $\theta : \mathbf{H} \to \bar{\mathbf{H}}$ can be viewed as a *branched covering* θ of $\mathcal{S}_g$ to the quotient orbifold $\mathcal{O}$ [3, 16]. Orbifolds come from geometry of manifolds, where they are defined in a more general setting. In our context a quotient orbifold is induced by an action of a discrete group of automorphisms of a manifold. Concretely, an orientable 2-dimensional (regular) *quotient orbifold* is an orientable surface of genus $\gamma \geq 0$ together with a finite set of distinguished points $\mathcal{B} = \{b_1, b_2, \dots, b_r\}$ (called *branch points*), where each point $b_i \in \mathcal{B}$ is endowed with an integer $m_i \geq 2$, $i = 1, 2, \dots r$, called *branch index.* The quotient orbifold is determined by its *signature* $(\gamma; \{m_1, m_2, \dots, m_r\})$ with the branch indices ordered in a non-decreasing sequence $m_1 \leq m_2 \leq \cdots \leq m_r$.

A morphism between two quotient orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$, is a covering (possibly branched) $\psi : \mathcal{O}_1 \to \mathcal{O}_2$ between the underlying surfaces that takes branch points of $\mathcal{O}_1$ into branch points of $\mathcal{O}_2$ and satisfy the following property: if $b_i \in \mathcal{O}_1$ is a branch point of index k and $\psi(b_i)$ is a branch point in $\mathcal{O}_2$ of index k', then $k \mid k'$.

A *map on an orbifold* $\mathcal{O}(\gamma; \{m_1, m_2, \dots, m_r\})$ is a map $\mathbf{M}$ on S_γ such that neither a face nor an edge contains more than one branch point. The free end of a semi-edge is either non-singular, or a branch point of index two. A vertex may, or may not, be a singular point. It follows that a map on an orbifold gives rise to a function

$$b : V(\mathbf{M}) \cup E(\mathbf{M}) \cup F(\mathbf{M}) \to \{1, m_1, m_2, \dots, m_r\},$$

taking the value $m_i \geq 2$ for some $x \in V(\mathbf{M}) \cup E(\mathbf{M}) \cup F(\mathbf{M})$ if either x is a vertex of branch index m_i; or x is a face containing a branch point of index m_i; or x is a semi-edge and its free end is a branch point of index $m_i = 2$. In all other cases $b(x) = 1$. Vice-versa, a pair $(\mathbf{M}, b)$, where $\mathbf{M}$ is a map on S_γ and $b\colon V(\mathbf{M}) \cup E(\mathbf{M}) \cup F(\mathbf{M}) \to \{1, m_1, m_2, \dots, m_r\}$, is a function satisfying the above conditions determines a map on orbifold $\mathcal{O}$, see [12].

Every vertex-transitive map $\mathbf{M}$ of genus g covers regularly a quotient map $\bar{\mathbf{M}} = \mathbf{M}/\mathrm{G}$, where $\mathrm{G} \leq \mathrm{Aut}^+(\mathbf{M})$. The quotient map $\bar{\mathbf{M}}$ is of genus $\gamma \leq g$ and it has at most two vertices. The underlying surface $\mathcal{S}_\gamma$ is a two-dimensional quotient orbifold $\mathcal{O}(\gamma; \{m_1, m_2, \dots, m_r\}) = \mathcal{S}_g/\mathrm{G}$ and the parameters γ, g, $m_1, \dots m_r$ are related by the Riemann-Hurwitz equation:

$$(1) \qquad 2 - 2g = |\mathrm{G}|(2 - 2\gamma - \sum_{i=1}^{r}(1 - \frac{1}{m_i})),\ m_i \mid |\mathrm{G}|, m_i \geq 2,\ i = 1, 2, \dots, r.$$

The Hurwitz bound $|\mathrm{G}| \leq 84(g-1)$, $g > 1$, is derived for any finite group G of orientation-preserving automorphisms of a map of genus g. The orbifold fundamental group $\pi_1(\mathcal{O})$ is a Fuchsian group with presentation

$$\pi_1(\mathcal{O}) = \langle a_1, b_1, \dots, a_\gamma, b_\gamma, x_1, \dots, x_r |\ \Pi_{i=1}^{\gamma}[a_i, b_i] x_1 x_2 \dots x_r = 1,$$
$$x_1^{m_1} = x_2^{m_2} = \cdots = x_r^{m_r} = 1\rangle.$$

By standard arguments G is the quotient group of $\pi_1(\mathcal{O})$ by some normal torsion-free subgroup of finite index Γ. A morphism $\mathcal{O}_1 \to \mathcal{O}_2$ induces an embedding $\pi_1(\mathcal{O}_1) \hookrightarrow \pi_1(\mathcal{O}_2)$.

Map homomorphisms and automorphisms, as well as external symmetries, naturally extend to coverings between the underlying surfaces. We shall often use the same notation to denote map homomorphisms (or external symmetries) and their continuous counterparts. Moreover, let $\mathbf{M}$ and $\mathbf{N}$ have same underlying surface $\mathcal{S}_g$ and let $\varphi : \mathcal{S}_g \to \mathcal{S}_\gamma$ be a covering between surfaces. Then the same notation φ will be used to denote the three distinct objects: φ, $\varphi|_{\mathbf{M}}$ and $\varphi|_{\mathbf{N}}$, where $\varphi|_{\mathbf{M}}$ and $\varphi|_{\mathbf{N}}$ are restrictions of φ onto the sets of darts of $\mathbf{M}$ or $\mathbf{N}$, respectively.

By a 3-*pointed sphere* we mean a Riemann sphere with exactly three singular points at 0, 1, and ∞. A map on a 3-pointed surface is defined in the same way as a map on an orbifold.

4. Archimedean operations

Let $\mathbf{H}$ be a hypermap and $W(\mathbf{H})$ be its associated dessin, or a bi-coloured bipartite map, on a surface $\mathcal{S}_g$. Vice-versa, if $\mathbf{M}$ is a dessin, then $W^{-1}(\mathbf{M})$ denotes the associated hypermap. As above, denote by $\omega_{\mathbf{H}} : W(\mathbf{H}) \to \mathbf{B}$ a Belyĭ function onto the trivial dessin $\mathbf{B}$ on the Riemann sphere Σ. By definition, $\omega_{\mathbf{H}}$ takes $W(\mathbf{H})$ onto the 3-pointed sphere Σ^* with the singular points 0, 1, and ∞. In what follows, we use the convention that the fibres over 0, 1, and ∞ represent respectively hypervertices, hyperedges, and hyperfaces of $\mathbf{H}$. Let $\mathbf{N}$ be a dessin on Σ^*. Then the lift $\omega_{\mathbf{H}}^{-1}(\mathbf{N})$ determines a dessin $W(\mathbf{H}')$, associated with a hypermap $\mathbf{H}'$. The combinatorial structure of $W(\mathbf{H}')$ is uniquely determined and the Belyĭ function $\omega_{\mathbf{N}}$ satisfying $\omega_{\mathbf{N}}(\{0,1,\infty\}) \subseteq \{0,1,\infty\}$. Setting

$$\mathbf{H}' = T_{\mathbf{N}}(\mathbf{H}) = W^{-1}(\omega_{\mathbf{H}}^{-1}(\mathbf{N})) = W^{-1}(\omega_{\mathbf{H}}^{-1} \circ \omega_{\mathbf{N}}^{-1}(\mathbf{B})) \tag{2}$$

we get an *operation* $T_{\mathbf{N}} : \mathbf{H} \mapsto \mathbf{H}'$ from the set of hypermaps into the set of hypermaps. Clearly, $T_{\mathbf{N}}$ is universally defined on the category of oriented hypermaps and it depends just on the choice of $\mathbf{N}$. For technical reasons we consider taking mirror image of a hypermap, taking $\mathbf{H} \mapsto \mathbf{H}^{-1}$, to be an operation. Moreover, any of the twelve α^e-dualities is an operation $T_{\mathbf{N}}$ for some $\mathbf{N}$. The definition of an operation is done in the similar fashion as it was introduced by Singerman and Syddall in [15, Section 5].

The following lemma gives a list of basic properties of operations.

Lemma 2. *Let T, U be the operations determined by maps $\mathbf{N}_1$ and $\mathbf{N}_2$, respectively and let $\mathbf{H}$ be any hypermap. Then*

(a) $\mathbf{H}$ and $T(\mathbf{H})$ share the same Riemann surface $\mathcal{R}(\mathbf{H}) = \mathcal{R}(T(\mathbf{H}))$;
(b) $\mathrm{Aut}^+(\mathbf{H}) \leq \mathrm{Aut}^+(T(\mathbf{H}))$;
(c) the composition $(U \circ T)(\mathbf{H}) = U(T(\mathbf{H}))$ is an operation,
(d) the composition of operations is associative.

Proof. Let $T = T_{\mathbf{N}}$ for some $\mathbf{N}$ on Σ^*. By (2), the Belyĭ function of $\mathbf{H}' = T_{\mathbf{N}}(\mathbf{H})$ is $\omega_{\mathbf{N}} \circ \omega_{\mathbf{H}}$. The conformal structure of $\mathcal{R}(\mathbf{H}') = \mathcal{R}(T(\mathbf{H}))$ is then the pull-back of the conformal structure of the Riemann sphere Σ using $\omega_{\mathbf{N}} \circ \omega_{\mathbf{H}}$. Since $\omega_{\mathbf{N}}$ is a Belyĭ function mapping the Riemann sphere onto the Riemann sphere, and the Riemann sphere Σ has a unique Riemann structure, then $\mathcal{R}(\mathbf{H}')$ is conformally equivalent to Riemann surface $\mathcal{R}(\mathbf{H})$, which conformal structure is the pull-back of the conformal structure of the Riemann sphere using the Belyĭ function $\omega_{\mathbf{H}}$.

Let $\psi \in \mathrm{Aut}^+(\mathbf{H})$. Then $\omega_{\mathbf{H}}(x) = \omega_{\mathbf{H}}(\psi(x))$, for every edge $x \in W(\mathbf{H})$. The automorphism ψ extends to an orientation-preserving self-homeomorphism ψ_* of $\mathcal{R}(\mathbf{H})$ such that $\omega_{\mathbf{H}} = \omega_{\mathbf{H}}\psi_*$. Denote $\widetilde{\mathbf{N}}_1 = W(T(\mathbf{H})) = \omega_{\mathbf{H}}^{-1}(\mathbf{N}_1)$. Taking the restrictions $\omega = \omega_{\mathbf{H}}|_{\widetilde{\mathbf{N}}_1}$ and $\varphi = \psi_*|_{\widetilde{\mathbf{N}}_1}$ we see $\omega(x) = \omega(\varphi(x))$, for every dart $x \in \widetilde{\mathbf{N}}_1$. By definition, φ is a colour-preserving automorphism of the dessin $\widetilde{\mathbf{N}}_1$. Therefore, it induces an automorphism Φ of $T(\mathbf{H}) = W^{-1}(\widetilde{\mathbf{N}}_1)$. The mapping $\psi \mapsto \Phi$ defines an inclusion of $\mathrm{Aut}^+(\mathbf{H})$ into $\mathrm{Aut}^+(T(\mathbf{H}))$.

To see (c) observe that

$$(U \circ T)(\mathbf{H}) = W^{-1}(\omega_{\mathbf{H}}^{-1}(\omega_{\mathbf{N}_1}^{-1}(\mathbf{N}_2))) = W^{-1}(\omega_{\mathbf{H}}^{-1}(\mathbf{N})), \tag{3}$$

where $\mathbf{N} = \omega_{\mathbf{N}_1}^{-1}(\mathbf{N}_2) = \omega_{\mathbf{N}_1}^{-1}(\omega_{\mathbf{N}_2}^{-1}(\mathbf{B}))$. Observe that the composition $\omega_{\mathbf{H}} \circ \omega_{\mathbf{N}_1}$ is a well-defined Belyĭ function and that $\omega_{\mathbf{N}_2} \circ \omega_{\mathbf{N}_1} = \omega_{\mathbf{N}}$ satisfies $\omega_{\mathbf{N}}(\{0, 1, \infty\}) \subseteq \{0, 1, \infty\}$. Associativity of composition is a direct consequence of (3). □

Lemma 3. *Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be two hypermaps, $\vartheta : \mathbf{H}_1 \to \mathbf{H}_2$ be a covering, and T be an operation. Then $\vartheta \circ T(\mathbf{H}_1) = T(\mathbf{H}_2) = T \circ \vartheta(\mathbf{H}_1)$.*

Proof. Let $\omega_{\mathbf{H}_2}$ be a Belyĭ function associated with $\mathbf{H}_2$ and let $\mathbf{N}$ be a dessin on Σ^* determining the operation T. Since ϑ sends hypervertices, hyperedges, and hyperfaces of $\mathbf{H}_1$ to hypervertices, hyperedges, and hyperfaces of $\mathbf{H}_2$, then $\omega_{\mathbf{H}_1} = \omega_{\mathbf{H}_2} \circ \vartheta$ is a Belyĭ function associated with $\mathbf{H}_1$. Then, by definition $T(\mathbf{H}_2) = W^{-1}(\omega_{\mathbf{H}_2}^{-1}(\mathbf{N}))$ and

$$\begin{aligned} T(\mathbf{H}_1) = W^{-1}(\omega_{\mathbf{H}_1}^{-1}(\mathbf{N})) &= W^{-1}(\vartheta^{-1} \circ \omega_{\mathbf{H}_2}^{-1}(\mathbf{N})) \\ &= \vartheta^{-1}(W^{-1} \circ \omega_{\mathbf{H}_2}^{-1}(\mathbf{N})) \\ &= \vartheta^{-1}(T(\mathbf{H}_2)). \end{aligned}$$

□

Let us remark that by means of Lemma 2 and Lemma 3, any operation can be seen as a functor of the category OHM.

A group G will be called *triangular* if G is a quotient of a triangle group by a torsion-free normal subgroup of finite index. It follows, that if G acts on a surface $\mathcal{S}_g$ as a group of self-homeomorphisms of $\mathcal{S}_g$, then the quotient orbifold $\mathcal{S}_g/\mathrm{G}$ has signature $(0; \{k, m, n\})$.

Definition. *We say that a non-degenerate map on an orientable surface is of* Archimedean class*, if its full group of automorphisms acts transitively on vertices and the group of orientation-preserving automorphisms contains a triangular group.*

Singerman investigated inclusions between triangle groups. Among others, he has proved the following proposition.

Proposition 4 ([7], [13, Proposition 1])**.** *The only Fuchsian groups which contain triangle groups are triangle groups.*

Proposition 5. *The group of orientation-preserving automorphisms of a map of Archimedean class is itself triangular.*

Proof. Let $\mathbf{M}$ be a map of Archimedean class of genus g. Let $\mathrm{G} = \mathrm{Aut}^+(\mathbf{M})$ and let H be an automorphism group of a regular hypermap of genus g such that $\mathrm{H} = \Delta(k, m, n)/\Gamma \leq \mathrm{G}$, where Γ is a torsion-free normal subgroup of finite index. Consider the quotient orbifolds $\mathcal{S}_g/\mathrm{H}$ and $\mathcal{S}_g/\mathrm{G}$. Since $\mathrm{H} \leq \mathrm{G}$ we have a covering $\mathcal{S}_g/\mathrm{H} \to \mathcal{S}_g/\mathrm{G}$. First, the underlying topological space for both orbifolds is the sphere $\mathcal{S}_0$. Further, let $\mathrm{fib}_{\mathrm{H}}(x)$ be an orbit over a singular point x in the action of H. Since H is a group of automorphisms of a map it preserves vertices, edges and centres of faces. Since $\mathrm{H} \leq \mathrm{G}$, then $\mathrm{fib}_{\mathrm{H}}(x)$ is contained in $\mathrm{fib}_{\mathrm{G}}(x)$ for any branch point $x \in \mathcal{S}_g/\mathrm{H}$. Assume $a \in H$ fixes point-wise a fibre $\mathrm{fib}_{\mathrm{H}}(x)$ over a branch point x of index k. Then $a \in \langle a' \rangle$ where $a' \in \mathrm{G}$ fixes point-wise $\mathrm{fib}_{\mathrm{G}}(x)$ and induces a branch index k' for some k', where $k|k'$. It follows that we have an embedding of the orbifold fundamental groups $\pi_1(\mathcal{S}_g/\mathrm{H}) \hookrightarrow \pi_1(\mathcal{S}_g/\mathrm{G})$. However $\pi_1(\mathcal{S}_g/\mathrm{H}) \cong \Delta(k,m,n)$ and by Köbe Theorem both groups $\Delta(k,m,n) < \mathrm{F} = \pi_1(\mathcal{S}_g/\mathrm{G})$ act on the universal hyperbolic plane, thus F is a Fuchsian group as well. Finally, by Proposition 4, $\mathrm{F} \cong \Delta(k',m',n')$ for some k',m',n'. It follows that G is an epimorphic image of a triangle group $\Delta(k',m',n')$ by an epimorphism with a torsion-free kernel. □

We say that a map of Archimedean class is *of type* $\{k,m,n\}$ if $\mathrm{Aut}^+(\mathbf{M})$ is a torsion-free quotient of $\Delta(k,m,n)$.

As it was already noted, $\mathrm{Aut}^+(\mathbf{H})$ of a regular hypermap is a quotient of a triangle group $\Delta(k,m,n)$ by a normal subgroup, for some $k,m,n > 1$. Vice-versa, any finite quotient G of a triangle group $\Delta(k,m,n)$ determines a regular hypermap defined as an algebraic hypermap $\mathbf{H} = (\mathrm{G}; x, y)$. In particular, if $\mathbf{H}$ is a regular hypermap, then $\omega_{\mathbf{H}}$ is a regular cover defined by the action of $\mathrm{Aut}^+(\mathbf{H})$ and $\omega_{\mathbf{H}} : \mathcal{S}_g \to \mathcal{S}_g/\mathrm{Aut}^+(\mathbf{H}) = \mathcal{O}(0;\{k,m,n\})$.

We call an operation given by $\mathbf{N}$ to be a *Platonic operation*, if there exist regular hypermaps $\mathbf{H}$ and $\mathbf{M}$, such that $T_{\mathbf{N}}(\mathbf{H}) = \mathbf{M}$. An operation given by $\mathbf{N}$ is an *Archimedean operation*, if there exist a map $\mathbf{M}$ of Archimedean class and a regular hypermap $\mathbf{H}$, such that $T_{\mathbf{N}}(\mathbf{H}) = \mathbf{M}$. Note that the mirror image of a hypermap and the dualities of all kinds are Platonic operations. Platonic operation on regular maps were investigated by Singerman and Syddall [15], while Girondo investigated operations between uniform dessins in [8]. Note that all Girondo's operations (called surgeries in [8]) give rise to Platonic operations. Observe that many Archimedean operations can be obtained as compositions $T(P(\mathbf{H}))$, where P is a Platonic operation and T is an Archimedean operation.

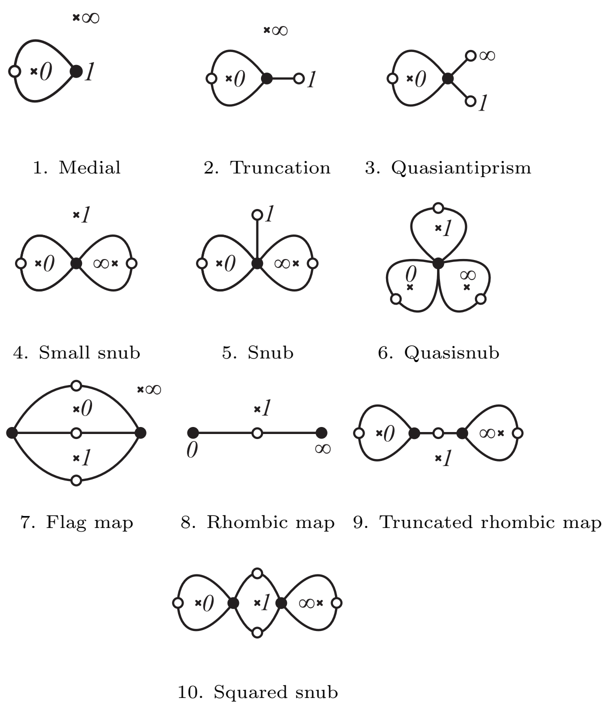


FIGURE 1. Dessins defining Archimedean operations

Figure 1 shows 10 dessins on the 3-pointed sphere defining 10 operations on hypermaps, which will be proved to be Archimedean. Most of operations described by dessins in Figure 1 are well-known, and were intensively employed in distinguished contexts. In particular, all but T_2, T_3, and T_5 give always maps, the operations T_2 (truncation), T_3 and T_5 (snub) produce maps if they are applied to oriented maps. The operations T_1,...,T_5, T_7, and T_8 represent the classical operations transforming Platonic solids onto Archimedean ones.

Till the end of this section, let $\mathrm{G} = \langle x, y \mid x^k = y^n = (x^{-1}y)^m, \ldots \rangle$ be a finite quotient of a triangle group, and $\mathbf{H}$ be a regular hypermap of genus g with $\mathrm{Aut}^+(\mathbf{H}) = \mathrm{G}$. Denote by $\mathbf{N}_j$, $j \in \{1, 2, \ldots, 10\}$ the dessins depicted on Figures 1.1-1.10, respectively. Furthermore, denote $\mathbf{H}_j = T_j(\mathbf{H}) = T_{\mathbf{N}_j}(\mathbf{H})$.

Lemma 6. *The group* $\mathrm{G} = \mathrm{Aut}^+(\mathbf{H})$ *acts transitively on the vertices of* $\mathbf{H}_j$ *for* $j = 1, \ldots, 6$*, and it acts with two orbits on the vertices of* $\mathbf{H}_j$ *for* $j = 7, \ldots, 10$*. In particular, hypermaps* $\mathbf{H}_j$ *are vertex-transitive, for* $j = 1, \ldots, 6$*.*

Proof. Since $\mathbf{H}$ is regular hypermap, the covering $\omega_{\mathbf{H}} : \mathcal{S}_g \to \mathcal{S}_g/\mathrm{G}$ is regular. Moreover, $\mathbf{N}_j = W(\mathbf{H}_j)/\mathrm{G}$, $j \in \{1, 2, \ldots, 6\}$, is a dessin with exactly one black vertex. It follows that G acts transitively on the black vertices of the dessin $W(\mathbf{H}_j)$. Consequently, G is transitive on the vertices of $\mathbf{H}_j$.

Similarly, the maps $\mathbf{H}_7$,...,$\mathbf{H}_{10}$ are lifts of dessins with two black vertices along the regular cover. It follows that G acts on $\mathbf{H}_j$ as a group of orientation-preserving automorphisms with two orbits on vertices of $\mathbf{H}_j$, for $j = 7, \ldots, 10$, respectively. □

Recall that by Lemma 2(b) we have $\mathrm{G} \leq \mathrm{Aut}^+(\mathbf{H}_j)$. Note that by Proposition 4, $\mathrm{Aut}^+(\mathbf{H}_j)$ is triangular. Hence, for the purpose of description of maps of Archimedean class we assume in the following statements that the groups $\mathrm{G} = \mathrm{Aut}^+(\mathbf{H})$ and $\mathrm{Aut}^+(\mathbf{H}_j)$ coincide.

Lemma 7 (Lifting condition)**.** *Let* $\mathbf{H}_j$*,* $j \in \{7, \ldots, 10\}$*, be a vertex-transitive hypermap and* $[\mathrm{Aut}(\mathbf{H}_j) : \mathrm{G}] = 2$*. Then there is a reflection* ϱ *transposing the two orbits of* G *on vertices. Moreover,* ϱ *projects onto a reflection of* $\mathbf{N}_j$*, transposing its two black vertices.*

Proof. Since $\mathrm{G} = \mathrm{Aut}^+(\mathbf{H}_j)$ and $\mathbf{H}_j = (D; R, L)$ is vertex-transitive, there is a reflection transposing the two orbits of G onto the vertices of $\mathbf{H}_j$. Since $\mathrm{G} \trianglelefteq \mathrm{Aut}(\mathbf{H}_j)$,

$$[g\varrho x]_{\mathrm{G}} = [\varrho g' x]_{\mathrm{G}} = [\varrho x]_{\mathrm{G}},$$

for every $g \in \mathrm{G}$ and every dart $x \in D$. It follows that $\bar{\varrho} : [x]_{\mathrm{G}} \mapsto [\varrho x]_{\mathrm{G}}$ is a well-defined reflection of the quotient map $\mathbf{H}_j/\mathrm{G}$. The reflection $\bar{\varrho}$ induces a reflection of $\mathbf{N}_j = W(\mathbf{H}_j/\mathrm{G})$, transposing its black vertices. □

In the following Lemmas we investigate reflections ϱ transposing the two orbits of G on vertices in detail.

Lemma 8. *Let* $\mathrm{G} = \mathrm{Aut}^+(\mathbf{H}_7)$*. Then* $\mathbf{H}_7 = \mathrm{T}_7(\mathbf{H})$ *is vertex-transitive, if and only if* $\mathbf{H}$ *is a reflexible regular hypermap.*

Proof. If $\mathbf{H}$ is a reflexible regular hypermap, then $\mathrm{Aut}(\mathbf{H})$ acts regularly on the vertices of $\mathbf{H}_7$, corresponding to the flags of $\mathbf{H}$ (a flag is a triple of the form vertex-edge-face, where all the three object are mutually incident).

Assume $\mathbf{H}_7$ is vertex-transitive. Then there exists an orientation-reversing automorphism ϱ swapping the two vertex orbits of the action of G. By Lemma 7, ϱ projects to an orientation-reversing automorphism of $\mathbf{N}_7$, transposing the two vertices. The map $\mathbf{N}_7$ admits exactly three such automorphisms. One of them generates a cyclic group $\langle\beta\rangle$ of order 6, the other two are β^{-1} and β^3. Moreover, β^3 fixes each edge and each face of $\mathbf{N}_7$. Therefore, the lift of β^3 acts as a reflection of $\mathbf{H}$. It follows that either ϱ or ϱ^3 is an orientation-reversing automorphism of $\mathbf{H}$. □

Lemma 9. *Rhombic map* $\mathbf{H}_8 = \mathrm{T}_8(\mathbf{H})$ *is bipartite. Let there exist an automorphism* ϱ *of* $\mathbf{H}_8$ *transposing the two orbits of* G *on vertices.*

Then either ϱ *is orientation-reversing and* $\mathbf{H}$ *is* $(0,\infty)^-$*-self-dual, or* ϱ *is orientation-preserving and* $\mathbf{H}$ *is* $(0,\infty)^+$*-self-dual, respectively.*

Proof. The group $\mathrm{G} = \mathrm{Aut}^+(\mathbf{H})$ acts on vertices of $\mathbf{H}_8$ preserving the two orbits on vertices. Since G is transitive on the edges of $\mathbf{H}_8$ we may assume that ϱ fixes an edge $e \in \mathbf{H}_8$. Either, ϱ is a reflection, or it is 180-degree rotation around the centre of e. In the first case ϱ is $(0,\infty)^-$-duality of $\mathbf{H}$. In the second case it is $(0,\infty)^+$-duality of $\mathbf{H}$. □

Lemma 10. *Let* $\mathbf{H}_9 = \mathrm{T}_9(\mathbf{H})$ *be the truncated rhombic map. Let there exist an automorphism* ϱ *of* $\mathbf{H}_9$ *transposing the two orbits of* G *on vertices.*

Then either ϱ *is orientation-reversing and* $\mathbf{H}$ *is* $(0,\infty)^-$*-self-dual, or* ϱ *is orientation-preserving and* $\mathbf{H}$ *is* $(0,\infty)^+$*-self-dual, respectively.*

Proof. Using the fact that $\mathrm{Aut}^+(\mathbf{H})$ acts transitively on the edges of $\mathbf{H}_9$ joining the two orbits, the proof can be done in a similar manner as the proof of Lemma 9 (see Figure 2).

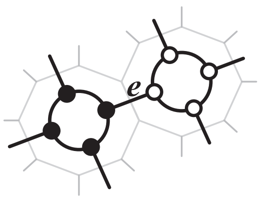


FIGURE 2. Proof of Lemma 10; original hypermap as a flag map is inscribed.

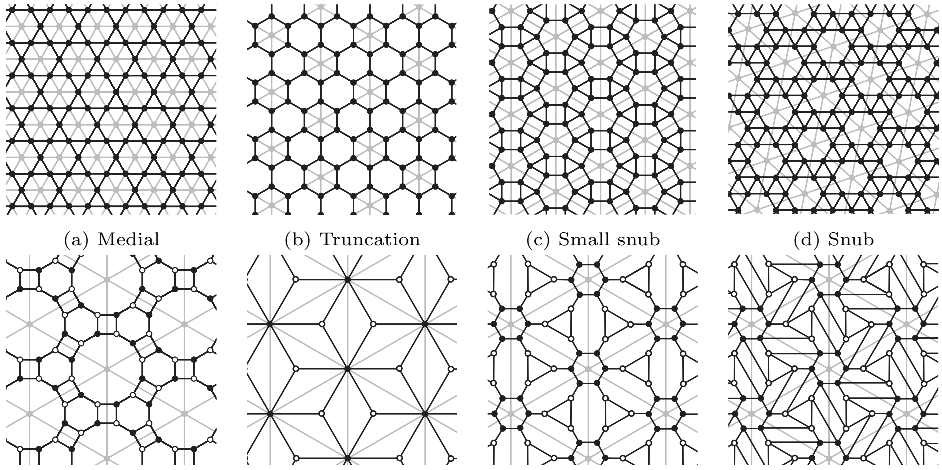
□

Lemma 11. *Let* $\mathbf{H}_{10} = \mathrm{T}_{10}(\mathbf{H})$ *be the squared snub map of* $\mathbf{H}$ *and let* $\mathrm{G} = \mathrm{Aut}^+(\mathbf{H}_{10})$.

The squared snub $\mathbf{H}_{10}$ *is vertex-transitive if and only if* $\mathbf{H}$ *is* $(0,\infty)^-$*-self-dual.*

Proof. Assume that $\mathbf{H}_{10}$ is vertex-transitive. Then there exist an orientation-reversing automorphism ϱ swapping the two orbits of vertices of G. By definition, $\mathbf{N}_{10}$ is embedded into the orbifold $\mathcal{O}(0;\{k,m,n\})$. It follows from Lemma 7 that ϱ projects and $k = m$. Inspecting the map $\mathbf{N}_{10}$ we conclude that there exist exactly one automorphism $\bar{\varrho}$ of $\mathbf{N}_{10}$ satisfying all the assumptions. It follows that $\bar{\varrho}$ fixes each of the two fibres over the two edges joining black and white vertex in $\mathbf{N}_{10}$. Also $\bar{\varrho}$ transposes the two singular points which are images of hypervertices and hyperfaces of $\mathbf{H}$. It follows that ϱ acts as a $(0,\infty)^-$-self-duality of $\mathbf{H}$. □

Corollary 12. *Let* $\mathrm{G} = \mathrm{Aut}^+(\mathbf{H}_j)$. *Then* $\mathbf{H}_j = \mathrm{T}_j(\mathbf{H})$, $j \in \{8, 9, 10\}$ *is vertex-transitive, if and only if* $\mathbf{H}$ *is a* $(0,\infty)^-$*-self-dual regular hypermap.*



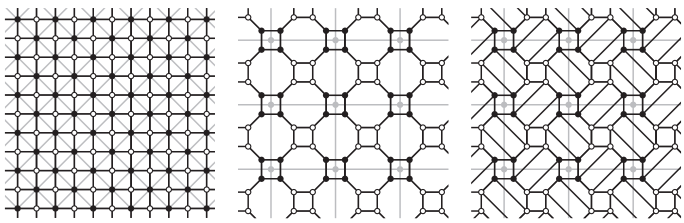
(a) Medial (b) Truncation (c) Small snub (d) Snub

(e) Flag map (f) Rhombic map (g) Trunc. rhombic map (h) Squared snub

FIGURE 3. Archimedean operations applied on Euclidean tiling $\{3, 6\}$

Example 13. Figure 3 shows the action of all the operations T_j, except T_3 and T_6, on the Euclidean tiling $\{3, 6\}$. The operation T_3 gives a map only for branch assignment $\{0, 1, \infty\} \to \{2, 2, n\}$. It follows that $T_3(\{3, 6\})$ is a pure hypermap. As concerns $T_6(\{3, 6\})$ it can be easily seen that the resulting map is degenerate.

The Euclidean tiling $\{3, 6\}$ is a 6-valent tessellation of Euclidean plane by equilateral triangles. It gives an example of the universal regular map with the automorphism group $\mathrm{G} = \Delta^+(2, 3, 6) = \langle x, y, z \mid x^2 = y^3 = z^6 = 1,\ xyz = 1\rangle$. The extended triangle group $\Delta(2, 3, 6)$ acts as a (full) group of automorphisms of $\{3, 6\}$, therefore $\{3, 6\}$ is a reflexible regular map. For $j = 7, 8, 9, 10$, the two orbits of G on vertices of $T_j(\{3, 6\})$ are distinguished by black and white colours, respectively (Figure 3(e)-(h)). By Lemma 6 and Lemma 8 the maps $T_j(\{3, 6\})$, $j \in \{1, 2, 4, 5, 7\}$, are vertex-transitive. Since $\{3, 6\}$ is not self-dual, by Corollary 12, the operations T_8, T_9, and T_{10} do not give vertex-transitive maps (see Figure 3). On the other hand, Corollary 12 implies that using T_8, T_9, and T_{10} on the tiling $\{4, 4\}$ we get the vertex-transitive maps depicted on Figure 4.



(a) Rhombic map (b) Trunc. rhombic map (c) Squared snub

FIGURE 4. Operations T_6, T_7, T_8 applied on Euclidean tiling $\{4, 4\}$

5. Completeness theorem

Let $\mathbf{M}$ be a map on a quotient orbifold and let x be either a vertex or a face. The *virtual valency* of x is the product $f(x).\mathrm{val}(x)$, where $\mathrm{val}(x)$ denotes the valency of x and $f(x)$ is the branch index of x.

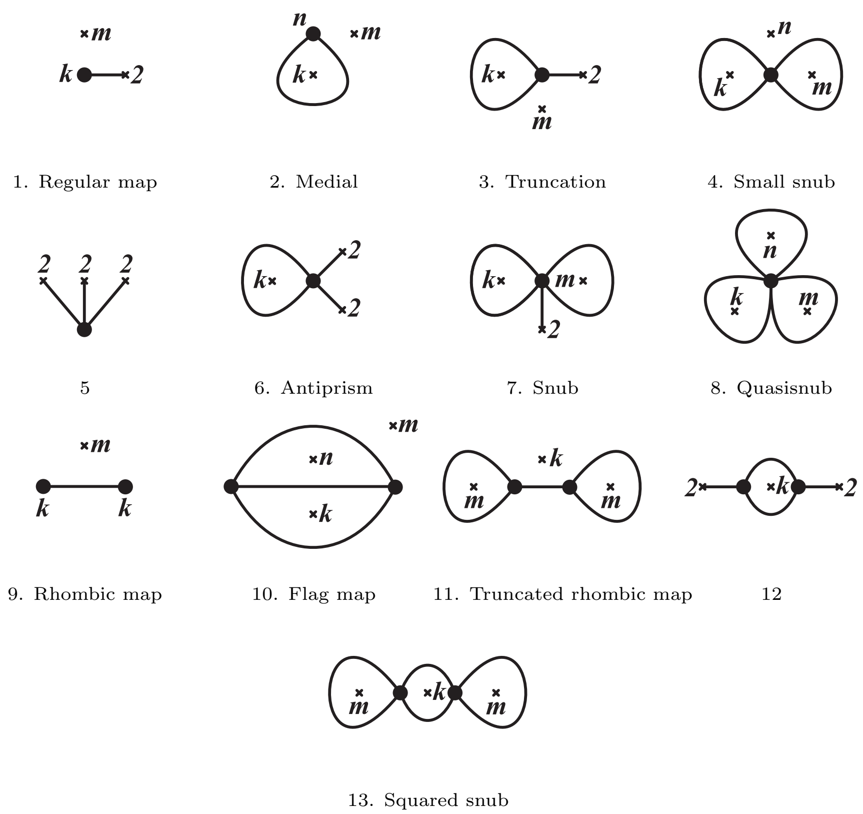


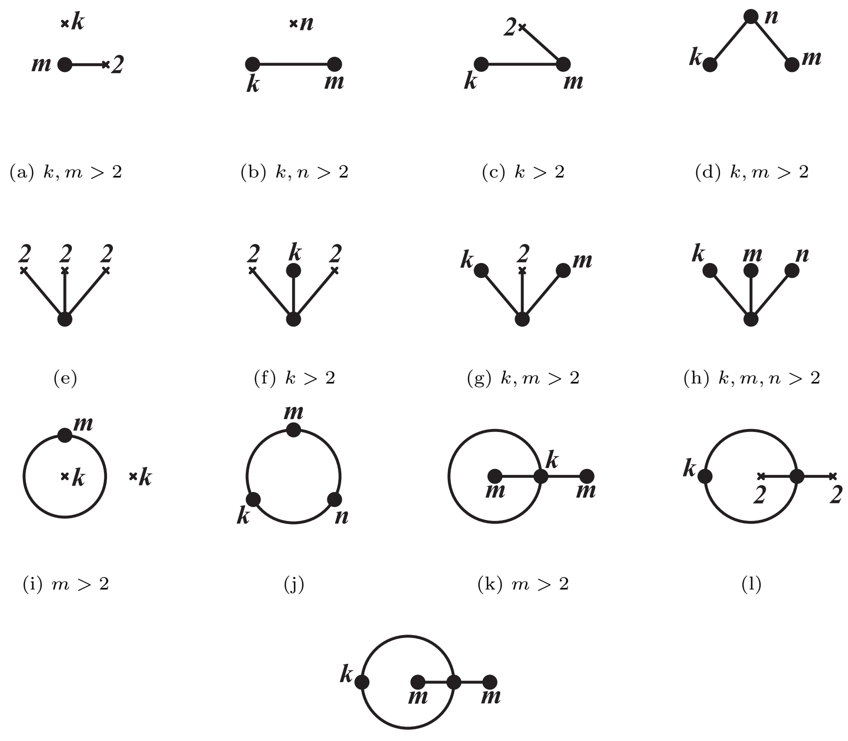
Figure 5. Maps satisfying assumptions of Proposition 14

Proposition 14. *Let $\mathbf{N}$ be a map on orbifold $\mathcal{O}(0;\{k,m,n\})$ satisfying the following conditions:*

(a) $\mathbf{N}$ *has at most* 2 *vertices;*
(b) the virtual valency of each vertex of $\mathbf{N}$ *is at least* 3*;*
(c) the virtual valency of each face of $\mathbf{N}$ *is at least* 3*;*
(d) if $\mathbf{N}$ *is a two-vertex map, then there exist a reflection transposing the two vertices;*
(e) free end of a semi-edge is a branch point of index 2.

Then $\mathbf{N}$ *is one of the maps depicted on Figure 5, up to permutation of branch indexes k, m, and n.*

Proof. Case I: $\mathbf{N}$ is a one-vertex map. Then its dual is a tree $\mathbf{T}=\mathbf{N}^*$, possibly with semi-edges. By conditions (b), (c) and (e), a pendant vertex of $\mathbf{T}$ is a branch point of index >2, a vertex of degree 2 in $\mathbf{T}$ is a branch point, and a free end of a semi-edge is a branch point of index 2. Since we have only 3 branch points to disposal, we end either with the trivial map, or with one of the eight tree-like maps, see Figures 6a-h.



(a) $k, m > 2$ (b) $k, n > 2$ (c) $k > 2$ (d) $k, m > 2$

(e) (f) $k > 2$ (g) $k, m > 2$ (h) $k, m, n > 2$

(i) $m > 2$ (j) (k) $m > 2$ (l)

(m) $m > 2$

FIGURE 6. Proof of Proposition 14

<u>CASE II</u>: $\mathbf{N}$ is two-vertex map. Then its dual is a spherical uni-cyclic map $\mathbf{U} = \mathbf{N}^*$ admitting an orientation reversing automorphism swapping the two faces (see condition (d)). Similarly as above, the pendant vertices are branch points of index > 2, vertices of degree two are branch points, and free ends of semi-edges are branch point of index 2. By conditions (b) and (c), a face of length 1 contains a branch point of index at least 3 and a face of length 2 contains a branch point. Since we have only three branch points to disposal, the number of pendant vertices in $\mathbf{U}$ is at most 3. Thus $0 \leq i \leq 3$, where i denotes the number of pendant vertices and semi-edges. By condition (d) the pendant vertices and semi-edges are equally distributed between the two faces separated by the unique cycle of $\mathbf{U}$. Thus, the integer $i \in \{0, 2\}$. If $i = 0$, then the map $\mathbf{U}$ is a cycle with at most 3 vertices. If $i = 2$, then $\mathbf{U}$ is a cycle with exactly two pendant vertices attached. We continue by a case to case analysis according to the pairs (i, j), where $i \in \{0, 2\}$ and $1 \leq j \leq 3$, where j is the number of vertices on the unique cycle of $\mathbf{U}$. Most of the 2-vertex duals $\mathbf{N} = \mathbf{U}^*$ do not satisfy the condition (d) and they were excluded. Finally, we end with the five maps depicted in Figures 6i-6m. $\square$

Remark 15. Recall that every map can be transformed to a dessin by applying the Walsh operator W. Observe that each map from Figure 5, except the map $\mathbf{I}$ on Figure 5a, has the respective dessin depicted on Figure 1. Observe that $W(\mathbf{I}) = \mathbf{B}$ is the trivial dessin. We use the same name for a map from Figure 5 and the respective dessin from Figure 1.

Now we are ready to prove the main result of the paper.

Theorem 16 (Completeness Theorem)**.** *Let* $\mathbf{M}$ *be a map of Archimedean class of type* $\{k, m, n\}$ *and of genus* g*. Then there exist a regular hypermap* $\mathbf{H}$ *of genus* g *with* $\mathrm{Aut}^+(\mathbf{H}) = \mathrm{Aut}^+(\mathbf{M})$ *such that one of the following cases happen:*

(1) $\mathbf{M} = \mathbf{H}$ *is a regular map;*
(2) $\mathbf{M} \cong \mathrm{T}_j(\mathbf{H})$ *for some* $j \in \{1, 4, 6\}$*, i.e.* $\mathbf{M}$ *is medial, small snub, or quasisnub of a regular hypermap* $\mathbf{H}$*, respectively;*
(3) $n = 2$ *and* $\mathbf{M} \cong \mathrm{T}_j(\mathbf{H})$ *for some* $j \in \{2, 5\}$*, i.e.* $\mathbf{M}$ *is truncation, or snub of a regular hypermap* $\mathbf{H}$*, respectively;*
(4) $k > 3$*,* $m = n = 2$*,* $\mathbf{H}$ *is a* k*-cycle in the sphere and* $\mathbf{M} \cong \mathrm{T}_3(\mathbf{H})$ *is a* k*-antiprism;*
(5) $\mathbf{H}$ *is reflexible and* $\mathbf{M} \cong \mathrm{T}_7(\mathbf{H})$ *is the flag map of a reflexible regular hypermap;*
(6) $k = m$*,* $\mathbf{H}$ *is* $(0, \infty)^-$*-self-dual and* $\mathbf{M} \cong \mathrm{T}_j(\mathbf{H})$*, for some* $j \in \{8, 9, 10\}$*, i.e.* $\mathbf{M}$ *is the rhombic, the truncated rhombic, or the squared snub map of* $\mathbf{H}$.

Proof. Denote $\mathrm{G} = \mathrm{Aut}^+(\mathbf{M})$. By Proposition 5, G is of orbifold type $(0; \{k, m, n\})$. Therefore, the essential assumption of Proposition 14 is fulfilled. We verify further assumptions of Proposition 14 for the quotient map $\mathbf{N} = \mathbf{M}/\mathrm{G}$. Because $\mathbf{M}$ is vertex-transitive, G acts on vertices of $\mathbf{M}$ with at most two orbits. Since $\mathbf{M}$ is non-degenerate, the virtual valency of a vertex, or of a face of $\mathbf{N}$ is at least 3. Hence, the conditions (a), (b), and (c) are satisfied. By Lemma 7, the condition (d) is satisfied, as well.

By Proposition 14, either $\mathbf{M} = \mathbf{H}$ is a regular map or $\mathbf{N} = \mathbf{M}/\mathrm{G}$ is one of the maps depicted on Figures 5:2-13. We first show that we can exclude the quotients $\mathbf{N}_1$ and $\mathbf{N}_2$, depicted on Figure 5:5 and Figure 5:12, respectively. Assume $\mathbf{N} = \mathbf{N}_1$. Then $\mathrm{G} = \Delta(2, 2, 2) \cong \mathrm{C}_2 \times \mathrm{C}_2$. Then $\mathbf{M}$ is the tetrahedral map with $|\mathrm{Aut}^+(\mathbf{M})| = |A_4| > |\mathrm{C}_2 \times \mathrm{C}_2|$, a contradiction. Assume $\mathbf{N} = \mathbf{N}_2$. Then $\mathrm{G} = \Delta(2, 2, k) \cong \mathrm{D}_{2k}$. Then $\mathbf{N}$ is the $2k$-prism with $|\mathrm{Aut}^+(\mathbf{M})| = |D_{4k}| > |\mathrm{D}_{2k}|$, a contradiction.

The remaining quotients give rise to the cases (2)-(6) of the statement. More precisely, if $\mathbf{M}$ is not regular, then $\mathbf{N} = T_j(\mathbf{B})$ for some $j \in \{1, 2, \ldots, 10\}$. The lift of $\mathbf{B}$ along the regular covering defined by the action of G is a regular hypermap $\mathbf{H}$ with $\mathrm{Aut}^+(\mathbf{H}) = \mathrm{G}$. By Lemma 3, the relation $\mathbf{N} = T_j(\mathbf{B})$ lifts to the relation $\mathbf{M} = T_j(\mathbf{H})$. In what follows, we discuss the cases according to $j \in \{1, 2, \ldots, 10\}$ in detail.

Dessins associated with the quotients depicted on Figures 5:2, 5:4, and 5:8 coincide with those determining the operations T_1 – medial, T_4 – small snub, and T_6 – quasisnub. Hence we end in case (2) of the statement.

Dessins associated with the quotients depicted on Figure 5:3 and Figure 5:7 define the operations T_2 – truncation and T_5 – snub. By Proposition 14, $n = 2$ and we end in case (3).

The dessin associated with the quotients depicted on Figure 5:6 defines the operation T_3 – quasiantiprism. By Proposition 14, $n = m = 2$, $\mathrm{G} = \Delta(2, 2, k) \cong \mathrm{D}_{2k}$ and $\mathbf{M}$ is a k-antiprism, for $k \geq 3$. If $k = 3$, $\mathbf{M}$ is the octahedral map with $|\mathrm{Aut}^+(\mathbf{M})| = |\mathrm{S}_4| > |\mathrm{D}_6|$, a contradiction. Hence, $k > 3$ and we end in case (4).

The dessin associated with the quotient depicted on Figure 5:10 defines the operation T_7 – flag map. By Lemma 8, $\mathbf{H}$ is reflexible and we end in case (5).

Dessins associated with the quotients depicted on Figures 5:9-13 define the operations T_8 – rhombic map, T_9 – truncated rhombic map and T_{10} – squared snub. By Corollary 12, $\mathbf{H}$ is $(0, \infty)^-$-self-dual and we end in case (6). □

Let $\mathbf{H}$ be a regular hypermap, or a map of Archimedean class and let $\mathbf{M} = T_j(\mathbf{H})$, $j \in \{1, 2, \ldots, 10\}$. Then the resulting hypermap $\mathbf{M}$ may not be a map of

Archimedean class for different reasons: $\mathbf{M}$ may be a pure hypermap, $\mathbf{M}$ may be a map, but not a vertex-transitive map, or $\mathbf{M}$ may be a map, but a degenerate map. If the surface is of negative Euler characteristic, then the number of vertices of a vertex-transitive map is bounded. Thus, a repeated composition of the above operations on regular (hyper)maps cannot preserve vertex-transitivity, simply because their application rise the number of vertices (except taking the mirror image). The situation changes for the torus and the sphere. For instance, we may apply the medial operation onto a Coxeter regular map of type $\{4,4\}$ arbitrarily many times and the resulting map is always a Coxeter regular map of type $\{4,4\}$.

Corollary 17. *Let $\mathbf{M}_1 = T_i(\mathbf{H}_1)$ and $\mathbf{M}_2 = T_j(\mathbf{H}_2)$, $i,j \in \{1,\ldots,10\}$, be two maps of Archimedean class, where $\mathbf{H}_1$ and $\mathbf{H}_2$ are regular hypermaps. Then $\mathcal{R}(\mathbf{M}_1) = \mathcal{R}(\mathbf{M}_2)$ if and only if $\mathcal{R}(\mathbf{H}_1) = \mathcal{R}(\mathbf{H}_2)$.*

Proof. By Lemma 2(a), operations preserve the underlying Riemann surfaces of hypermaps. In particular, if $\mathcal{R}(\mathbf{H}_1) = \mathcal{R}(\mathbf{H}_2)$ for two regular hypermaps, then all the maps of Archimedean class, derived from $\mathbf{H}_1$ and $\mathbf{H}_2$, share the same underlying Riemann surface. Vice-versa, the structure of the underlying Riemann surface for $\mathbf{M}_1$ and $\mathbf{M}_2$ is determined by the respective regular hypermaps. Therefore $\mathcal{R}(\mathbf{M}_1) = \mathcal{R}(\mathbf{M}_2)$ implies $\mathcal{R}(\mathbf{H}_1) = \mathcal{R}(\mathbf{H}_2)$. □

It follows that the problem to decide whether two maps of Archimedean class share the same Riemann surface transfers to regular hypermaps. The latter problem can be solved using the results by Girondo [8] and Singerman and Syddall [15].

Note that the families of maps of Archimedean class obtained by distinct operations are not disjoint. For instance, if $\mathbf{H}$ is a regular hypermap admitting a $(0,1,\infty)^-$-self-duality, then $\mathrm{T}_7(\mathbf{H})$ is a cubic regular map, which belongs to Case (1) of Theorem 16. The smallest example of genus $g > 1$ is the regular hypermap $\{4,4,4\}$ of genus 2 with 8 darts (RPH2.4 in [2]).

Example 18. It is well-known that regular maps on the sphere are m-cycles, m-dipoles and the 2-skeletons of the five Platonic solids. In particular, we have

(1) the infinite family of *m-dipoles*, $m \geq 1$ of local type (2^m);
(2) the infinite family of *m-cycles*, $m \geq 1$ of local type (m^2);
(3) *tetrahedron* of local type (3^3);
(4) *cube* of local type (4^3);
(5) *octahedron* of local type (3^4);
(6) *dodecahedron* of local type (5^3);
(7) *icosahedron* of local type (3^5).

The 2-skeletons of Archimedean solids are by definition polyhedral vertex-transitive maps. By Riemann-Hurwitz equation the following orbifold types of actions of discrete groups on the sphere are admissible: $\mathcal{O}(0;\{n,n\})$, $n \geq 2$, $\mathcal{O}(0;\{2,2,n\})$, $n \geq 2$, $\mathcal{O}(0,\{2,3,3\})$, $\mathcal{O}(0,\{2,3,4\})$ and $\mathcal{O}(0,\{2,3,5\})$. Except type $\mathcal{O}(0;\{n,n\})$ we may apply operations from Theorem 16 to construct all classical Archimedean solids (maps on sphere). The results are shown in Table 2

Columns of Table 2 represent regular maps on the sphere indicated by the local type, while the rows correspond to the operations. The (i,j)-entry gives an information of the resulting map $\mathrm{T}_i(\mathbf{M}_j)$, where $\mathbf{M}_j$ is the regular map represented by j-th column. All the maps are determined by their local types. Since the map $\mathrm{T}_i(\mathbf{M}_j)$ may not be polyhedral, we mark this fact by '$*$' in the table. If an operation T_i applied on the map $\mathbf{M}_j$ does not give a map or the resulting map is not vertex-transitive (see Theorem 16), we mark this fact by the symbol '$\times$'. In Table 2 we also exclude the column representing the infinite family of cycles, since

	(2^k), $k>2$	(3^3)	(4^3)	(3^4)	(5^3)	(3^5)
T_1	$*$	(3^4)	(3.4.3.4)	(3.4.3.4)	(3.5.3.5)	(3.5.3.5)
T_2	$(4.4.k)$	(3.6.6)	(3.8.8)	(4.6.6)	(3.10.10)	(5.6.6)
T_3	$(3.3.3.k)$	$\times$	$\times$	$\times$	$\times$	$\times$
T_4	$*$	(3.4.3.4)	(3.4^3)	(3.4^3)	(3.4.5.4)	(3.4.5.4)
T_5	$*$	(3^5)	$(3^4.4)$	$(3^4.4)$	$(3^4.5)$	$(3^4.5)$
T_6	$*$	$*$	$*$	$*$	$*$	$*$
T_7	$*$	(4.6.6)	(4.6.8)	(4.6.8)	(4.6.10)	(4.6.10)
T_8	$\times$	(4^3)	$\times$	$\times$	$\times$	$\times$
T_9	$\times$	(3.8.8)	$\times$	$\times$	$\times$	$\times$
T_{10}	$\times$	(3.4^3)	$\times$	$\times$	$\times$	$\times$

TABLE 2. Archimedean solids arising from Platonic ones

no operation applied on a cycle gives rise to a polyhedral, vertex-transitive map or does not give a map. Since Archimedean solids are uniquely determined by its local type (up to taking the mirror image), we can identify them in Table 2 by displaying their respective local type. Comparing the maps given in Table 2 with the classification of Archimedean solids we get complete census. List of Archimedean solid (see e.g. [4]) includes:

(1) *truncated tetrahedron* of local type (3.6.6);
(2) *truncated cube* of local type (3.8.8);
(3) *truncated dodecahedron* of local type (3.10.10);
(4) infinite series of *n-prisms*, $n > 2$ of local type $(4.4.n)$;
(5) *truncated octahedron* of local type (4.6.6);
(6) *truncated cuboctahedron* of local type (4.6.8);
(7) *truncated icosidodecahedron* of local type (4.6.10);
(8) *truncated icosahedron* ("soccer ball") of local type (5.6.6);
(9) infinite series of *n-antiprisms*, $n > 2$ of local type $(3^3.n)$;
(10) *cuboctahedron* of local type (3.4.3.4);
(11) *rhombicuboctahedron* of local type (3.4^3);
(12) *rhombicosidodecahedron* of local type (3.4.5.4);
(13) *icosidodecahedron* of local type (3.5.3.5);
(14) *snub cube* of local type $(3^4.4)^{\pm}$;
(15) *snub dodecahedron* of local type $(3^4.5)^{\pm}$.

Note that the *snub cube* and the *snub dodecahedron* appears in two chiral forms.

Acknowledgements

The work of the first two authors was supported by Scientific and Technological Cooperation Agreement FCT/ESLOVAQUIA - 2011/2012 and by *FEDER* founds through *COMPETE*–Operational Programme Factors of Competitiveness ("Programa Operacional Factores de Competitividade") and by Portuguese founds through the *Center for Research and Development in Mathematics and Applications* (University of Aveiro) and the Portuguese Foundation for Science and Technology ("FCT–Fundação para a Ciência e a Tecnologia"), within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690.

The work of the third and the fourth author was supported by APVV, Projects SK-PT-0004-12, APVV-0223-10, and VEGA, Grant No. 1/0150/14.

This work was supported by the APVV project ESF-EC-0009-10 within the EUROCORES Programme EUROGIGA (project GReGAS) of the European Science Foundation. This work was partially supported by the Agency of the Slovak Ministry of Education for the Structural Funds of the EU, under project ITMS:26220120007.

References

1. G. V. Belyĭ, *Galois extensions of a maximal cyclotomic field*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 2, 267–276, 479. MR 534593 (80f:12008)
2. M. D. E. Conder, *Orientable proper regular hypermaps of genus 2 to 101*, `http://www.math.auckland.ac.nz/~conder/OrientableProperHypermaps101.txt`, 2012.
3. J. H. Conway, *The orbifold notation for surface groups*, Groups, Combinatorics & Geometry (Martin W. Liebeck and Jan Saxl, eds.), Cambridge University Press, 1992, Cambridge Books Online, pp. 438–447.
4. J. H. Conway, H. Burgiel, and C. Goodman-Strauss, *The symmetries of things*, A K Peters Ltd., Wellesley, MA, 2008. MR 2410150 (2009c:00002)
5. D. Corn and D. Singerman, *Regular hypermaps*, European J. Combin. **9** (1988), 337–351.
6. J. D. Dixon and B. Mortimer, *Permutation groups*, Graduate Texts in Mathematics, vol. 163, Springer-Verlag, New York, 1996. MR 1409812 (98m:20003)
7. C. J. Earl, *Reduced Teichmüller Spaces*, Trans. Amer. Math. Soc. **126** (1967), no. 1, 54–63.
8. E. Girondo, *Multiply quasiplatonic Riemann surfaces*, Experiment. Math. **12** (2003), no. 4, 463–475. MR 2043996 (2005h:30078)
9. L. D. James, *Operations on hypermaps and outer automorphism*, European J. Combin. **9** (1988), 551–560.
10. G. A. Jones and D. Singerman, *Theory of maps on orientable surfaces*, Proc. London Math. Soc. (3) **37** (1978), no. 2, 273–307. MR 0505721 (58 #21744)
11. N. Magot and A. K. Zvonkin, *Belyi functions for Archimedean solids*, Discrete Mathematics **217** (2000), no. 1â3, 249–271.
12. A. D. Mednykh and R. Nedela, *Enumeration of unrooted maps of a given genus*, J. Combin. Theory Ser. B **96** (2006), no. 5, 706–729. MR 2236507 (2007g:05088)
13. D. Singerman, *Finitely maximal Fuchsian groups*, J. London Math. Soc. (2) **6** (1972), 29–38. MR 0322165 (48 #529)
14. ______, *Riemann surfaces, Belyi functions and hypermaps*, Topics on Riemann surfaces and Fuchsian groups (Madrid, 1998), London Math. Soc. Lecture Note Ser., vol. 287, Cambridge Univ. Press, Cambridge, 2001, pp. 43–68. MR 1842766 (2002g:14047)
15. D. Singerman and R. I. Syddall, *The Riemann surface of a uniform dessin*, Beiträge Algebra Geom. **44** (2003), no. 2, 413–430. MR 2017042 (2004k:14053)
16. W. P. Thurston and S. Levy, *Three-dimensional geometry and topology*, Luis A.Caffarelli, no. 1, Princeton University Press, 1997.

E-mail address, A. Breda d'Azevedo: `breda@ua.pt`

E-mail address, D. Catalano: `domenico@ua.pt`

E-mail address, J. Karabáš: `karabas@savbb.sk`

E-mail address, R. Nedela: `nedela@savbb.sk`

(A. Breda d'Azevedo, D. Catalano) Department of Mathematics, University of Aveiro, Aveiro, Portugal

(J. Karabáš, R. Nedela) Faculty of Natural Sciences, Matej Bel University, Tajovského 40, 974 01 Banská Bystrica, Slovakia

(J. Karabáš, R. Nedela) Mathematical Institute, Slovak Academy of Sciences, Ďumbierska 1, 974 11 Banská Bystrica, Slovakia

DISCRETE GROUP ACTIONS AND EDGE-TRANSITIVE MAPS ON A GIVEN SURFACE

JÁN KARABÁŠ AND ROMAN NEDELA

Abstract. A map $\mathbf{M}$ is edge-transitive if its group of automorphisms, $\mathrm{Aut}(\mathbf{M})$, acts transitively on the edges of the underlying graph of $\mathbf{M}$. The group of orientation-preserving automorphisms, is a subgroup $\mathrm{Aut}^+(\mathbf{M})$ of index at most two in $\mathrm{Aut}(\mathbf{M})$. It follows that the quotient map of an edge-transitive map, $\bar{\mathbf{M}} = \mathbf{M}/\mathrm{Aut}^+(\mathbf{M})$, is a map on an (quotient) orbifold with at most two edges. There are exactly 8 such quotient maps sitting on orbifolds with at most 4 singular points, seven are spherical and one is toroidal. The paper gives a classification of edge-transitive maps on an orientable surface of genus $g > 1$. More precisely, we show that for each of the 8 families the classification reduces to the problem of determining of normal subgroups of bounded index in the associated Fuchsian/NEC group. We modify the well-known techniques of voltage assignments and regular covers (lifts) to reconstruct all edge-transitive maps with a given supporting orientable surface. Compared to the method used by Orbanić et al. (2011) we control the genus g of the underlying surface by choosing a proper g-admissible orbifold.

A map is a 2-cell decomposition of a closed surface. In this paper we exclusively consider orientable surfaces. Highly symmetrical maps attract interest of mathematicians since ancient time. It is well-known that the group of orientation-preserving automorphisms G of a map $\mathbf{M}$ acts semiregularly on the set of darts D (pairs of the form (v, e), where e is an edge and v a vertex incident e). In the extremal case, the action of G is regular on D and the map $\mathbf{M}$ itself is called regular. The well-known examples of regular maps are the 2-skeletons of the five Platonic solids, or the family of triangular, 4-gonal and hexagonal regular maps on the torus, the maps $\{6, 3\}_{b,c}$, $\{4, 4\}_{b,c}$ $\{3, 6\}_{b,c}$ in Coxeter notation, see [8].

As concerns regular maps the classification was done by Conder [6, 5, 4] by for genera up genus 301 and for some infinite sequences of genera, for instance if $g = p + 1$, where p is a prime [7]. In general, the classification seems to be hopeless, in particular for surfaces which Euler characteristic has many divisors. The reason is that the classification requires to understand all 2-generator discrete groups acting on a surface of genus g, and this problem seems to be intractable for a large g such that $2g - 2$ has many divisors.

A slightly more general family of highly symmetrical maps are the edge-transitive maps. A map is *edge-transitive* if the full automorphism group is transitive on the set of edges. If the group of orientation-preserving automorphisms of a map $\mathbf{M}$ acts on edges with one orbit, then the map $\mathbf{M}$ is either regular, or it is a bipartite realisation of a regular hypermap, or its dual. Both families were classified by Conder up to genus 301 or 101, respectively. We include classification of these maps for sake of completeness. Edge-transitive, but not necessarily regular maps were

---

2010 *Mathematics Subject Classification.* 05C10; 57M60, 57M15.

Key words and phrases. edge-transitive map, discrete group, orbifold, group action.

investigated by several authors including Graver and Watkins [10] and Orbanic et al. [16], where a characterisation in terms of one-edge quotients on surfaces with boundary induced by the action of the full automorphism group is given.

Our approach is based on investigation of the group of orientation-preserving automorhisms. One can easily seen that in an edge-transitive map on an orientable surface the group of orientation-preserving automorphisms of an edge-transitive map acts with at most two orbits. It follows that the quotient map of an edge-transitive map, $\bar{\mathbf{M}} = \mathbf{M}/\mathrm{Aut}^+(\mathbf{M})$, is a map on an (quotient) orbifold with at most two edges. There are exactly 8 such quotient maps sitting on orbifolds with at most 4 singular points, seven are spherical and one is toroidal. The paper gives a classification of edge-transitive maps on an orientable surface of genus $g > 1$. More precisely, we show that for each of the 8 families the classification reduces to the problem of determining of normal subgroups of bounded index in the associated Fuchsian/NEC group. Similarly, as in Conder's list of regular maps, one describe the edge-transitive maps of genus g in terms of a presentation of a quotient of the corresponding Fuchsian/NEC group acting on S_g. For small genera these quotients can be effectively generated by using the low-index-normal subgroup procedure in Magma [3].

In order to get the permutation representation of the maps, we modify the well-known techniques of voltage assignments and regular covers (lifts) to reconstruct all edge-transitive maps with supporting orientable surfaces up to genus 101. Compared to the method used by Orbanić et al., we control the genus g of the underlying surface by choosing a proper g-admissible orbifold. Moreover, the list of g-admissible groups of proper signatures can be processed independently on the problem we consider. Actually it can be used to solve many other problems. Using the idea of the antipodal double cover our approach can be adapted to solve the classification problem for non-orientable surfaces as well.

1. Maps and their symmetries

Map. A (topological) *map* $\mathbf{M}$ on a closed surface $\mathcal{S}$ is a two-cell decomposition of $\mathcal{S}$. Maps can be viewed as two-cell embeddings of (connected) graphs. A map automorphism is an automorphism of the underlying graph which extends into a self-homeomorphism of the underlying surface. Map automorphisms of a map on an orientable surface split into two classes - orientation preserving and orientation reversing.

Given a two-cell embedding $\Gamma \hookrightarrow \mathcal{S}$ of a graph Γ into a closed orientable surface $\mathcal{S}$ we can derive an associated *combinatorial map* $\mathbf{M} = (D; R, L)$ as follows. Set D to be the set of directed edges (edges endowed with an orientation) called *darts*, R to be a permutation in the symmetry group $\mathrm{Sym}(D)$, called *rotation*, permuting cyclically the darts based at the same vertex of Γ following a chosen global orientation of $\mathcal{S}$ and $L \in \mathrm{Sym}(D)$, $L^2 = 1$, to be the *dart-reversing involution* transposing the two darts coming from the same edge of Γ. The underlying graph of a map is connected if and only if $\langle R, L\rangle$ is transitive on D. We assume that the group $\langle R, L\rangle$ has a right action on D. Vice-versa, given a combinatorial map $(D; R, L)$ based on an abstract set of darts D we can reconstruct the corresponding graph Γ with the set of darts D embedded into an orientable surface $\mathcal{S}$ as follows. The vertices of Γ are the cycles of R, the edges of Γ are the cycles of L and the boundaries of the faces are defined by the cycles of RL. The incidence relations between the three objects (vertices, edges, faces) are given by non-empty intersections of the corresponding sets of darts. The surface S is obtained first by considering Γ as a 1-CW complex and then by gluing a 2-cell to each closed walk in Γ defined by a cycle of RL. We allow fixed points of L giving rise to *semi-edges* of the map. The free end of a semi-edge represents a singular point of index two. Graphs considered in this paper may have loops, multiple edges and even semi-edges.

Map homomorphisms. A *map homomorphism* $\vartheta : \mathbf{M}_1 \to \mathbf{M}_2$ is a function $\vartheta : D_1 \to D_2$ such that $R_1\vartheta = \vartheta R_2$ and $L_1\vartheta = \vartheta L_2$. If ϑ is a bijection, the maps $\mathbf{M}_1$ and $\mathbf{M}_2$ are isomorphic, $\mathbf{M}_1 \cong \mathbf{M}_2$. A *map automorphism* φ is a map homomorphism, where $\varphi \in \mathrm{Sym}(D)$ is a permutation of the set of darts. The automorphisms of a map $\mathbf{M}$ form the *group of orientation-preserving automorphisms* of the map, $\mathrm{Aut}^+(\mathbf{M}) = C_{\mathrm{Sym}(D)}(\langle R, L\rangle)$, where $C_{\mathrm{Sym}(D)}(\langle R, L\rangle)$ is the centraliser of $\langle R, L\rangle$ in $\mathrm{Sym}(D)$. Let $\varrho \in \mathrm{Sym}(D)$ be a permutation satisfying $R^\varrho = R^{-1}$ and $L^\varrho = L$ will be called a *reflection* of $\mathbf{M}$. The orientation-preserving automorphisms and reflections form the (full) *automorphism group* $\mathrm{Aut}(\mathbf{M})$ of the map $\mathbf{M}$. A reflection fixing a dart of D is uniquely determined. Thus $\mathrm{Aut}^+(\mathbf{M}) \leq \mathrm{Aut}(\mathbf{M})$ is a subgroup of index at most 2. The map $\mathbf{M}$ is said to be *edge-transitive*, if $\mathrm{Aut}(\mathbf{M})$ has a transitive action on edges of the underlying graph Γ. Given map $\mathbf{M} = (D; R, L)$ the map $\mathbf{M}^{-1} = (D; R^{-1}, L)$ is called the mirror image of $\mathbf{M}$. The maps $\mathbf{M}$ and $\mathbf{M}^{-1}$ may, or may not, be isomorphic as combinatorial maps, although the particular topological maps coincide.

2-dimensional orbifolds. Let G be a finite group acting on an orientable closed surface $\mathcal{S}_g$ of genus g. Then we can form a quotient surface $\mathcal{S}_g/\mathrm{G}$ which is known to be isomorphic to a closed orientable surface $\mathcal{S}_\gamma$ of genus $\gamma \leq g$. There is a natural branched regular covering $\varphi : \mathcal{S}_g \to \mathcal{S}_\gamma$, where the *fibers* $\varphi^{-1}(x)$, over each point $x \in \mathcal{S}_\gamma$ are the orbits of the action of G. The size of each fiber divides the order of G and except finitely many singular points $\{x_1, x_2, \ldots, x_r\} \in \mathcal{S}_\gamma$ we have $|\varphi^{-1}(x)| = |\mathrm{G}|$. The *branch index* m_i of a singular point x_i is a divisor of $|\mathrm{G}|$, $m_i \geq 2$. All the parameters are related by the well-known Riemann-Hurwitz equation

$$2 - 2g = |\mathrm{G}| \left(2 - 2\gamma - \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)\right); m_i \geq 2, m_i \mid |\mathrm{G}|. \tag{1.1}$$

It is convenient to assign the branch indexes to the singular points of $\mathcal{S}_\gamma$ thus forming a *quotient orbifold* $\mathcal{S}_g/\mathrm{G} = \mathcal{O}$ with the *signature* $(\gamma; \{m_1, m_2, \ldots, m_r\})$. Two orbifolds with the same signature are homeomorphic and invariant under permutation of branch indices. Given orbifold $(\gamma; \{m_1, m_2, \ldots, m_r\})$ determines its *fundamental group* $\pi_1(\mathcal{O})$ isomorphic to the Fuchsian group with presentation

$$\begin{aligned} \pi_1(\mathcal{O}) = \langle x_1, x_2, \ldots, x_r, a_1, b_1, \ldots, a_\gamma, b_\gamma \mid x_1^{m_1} = \cdots = x_r^{m_r} = 1, \\ \prod_{i=1}^{\gamma}[a_i, b_i] \prod_{j=1}^{r} x_j = 1\rangle. \end{aligned} \tag{1.2}$$

Universal covers and uniformisation. Let G be a finite group acting on a surface of genus g. Let $\mathcal{O} = S_g/G$ be the quotient orbifold. By a theorem of Koebe [18] there is a universal orbifold $\widetilde{\mathcal{S}}$, such that $\widetilde{\mathrm{G}} \cong \pi_1(\mathcal{O})$ acts on $\widetilde{\mathcal{S}}$ as a discrete group of automorphisms. Moreover, there is a regular covering $\widetilde{\mathcal{S}} \to \mathcal{S}_g$ with a group of covering transformations $\mathrm{K} \trianglelefteq \widetilde{\mathrm{G}}$ such that $\mathrm{G} = \widetilde{\mathrm{G}}/\mathrm{K}$. Hence the group G is a quotient of $\pi_1(\mathcal{O})$ by some torsion-free normal subgroup K of finite index $|\mathrm{G}|$.

(1.3)
$$
\begin{tikzcd}
(\widetilde{\mathcal{S}}, \widetilde{\mathrm{G}}) \arrow[dd, "\widetilde{\mathrm{G}}"'] \arrow[dr, "K"] & \\
& (\mathcal{S}_g, \mathrm{G}) \arrow[dl, "\mathrm{G}"] \\
(\mathcal{S}_g/\mathrm{G}, 1) &
\end{tikzcd}
$$

Maps on orbifolds. A (topological) map $\mathbf{M}$ on an orbifold $\mathcal{O}(\gamma; \{m_1, m_2, \dots, m_r\})$ is a map on S_γ such that neither a face nor an edge contains more than one branch point. The free end of a semiedge is either non-singular, or a branch point of index two. A vertex may, or may not, be a singular point. It follows that a map on an orbifold gives rise to a function

$$b : V(\mathbf{M}) \cup E(\mathbf{M}) \cup F(\mathbf{M}) \to \{1, m_1, m_2, \dots, m_r\},$$

taking the value $m_i \geq 2$ for some $x \in V(\mathbf{M}) \cup E(\mathbf{M}) \cup F(\mathbf{M})$ if either x is a vertex of branch index m_i; or x is a face containing a branch point of index m_i; or x is a semiedge and its free end is a branch point of index $m_i = 2$. In all other cases $b(x) = 1$. Vice-versa, a pair $(\mathbf{M}, b)$, where $\mathbf{M}$ is a map on S_γ and $b : V(\mathbf{M}) \cup E(\mathbf{M}) \cup F(\mathbf{M}) \to \{1, m_1, m_2, \dots, m_r\}$, is a function satisfying the above conditions determines a map on $\mathcal{O}$. A walk W of a map on an orbifold is a sequence $v_0^{e_0} x_0 v_1^{e_1} x_1 \dots v_k^{e_k} x_k v_{k+1}^{e_{k+1}}$, where the darts x_i and Lx_{i-1} originate at the same vertex v_i, for $i = 1, 2, \dots k$, x_0 originates at v_0 and the exponents e_i are integers for $i = 0, 1, \dots, k$. Image $\psi(W)$ of W in an orientation preserving map automorphism ψ is a walk $w_0^{e_0} y_0 w_1^{e_1} y_1 \dots w_k^{e_k} y_k w_{k+1}^{e_{k+1}}$, where $w_i = \psi(v_i)$ and $y_i = \psi(x_i)$ for $i = 0, 1, \dots, k+1$. If ψ is an orientation reversing automorphism, then all the exponents e_i are in the image multiplied by -1. In what follows we make an agreement that v^0 will be omitted from a sequence determining a walk, and v^1 for a vertex v will be replaced just by v.

Reconstruction of maps from voltage assignments. Given a quotient map $\bar{\mathbf{M}} = (D, R, L)$ and a group G we can reconstruct each map $\mathbf{M}$ such that $\bar{\mathbf{M}} \cong \mathbf{M}/\mathrm{G}$ employing the idea of voltage assignments as follows (see [14, 15, 17]). Let T be a spanning tree of the underlying graph Γ of $\bar{\mathbf{M}}$ with one distinguished dart x_0, based at a vertex v_0, which will be called the *root*. Clearly, for every vertex v in Γ there exist a unique dart based at v on a shortest path in T joining $v \neq v_0$ to v_0. Form a set $D^+(T)$ as follows. By definition set $x_0 \in D^+(T)$. For a dart x set $x \in D^+(T)$ if x is a dart on the unique shortest path on T joining a vertex v to the root. Observe that for each vertex v there is exacly one dart in $D^+(T)$ originating at v.

By a *T-reduced voltage assignment* on $\mathbf{N}$ we mean a mapping $\xi : V \cup D \to G$ taking values in a group G satisfying the following conditions:

(1) all darts on the rooted spanning tree (T, x_0) receive trivial voltages,
(2) $\xi_{xL} = \xi_x^{-1}$ for all $x \in D$,
(3) $G = \langle \{\xi_x : x \in D \cup V\} \rangle$.

The *derived map* $\mathbf{M} = \mathbf{N}^\xi = (D^\xi, R^\xi, L^\xi)$ is defined as follows: $D^\xi = D \times G$ and

$$(x,g)R^\xi = \begin{cases} (xR, g\cdot\xi_v), & x \in D^+(T), \\ (xR, g), & \text{otherwise} \end{cases} \tag{1.4}$$

$$(x,g)L^\xi = (xL, g\cdot\xi_x) \tag{1.5}$$

If $\xi|_V = id$, then $\mathbf{N}^\xi$ coincides with the classic construction by Gross & Tucker [11]. It is easy to see that the *natural projection* $\pi_\xi : \mathbf{N}^\xi \to \mathbf{N}$ erasing the second coordinate is a map covering. Observe that for each element $a \in \mathrm{G}$ the mapping $\psi_a : (x,g) \mapsto (x, ag)$ is a fibre-preserving automorphism of $\mathbf{N}^\xi$ and that the group $\widetilde{\mathrm{G}} = \{\psi_a;\, a \in \mathrm{G}\}$ is isomorphic to G. Moreover, the projection $\pi_{\widetilde{\mathrm{G}}} : \mathbf{M} \to \mathbf{M}/\widetilde{\mathrm{G}} = \bar{\mathbf{M}}$ is clearly equivalent to π_ξ. Therefore, π_ξ is a regular map homomorphism. The converse holds as well, see [15, Theorem 5.1] for the special case when $\xi(v) = 1$ for each vertex v of $\bar{\mathbf{M}}$. Observe that a T-reduced voltage assignment defined on $\mathbf{N}$ naturally extends from vertices and darts onto walks by setting

$$\xi_W = \xi(v_0^{e_0} x_0 v_1^{e_1} x_1 \dots v_k^{e_k} x_k v_{k+1}^{e_{k+1}}) = \xi_{v_0}^{e_0} \prod_{i=0}^{k} \xi_{x_i} \xi_{v_{i+1}}^{e_{i+1}}.$$

Given a T-reduced voltage assignment on a map $\mathbf{N}$ on a surface $\mathcal{S}_\gamma$ implicitly determines branch points at vertices and centres of faces. The index of a vertex v is the order of an element assigned to v and the index of a face is the order of the voltage of the boundary walk.

Projection and lifting of orientation reversing automorphisms. In what follows, the following two statements will be useful.

Lemma 1. *Let* $\mathbf{M}$ *be an edge-transitive map such that* $\mathrm{G} = \mathrm{Aut}^+(\mathbf{M})$ *acts with two orbits. Let* φ *be an automorphism of* $\mathbf{M}$ *transposing the two orbits in the action of* G*. Then* φ *projects.*

Proof. The following diagram has to commute

$$\begin{tikzcd} x \arrow[r, "\varphi"] \arrow[d, "p"'] & \varphi x \arrow[d, "p"] \\ \bar{x} \arrow[r, "\bar{\varphi}"] & \bar{\varphi}\bar{x} \end{tikzcd}$$

Since G is maximal and φ^2 is an orientation-preserving automorphism fixing the two orbits, we have $\varphi^2 \in \mathrm{G}$. In particular G is an index 2 subgroup of $\langle \mathrm{G}, \varphi\rangle$. This means that for every $g \in \mathrm{G}$ there exist $g' \in \mathrm{G}$ such that $g\varphi = \varphi g'$. Set $p : \mathbf{M} \to \bar{\mathbf{M}} = \mathbf{M}/\mathrm{G}$, $p : x \mapsto [x]_\mathrm{G}$, where $[x]_\mathrm{G}$ denotes the orbit of G containing a dart x of . It follows that

$$[g\varphi x]_\mathrm{G} = [\varphi g' x]_\mathrm{G} = [\varphi x]_\mathrm{G}. \tag{1.6}$$

Thus $\bar{\varphi} : [x]_\mathrm{G} \mapsto [\varphi x]_\varphi$ is well-defined. □

Following [15, p. 455–456, Theorem 6.1] we get the following lifting condition.

Lemma 2. *Let* $\mathbf{N}$ *be a map with two edges. Let* φ *be an automorphism of* $\mathbf{N}$ *transposing the two edges of* $\mathbf{N}$*. Let* ξ *be a* T*-reduced voltage assignment in* G*.*

Then φ has a lift if and only if the assignment $\xi_W \mapsto \xi_{\varphi W}$, where W ranges through the fundamental closed walks, extends to a well-defined automorphism $\varphi^{\#}$ of the voltage group G*. In this case $\mathbf{N}^{\xi}$ is an edge-transitive map.*

Proof. By [15, Theorem 6.1] it follows that in case when $\mathbf{N} = \mathbf{M}/\mathrm{Aut}^+(\mathbf{M})$ is a two-edge map, the edge-transitive map $\mathbf{M}$ can be reconstructed using a T-reduced voltage assignment $\xi : D(\mathbf{N}) \to \mathrm{G} \cong \mathrm{Aut}^+(\mathbf{M})$ such that some reflection φ of $\mathbf{N}$ transposing the two edges determines an automorphism of G given by $\varphi^{\#} \colon \xi_W \mapsto \xi_{\varphi W}$, where W is a closed angle-walk based at a dart of $\mathbf{N}$. Clearly, the automorphism $\varphi^{\#}$ of G is determined by the images of voltages generating G. ☐

2. Classification of edge-transitive maps

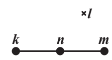
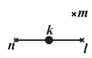
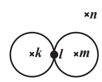
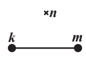
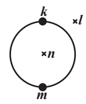
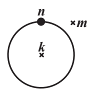
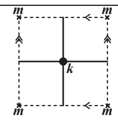
Fam.	Map	Conditions	$\bar{\mathbf{M}}$	Fam.	Map	Conditions	$\bar{\mathbf{M}}$
E1	$\mathbf{M}_1$	$n \leq 2$		E4	$\mathbf{M}_5$	$k = m$	
E2	$\mathbf{M}_2$	$n \leq 2, n = l$		E4*	$\mathbf{M}_6$		
E3	$\mathbf{M}_3$	none		E5	$\mathbf{M}_7$	none	
E3*	$\mathbf{M}_4$			E6	$\mathbf{M}_8$	none	

Table 1

Proposition 3. *Let $\mathbf{M}$ be an edge-transitive map. Then $\bar{\mathbf{M}} = \mathbf{M}/\mathrm{Aut}^+(\mathbf{M})$ is one of the 8 maps, $\mathbf{M}_1$, $\mathbf{M}_2$,…,$\mathbf{M}_8$, on the orbifolds depicted in Table 1, where some of the branch indices may be trivial (taking value 1). We have $\mathbf{M}_3^* = \mathbf{M}_4$ and $\mathbf{M}_5^* = \mathbf{M}_6$. The remaining maps are self-dual.*

Proof. $\mathrm{Aut}^+(\mathbf{M})$ acts with at most two orbits on edges of an edge-transitive map $\mathbf{M}$. With respect to Lemma 1 we do not consider maps with two edges not admitting a reflection transposing the two edges. Hence $\bar{\mathbf{M}} = \mathbf{M}/\mathrm{Aut}^+(\mathbf{M})$ is a one-edge, or a two-edge map admitting a reflection swapping the two edges. Assume $\bar{\mathbf{M}}$ has one edge. The single edge of $\bar{\mathbf{M}}$ is either a semi-edge, a loop, or a link, giving rise to families E1, E3 and E3*.

Assume $\bar{\mathbf{M}}$ has two edges. We have 6 combinations of kinds of edges: semiedge-semiedge, semiedge-loop, semiedge-link, loop-loop, loop-link, link-link. By Lemma 1 the two edges have to be of the same kind. Hence we get the following three cases.

(a) **semiedge–semiedge** In this case we have exactly one quotient spherical map, corresponding to the family E2;

(b) link–link We get two spherical quotient maps related with families E4 and E5;
(c) loop-loop There are two quotient maps associated with this case: one is spherical, related with the family E4* and the other one is toroidal, giving rise to the family E6.

The distribution of branch points reflects the fact that a branch point may be placed either into a vertex, or at a face, or at the free end of the semiedge. In the latter case, the branch point is of index at most 2. The conditions on branch indices in families E2, E4, E4* are forced by Lemma 1. □

Checking the orbifolds in Table 1 we get the following corollary.

Corollary 4. *The admissible Fuchsian groups are listed as follows:*
(a) $\mathrm{F}(0;k,k) = \langle x_1, x_2 \mid x_1^k = x_2^k = x_1x_2 = 1\rangle \cong \mathbb{Z}_k$,
(b) $\mathrm{F}(0;k,m,n) = \langle x_1, x_2, x_3 \mid x_1^k = x_2^m = x_3^n = x_1x_2x_3 = 1\rangle$,
(c) $\mathrm{F}(0;k,m,n,l) = \langle x_1, x_2, x_3, x_4 \mid x_1^k = x_2^m = x_3^n = x_4^l = x_1x_2x_3x_4 = 1\rangle$,
(d) $\mathrm{F}(1;\emptyset) = \langle a_1, b_1 \mid [a_1, b_1] = 1\rangle$,
(e) $\mathrm{F}(1;k) = \langle x_1, a_1, b_1 \mid x_1^k = [a_1, b_1]x_1 = 1\rangle$,
(f) $\mathrm{F}(1;k,m) = \langle x_1, x_2, a_1, b_1 \mid x_1^k = x_2^m = [a_1, b_1]x_1x_2 = 1\rangle$.

Corollary 5. *For a fixed genus $g > 1$ there are finitely many edge-transitive maps of genus g.*

Proof. Let $\mathbf{M}$ be an edge-transitive map of genus $g > 1$. By Hurwitz bound $|\mathrm{Aut}(\mathbf{M})| \leq 168(g-1)$, thus there are just finitely many discrete groups acting on $\mathcal{S}_g$. From the transitivity of the action of $\mathrm{Aut}(\mathbf{M})$ the number of edges $e(\mathbf{M})$ is bounded by $e(\mathbf{M}) \leq |\mathrm{Aut}(\mathbf{M})| \leq 168(g-1)$. The statement follows. □

Now we are ready to formulate the main result of the paper.

Theorem 6 (Projection Theorem)**.** *Let $\mathbf{M}$ be an edge-transitive map on an orientable surface of genus $g > 1$. Then, up to duality $\mathbf{M}$ is isomorphic to one of the maps described further. In particular, there are 8 families of non-degenerate edge-transitive maps distinguished by the quotients $\mathbf{M}/\mathrm{Aut}^+(\mathbf{M})$ described in Table 1.*

In what follows, we describe voltage assignments on each of the eight quotients (see Table 1) determining completely the corresponding families of edge-transitive maps. In concrete computations we will use the following convention. The vertices will be denoted by small-case Latin letters, while darts will take values in positive integers.

Family E1. Consider the quotient map $\mathbf{M}_1$ on the orbifold with signature $(0;\{k,m,n\})$. Since $g > 1$, $n = 2$. Thus $\mathrm{G} = \mathrm{Aut}^+(\mathbf{M})$ is a finite quotient of the Fuchsian group $\mathrm{F}(0;2,k,m)$, $2 \leq k \leq m$, where $\frac{1}{k} + \frac{1}{m} < \frac{1}{2}$.

$$\mathrm{G} = \langle x_1, x_2 \mid x_1^k = x_2^2 = (x_1^{-1}x_2) = 1, \ldots\rangle, \tag{2.1}$$

x

Set the voltage assignment ξ on the base map $\mathbf{M}_1$ (Figure 1(b)) as follows

$$\xi_v := x_1,\ \xi_1 := x_2. \tag{2.2}$$

The derived map $\mathbf{M}^\xi$ is an edge-transitive map, in fact a regular map. The bold dart in the Figure 1(b) means that it is the 'root dart' in the corresponding T-reduced voltage assignment. We shall use this convention in the following drawings.

(a) (b)

FIGURE 1

Proposition 7. *Let* $\mathrm{G} = \langle x_1, x_2 \mid x_1^k = x_2^2 = (x_1^{-1}x_2) = 1, \ldots\rangle$ *be a finite quotient of the Fuchsian group* $\mathrm{F}(0; 2, k, m)$ *for some integers* $m \geq k \geq 2$ *such that* $\frac{1}{k} + \frac{1}{m} < \frac{1}{2}$.

Then the derived map $\mathbf{M}_1^\xi$, *where* ξ *is determined by* (2.2), *is an edge-transitive map with the* $\mathrm{Aut}^+(\mathbf{M}) \cong \mathrm{G}$.

Family E2. Consider the quotient map $\mathbf{M}_2$ on the orbifold with signature $(0; \{k, m, n, l\})$, see Figure 2. Since $g > 1$, $n = l = 2$. Let $\mathrm{G} = \mathrm{Aut}^+(\mathbf{M})$ be a finite quotient of the Fuchsian group $\mathrm{F}(0; 2, 2, k, m)$, $m \geq k \geq 2$, where $\frac{1}{k} + \frac{1}{m} < 1$. The presentation of G is of the following form

$$\mathrm{G} = \langle x_1, x_2, x_3 \mid x_1^k = x_2^2 = x_3^2 = (x_1^{-1}x_2x_3)^m = 1, \ldots\rangle. \tag{2.3}$$

In the notation introduced on Figure 2(b), set the voltage assignment in G as follows

$$\xi_v := x_1,\ \xi_1 := x_2,\ \xi_2 := x_3. \tag{2.4}$$

The lifting condition (see Lemma 1 and Lemma 2) requires that the group G admits the automorphism

$$\varphi^\# : x_1 \mapsto x_1^{-1},\ x_2 \mapsto x_3,\ x_3 \mapsto x_2. \tag{2.5}$$

There is only one reflection φ of $\mathbf{M}_2$ transposing its two semiedges which induces the group automorphism $\varphi^\#$.

(a) (b)

FIGURE 2

We summarise the constructive part of our analysis in the following proposition.

Proposition 8. *Let* $\mathrm{G} = \langle x_1, x_2, x_3 \mid x_1^k = x_2^2 = x_3^2 = (x_1^{-1}x_2x_3)^m = 1, \ldots\rangle$ *be a finite quotient of the Fuchsian group* $\mathrm{F}(0; 2, 2, k, m)$ *for some integers* $m \geq k \geq 2$ *such that* $\frac{1}{k} + \frac{1}{m} < 1$. *Let* $\varphi^\#$ *defined in* (2.5) *extend to a group automorphism.*

Then the derived map $\mathbf{M}_2^{\xi}$, *where* ξ *is determined by* (2.4), *is an edge-transitive map with the* $\mathrm{Aut}^+(\mathbf{M}) \cong \mathrm{G}$.

Families E3 and E3*. Consider the quotient map $\mathbf{M}_3$ ($\mathbf{M}_4$, respectively) on the orbifold with signature $(0; \{k, m, n\})$, see Figure 3. Let $\mathrm{G} = \mathrm{Aut}^+(\mathbf{M})$ be a finite quotient of the Fuchsian group $\mathrm{F}(0; k, m, n)$, $2 \le k \le m \le n$, where $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} < 1$.

$$\mathrm{G} = \langle x_1, x_2 \mid x_1^k = x_2^m = (x_2^{-1} x_1^{-1})^n = 1, \ldots \rangle,$$
$$\mathrm{G}^* = \langle x_1, x_2 \mid x_1^k = x_2^n = (x_1^{-1} x_2^{-1})^m = 1, \ldots \rangle.$$

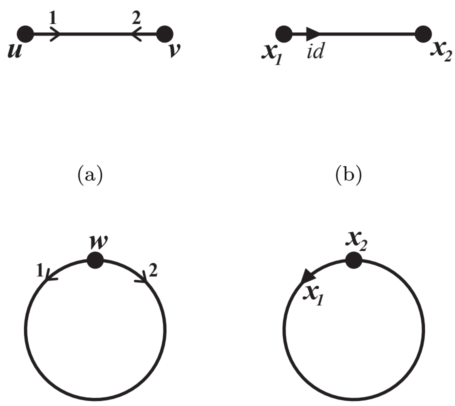
Since the quotient map for the maps of this family is not self-dual, we have to set two different voltage assignments ξ and ξ^* from G and from G^*, respectively, for two different quotients which are dual maps each to the other. We set the voltage assignments ξ, ξ^* on the base maps $\mathbf{M}_3$ (cf. Figure 3(b)) and $\mathbf{M}_4$ (cf. Figure 3(d)), respectively, as follows:

(2.6) $$\xi_u := x_1, \ \xi_v := x_2,$$

and

(2.7) $$\xi_1^* := x_1, \ \xi_w^* := x_2.$$

In both cases we obtain edge-transitive derived maps, namely the families E3 and E3*.



(a) (b)

(c) (d)

Figure 3

Note that E3 is the family of bipartite maps with a transitive action of $\mathrm{Aut}^+(\mathbf{M})$ on the edges, but non-transitive action on the vertices. The two orbits on the vertices form the bipartition of the vertex set. It can be easily seen that maps of family E3 correspond to Walsh representations of regular hypermaps.

Proposition 9. *Let* $\mathrm{G} = \langle x_1, x_2 \mid x_1^k = x_2^m = (x_2^{-1}x_1^{-1})^n = 1, \ldots\rangle$ *and* $\mathrm{G}^* = \langle x_1, x_2 \mid x_1^k = x_2^n = (x_1^{-1}x_2^{-1})^m = 1, \ldots\rangle$ *be finite quotients of the Fuchsian group* $\mathrm{F}(0; k, m, n)$ *for some integers* $2 \le k \le m \le n$*, where* $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} < 1$.

Then the derived maps $\mathbf{M}_3^{\xi}$ *and* $\mathbf{M}_4^{\xi^*}$*, where* ξ, ξ^* *are determined by* (2.6), (2.7)*, are edge-transitive maps with* $\mathrm{Aut}^+(\mathbf{M}) \cong \mathrm{Aut}^+(\mathbf{M}^*) \cong \mathrm{G}$.

Families E4 and E4*. Let G be a finite quotient of the Fuchsian group $\mathrm{F}(0; k, k, m, n)$, such that $\frac{2}{k} + \frac{1}{m} + \frac{1}{n} < 2$

$$\mathrm{G} = \langle x_1, x_2, x_3 \mid x_1^k = x_2^k = x_3^m = (x_1^{-1}x_2^{-1}x_3^{-1})^n = 1, \ldots\rangle,$$
$$\mathrm{G}^* = \langle x_1, x_2, x_3 \mid x_1^k = x_2^k = x_3^n = (x_1^{-1}x_3^{-1}x_2^{-1})^m = 1, \ldots\rangle.$$

Edge-transitive map in the family E4 is obtained by setting the voltage assignment on $\mathbf{M}_5$ (see Figure 4(b)) as follows

$$\xi_u := x_1,\ \xi_v := x_2,\ \xi_w := x_3. \tag{2.8}$$

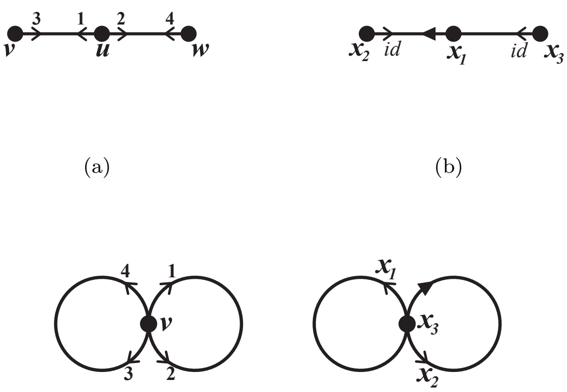
All darts of $\mathbf{M}_5$ receive trivial voltage, as the voltage assignment ξ is T-reduced.

We use the voltage map $(\mathbf{M}_6, \xi^*)$ (see Figure 4(d)) to construct the family E4* dual to E4 by setting ξ^* as

$$\xi_2^* := x_2,\ \xi_4^* := x_1,\ \xi_v^* := x_3. \tag{2.9}$$

Applying Lemma 2 to the voltage map $(\mathbf{M}_5, \xi)$ we deduce that a reflection swapping the two edges lifts. Since the reflection is uniquely determined, it follows that both the group G and G* admit the automorphism

$$\varphi^{\#} : x_1 \mapsto x_2^{-1},\ x_2 \mapsto x_1^{-1},\ x_3 \mapsto x_3^{-1}. \tag{2.10}$$



(a) (b)

(c) (d)

Figure 4

Proposition 10. *Let* $\mathrm{G} = \langle x_1, x_2, x_3 \mid x_1^k = x_2^k = x_3^m = (x_1^{-1}x_3^{-1}x_2^{-1})^n = 1, \ldots\rangle$ *and* $\mathrm{G}^* = \langle x_1, x_2, x_3 \mid x_1^k = x_2^k = x_3^n = (x_1^{-1}x_3^{-1}x_2^{-1})^m = 1, \ldots\rangle$ *be finite quotients of* *the Fuchsian group* $\mathrm{F}(0; k, k, m, n)$, *where* $\frac{2}{k} + \frac{1}{m} + \frac{1}{n} < 2$. *Let* $\varphi^{\#}$ *defined by* (2.10) *extend to a group automorphism in both* G *and* G*.

Then the derived maps $\mathbf{M}_5^{\xi}$ *and* $\mathbf{M}_6^{\xi^*}$, *where* ξ, ξ^* *are determined by* (2.8), (2.9), *are edge-transitive maps with* $\mathrm{Aut}^+(\mathbf{M}) \cong \mathrm{Aut}^+(\mathbf{M}^*) \cong \mathrm{G}$.

2.1. **Family E5.** Let G be a finite quotient of the Fuchsian group $\mathrm{F}(0; k, m, n, l)$, for some integers $1 \le k \le m \le n \le l$, where $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} + \frac{1}{l} < 2$.

$$\mathrm{G} = \langle x_1, x_2, x_3 \mid x_1^k = x_2^m = (x_3 x_2^{-1})^n = (x_3^{-1} x_1^{-1})^l = 1, \ldots \rangle.$$

Consider the quotient map $\mathbf{M}_7$ depicted on Figure 5. With the notation taken from Figure 5, $\mathbf{M}_7$ admits two reflections swapping the two edges defined by permutations of darts and vertices as follows

$$\varphi = (u, u^{-1})(v, v^{-1})(1, 2)(1^{-1}, 2^{-1}), \tag{2.11}$$

and

$$\psi = (u, v^{-1})(v, u^{-1})(1, 2^{-1})(2, 1^{-1}). \tag{2.12}$$

Let T be the spanning tree containing the darts 2 and 2^{-1}. Then the fundamental u-based walks with respect to T are: u^1, $1^{-1}2$, and $2^{-1}v^1 2$. Hence $\varphi(u^1) = u^{-1}$, $\varphi(1^{-1}2) = 2^{-1}1$, $\varphi(2^{-1}v^1 2) = 1^{-1}v^{-1}1$, and $\psi(u^1) = v^{-1}$, $\psi(1^{-1}2) = 21^{-1}$, and $\psi(2^{-1}v^1 2) = 1u^{-1}1^{-1}$.

Set the T-reduced voltage assignment (see Figure 5(b)) to be

$$\xi_u = x_2,\ \xi_v = x_1,\ \xi_1 = x_3. \tag{2.13}$$

By Lemma 2 either

$$\varphi^{\#}\colon x_1 \mapsto x_1^{-1},\ x_2 \mapsto x_2^{-1},\ x_3 \mapsto x_2 x_3^{-1} x_2^{-1}, \tag{2.14}$$

or

$$\psi^{\#}\colon x_1 \mapsto x_2^{-1},\ x_2 \mapsto x_1^{-1},\ x_3 \mapsto x_1 x_3 x_1^{-1}, \tag{2.15}$$

extends to a group automorphism of G.

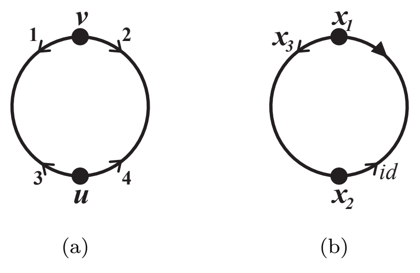


FIGURE 5

Let us note that the self-homeomorphism of the underlying surface $\mathcal{S}$ of an edge-transitive map $\mathbf{M}$, induced by φ is a reflection, while the self-homeomorphism of $\mathcal{S}$ induced by ψ is a glide reflection. Thus, if $\mathrm{Aut}^+(\mathbf{M}) \cong \mathrm{G}$ admits the automorphimsm $\varphi^{\#}$, then the corresponding signatures are $(0; \{k, k, m\})$ or $(0; \{k, k, m, n\})$, $k, m, n > 1$. If $\mathrm{Aut}^+(\mathbf{M})$ admits the automorphimsm $\varphi^{\#}$, then the signature of g-admissible orbifold must be $(0; \{k, k, m, m\})$, $k, m > 1$. In what follows, we shall distinguish maps of family E5 possessing automorphisms $\varphi^{\#}$ and $\psi^{\#}$. We talk about (sub)families E5a and E5b, respectively. However, the intersection of E5a and E5b is non-trivial.

Proposition 11. *Let* $\mathrm{G} = \langle x_1, x_2, x_3 \mid x_1^k = x_2^m = (x_3x_2^{-1})^n = (x_3^{-1}x_1^{-1})^l = 1, \ldots\rangle$ *be a finite quotient of the Fuchsian group* $\mathrm{F}(0; k, m, n, l)$ *for some integers* $k, m, n, l \geq 1$*, such that* $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} + \frac{1}{l} < 2$*. Let at least one of the assignments* $\varphi^{\#}$ *or* $\psi^{\#}$ *defined by* (2.14)*, or* (2.15)*, respectively extend to a group automorphism.*

Then the derived map $\mathbf{M}_7^{\xi}$*, where* ξ *is determined by* (2.13)*, is an edge-transitive with the* $\mathrm{Aut}^+(\mathbf{M}) \cong \mathrm{G}$.

2.2. **Family E6.** Let G be a finite quotient of the Fuchsian group $\mathrm{F}(1; k, m)$, where $mk > 1$. By Corollary 4, G has the presentation

$$\mathrm{G} = \langle x_1, a, b \mid x_1^k = (x_1^{-1}ab^{-1}a^{-1}b)^m = 1, \ldots\rangle.$$

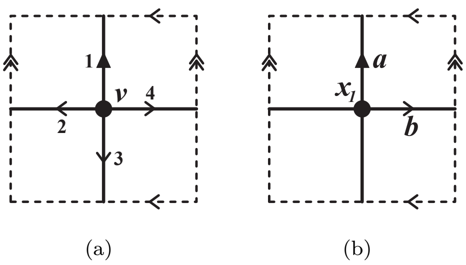


Figure 6

Let $\mathbf{M}_8$ be the unique one-vertex quotient of an edge-transitive map from the Family E6 by its orientation-preserving group of automorphisms depicted on Figure 6. In the notation taken from Figure 6, let the voltage map $(\mathbf{M}_8, \xi)$ be determined by the assignment

$$\xi_v := x_1,\ \xi_1 := a,\ \xi_2 := b^{-1}. \tag{2.16}$$

By Lemma 1 and Lemma 2, $\mathbf{M}_8$ admits a voltage assignment in G such that a reflection of $\mathbf{M}_8$ transposing the two edges lifts. There are exactly two possibilities for the reflection, namely $\varphi = (1,2)(1^{-1},2^{-1})(v,v^{-1})$ or $\psi = (1,2^{-1})(1^{-1},2)(v,v^{-1})$. The fundamental walks in $\mathbf{M}_8$ are v, 1, and 2. Following Lemma 2, the lifting condition requires that the group G admits one of the induced automorphisms

$$\varphi^{\#} \colon x_1 \mapsto x_1^{-1},\ a \mapsto b^{-1},\ b \mapsto a^{-1}, \tag{2.17}$$

or

$$\psi^{\#} \colon x_1 \mapsto x_1^{-1},\ a \mapsto b,\ b \mapsto a. \tag{2.18}$$

Subfamilies E6a (admitting $\varphi^{\#}$) and E6b (admitting $\psi^{\#}$) are introduced in aforementioned manner.

We classify edge-transitive maps of family E6 in the following proposition.

Proposition 12. *Let* $\mathrm{G} = \langle x_1, a, b \mid x_1^k = (x_1^{-1}ab^{-1}a^{-1}b)^m = 1, \ldots \rangle$ *be a finite quotient of the Fuchsian group* $\mathrm{F}(1; k, m)$ *for some integers* k, m *such that* $k.m > 1$. *Let either* $\varphi^{\#}$ *defined by* (2.17), *or* $\psi^{\#}$ *defined by* (2.18), *extend to a group automorphism.*

Then the derived map $\mathbf{M}_8^{\xi}$, *where* ξ *is determined by* (2.16) *is an edge-transitive map.*

Theorem 13 (Reconstruction Theorem)**.** *The above propositions give a complete information for the reconstruction of the families of edge-transitive maps E1-E6.*

Let two edge-transitive maps $\mathbf{M}$ and $\mathbf{M}'$ be isomorphic. Then both $\mathbf{M}$ and $\mathbf{M}'$ can be respectively reconstructed as derived maps $\mathbf{N}^{\xi}$ and $\mathbf{N}^{\xi'}$ over the uniquely determined quotient $\mathbf{N}$ isomorphic to the one of the map $\mathbf{M}_i$, $i \in \{1, 2, \ldots, 8\}$. The two voltage assignments ξ, ξ' take values in the same voltage group $\mathrm{G} \cong \mathrm{Aut}^+(\mathbf{M}) \cong \mathrm{Aut}^+(\mathbf{M}')$.

Set $\mathbf{N} = \mathbf{M}_i$ for some $i \in \{1, 2, \ldots, 8\}$. Let $\mathbf{N}^{\xi}$ be the derived map determined by $(\mathbf{M}_i, \xi)$ and let $\mathbf{N}^{\xi'}$ be the derived map determined by $(\mathbf{M}_i, \xi')$, where ξ and ξ' take values from the same voltage group G. To solve the isomorphism problem we need to check whether $\mathbf{N}^{\xi} \cong \mathbf{N}^{\xi'}$. We claim that it can be reduced to a problem whether a natural mapping between the corresponding voltage assignments defined on the darts and the vertices of $\mathbf{N} = \mathbf{M}_i$ extends to an automorphism of G. For instance, if $i = 2$, then $\mathrm{G} = \langle \xi_1, \xi_2, \xi_v \rangle = \langle \xi_1', \xi_2', \xi_v' \rangle$. It follows that $\mathbf{N}^{\xi} \cong \mathbf{N}^{\xi'}$ if and only if the mapping $\xi_1 \mapsto \xi_1', \xi_2 \mapsto \xi_2', \xi_v \mapsto \xi_v'$ extends to an automorphism of G. We summarise the above discussion in the following statement.

Theorem 14 (Isomorphism Theorem)**.** *Let* $\mathbf{M}$ *and* $\mathbf{M}'$ *be two edge-transitive maps defined by two* T*-reduced voltage assignments* ξ, ξ' *in* G, *defined on the quotient map* $\mathbf{M}_i$, $i \in \{1, 2, \ldots, 8\}$. *Then the isomorphism problem is equivalent to the problem whether the natural correspondence between the voltages on darts and vertices of* $\mathbf{M}_i$ *extends to an automorphism of* G.

Proof. Let $\mathbf{N} = \mathbf{M}_i$, for some $i \in \{1, 2, \ldots, 8\}$. Let the natural assignment $\xi \mapsto \xi'$ between the vectors of voltages defined on $\mathbf{N}$ extend to an automorphism of voltage groups. By [15, p. 455–456, Theorem 6.1] the identity mapping lifts to an isomorphism between $\mathbf{N}^{\xi}$ and $\mathbf{N}^{\xi'}$.

Assume $\theta \colon \mathbf{N}^{\xi} \to \mathbf{N}^{\xi'}$ is an orientation-preserving isomorphism between the derived maps. Since the dart sets of $\mathbf{N}^{\xi}$ and $\mathbf{N}^{\xi'}$ coincide, θ can be viewed as an automorphism of $\mathbf{N}^{\xi}$. Our aim is to show that θ preserves fibers over darts on $\mathbf{N}$. If G acts with one orbit on the dart set, there is nothing to prove. Let G act with two orbits on darts. Assume G acts transitively on edges. Then θ projects to the 180-degree rotation fixing an edge, contradicting the maximality of G. If θ does not fix the orbits over edges, then some 'red' dart is mapped onto a 'blue' dart. This would mean that $\langle \mathrm{G}, \theta \rangle$ is an extension of G by an orientation-preserving automorphism, a contradiction to the maximality of G. It follows that θ projects either to an identity or to the 180-degree rotation fixing an edge. In the latter case we get a contradiction with the maximality of G. It follows that θ is the lift of identity and the result comes from [15, p. 455–456, Theorem 6.1]. □

3. Generation of edge-transitive maps of small genera

We apply the theory developed in this paper to derive the list of edge-transitive maps of small genera. The census is published on-line on the author's web-page [13]

E1	$\langle x_1, x_2 \mid x_1^k = x_2^2 = (x_1^{-1}x_2)^m = 1\rangle$
E2	$\langle y_1, y_2, y_3, r \mid y_1^k = y_2^2 = y_3^2 = (y_1^{-1}y_2y_3)^m = 1,$ $r^2 = 1, y_1^r = y_1^{-1}, y_2^r = y_3, y_3^r = y_2\rangle$
E3	$\langle x_1, x_2 \mid x_1^k = x_2^l = (x_2^{-1}x_1^{-1})^m = 1\rangle$
E4	$\langle y_1, y_2, y_3, r \mid y_1^k = y_2^l = y_3^m = (y_1^{-1}y_2^{-1}y_3^{-1})^n = 1, \rangle$ $r^2 = 1, y_1^r = y_2^{-1}, y_2^r = y_1^{-1}, y_3^r = y_3^{-1}\rangle$
E5a	$\langle y_1, y_2, y_3, r \mid y_1^k = y_2^l = (y_3y_2^{-1})^m = (y_3^{-1}y_1^{-1})^n = 1, \rangle$ $r^2 = 1, y_1^r = y_1^{-1}, y_2^r = y_2^{-1}, y_3^r = (y_2y_3y_2^{-1})^{-1}\rangle$
E5b	$\langle y_1, y_2, y_3, r \mid y_1^k = y_2^l = (y_3y_2^{-1})^m = (y_3^{-1}y_1^{-1})^n = 1, \rangle$ $r^2 = 1, y_1^r = y_2^{-1}, y_2^r = y_1^{-1}, y_3^r = (y_1y_3^{-1}y_1^{-1})^{-1}\rangle$
E6a	$\langle z, a, b, s \mid z^k = (z^{-1}ab^{-1}a^{-1}b)^m = 1,$ $s^2 = 1, z^s = z^{-1}, a^s = b^{-1}, b^s = a^{-1}\rangle$
E6b	$\langle z, a, b, s \mid z^k = (z^{-1}ab^{-1}a^{-1}b)^m = 1,$ $s^2 = 1, z^s = z^{-1}, a^s = b, b^s = a\rangle$

TABLE 2

We shall briefly describe the method how to obtain all edge-transitive maps of prescribed genus $g > 1$.

Step 1: Determining g-admissible groups. We need to generate g-admissible quotients of Fuchsian groups of types $\mathrm{F}(0; k, m, n)$, $\mathrm{F}(0; k, m, n, l)$, $\mathrm{F}(1; k)$, and $\mathrm{F}(1; k, m)$ – see Corollary 4. Generation of lists of actions of discrete groups of a given genus is an interesting problem on its own. Its solution for triangular and quadrangular signatures and for the genera up to the genus 101 can be found on Marston Conder's web page [4]. The list of discrete groups acting on surfaces of small genera (including those with signatures $(1; \{k\})$ and $(1; \{k, m\})$ up to genus 25 can also be found at the web page [12]. The latter list is adapted to suit the needs of the algorithm determining edge-transitive maps. As the result of this step we have a list consisting of g-admissible signatures and the corresponding finite groups. This step can be done independently and may serve for other purposes as well.

Step 2: Setting the distribution of branch indices and a universal group. Choose a g-admissible pair of the form (σ, G), where G is a discrete group acting on a surface of genus g and σ the corresponding orbifold signature. In order to construct the edge-transitive map of a given family we take the quotient map with respect to the family, see Proposition 3 and Table 1 for details. We take σ and we distribute branch indices into branch points. In the case when the number of branch points is larger than the number of branch indices, we extend the vector of branch indices by including 'trivial' branch indices to the proper length. Given the list of branch indices and the quotient map on the orbifold we consider all admissible assignments of branch indices.

Given the quotient map on the orbifold, we determine the (extended) universal group U in such a fashion that it is an *extended* Fuchsian group of given type (see Corrolary 4). The relators of the Fuchsian group are determined using walk-calculus, described in Section 1 (Maps on orbifolds). Every relator arises as a word determined by a fundamental walk in the quotient map raised to a power given by the assigned branch index of a face or of a vertex. We construct the extension of a Fuchsian group in such a way that we add a 'reflection' generator and relators induced by the orientation-reversing automorphism into the presentation of the Fuchsian group. We derive these relators by employing Proposition 7–12. The corresponding presentations of (NEC) universal groups used in the computations are in the Table 2. Let us recall that families E5 and E6 give rise to two different presentations (subfamilies) since each of them possesses two different reflections.

Step 3: Determining quotient actions. The group of orientation-preserving automorphisms G of an edge transitive map arises as a factor group A = U/K of the universal group U by a torsion-free normal subgroup K of given index. The index of K is $k.|\mathrm{G}|$. The parameter $k \in \{1, 2\}$ is given by the number of edges of the quotient map (see Step 2). We derive the list of subgroups of the group U by using Magma function `LowIndexNormalSubgroups` [2]. Then we check whether the group G embeds into the group A as an index k subgroup. Having the group G we derive the corresponding voltage assignment in G by using Magma function `CosetAction` [2].

Step 4: Computation of permutation representation of the map. As the result of Step 3 we determined a T-reduced voltage assignment on the base quotient map. By using formula (1.6) we compute the rotation R and the dart reversing involution L of the derived edge-transitive map M on the surface of genus g.

Given the permutation representation of $\mathbf{M}$ in terms of the two permutations R and L we compute the automorphism group of $\mathrm{Aut}^+(\mathbf{M})$. It may happen that $|\mathrm{G}| \neq |\mathrm{Aut}^+(\mathbf{M})|$. This means that G embeds as a subgroup of index 2 or 4 and so $\mathbf{M}$ belongs to another family of edge-transitive maps. Thus, with respect to Theorem 13 and 14 we accept the map $\mathbf{M}$ only if $|\mathrm{G}| = |\mathrm{Aut}^+(\mathbf{M})|$.

Step 5: Checking isomorphism classes. The maps obtained from different base maps are by definition non-isomorphic, see Theorem 14. However, two maps of the same family may arise from different voltage assignments, since they can be obtained one from the other by an outer automorphism (duality). We filter the list of maps of the same family by determining isomorphism classes of maps. Chiral maps forming a chiral twin pair are considered to be different, as well as a map and its dual.

In the Table 3 displayed below there are enumerations of number of actions and corresponding maps with regard to given genus. Let us note that two (sub)columns in family expresses the respective numbers of actions giving rise to edge-transitive map and numbers of non-isomorphic maps written into the output data (see below). The number in the least column is the number of non-isomorphic edge-transitive maps of given genus.

Genus	Family																Maps
	E1		E2		E3		E4		E5a		E5b		E6a		E6b		
2	10	10	2	1	36	18	4	2	148	44	0	0	2	1	2	0	76
3	20	20	4	2	92	46	12	6	358	108	0	0	2	1	3	1	184
4	20	20	14	7	106	53	26	13	448	137	0	0	12	6	12	2	238
5	26	26	22	11	108	54	40	20	572	177	0	0	10	5	12	4	297
6	23	23	18	9	140	70	32	16	733	221	4	2	14	7	14	4	352

TABLE 3

How to read published data. The list of isomorphism classes of edge-transitive maps is long, even for genus 2. Therefore we published data at the webpage [13]. The census consists of plain text files, one file for each genus. The format of a record is provided with hope that the data are self-explanatory. We briefly explain the structure of a record. It consists of four lines:

```
E2.2.6 (0; {2, 2, 3, 3}) :: reflexible, selfdual  :: v = 2 e = 6 f = 2
Aut^+ = S3 :: [ X2^2, X3^2, R^2, X3 * X2, X1^3, (R * X1)^2, R * X2 * R * X3, (X2 * X1^-1)^2 ]
R = (1, 9, 3, 10, 4, 7)(2, 11, 5, 12, 6, 8)
L = (1, 2)(3, 6)(4, 5)(7, 8)(9, 12)(10, 11)
```

The first line consists of several fields delimited by double semicolons:

1. The identifier: `E2.2.6` uniquely determines a map. It consists from the family name, the genus and the unique number of a map in the catalogue file;
2. Properties of the map: we have checked whether the map is reflexible, (positively-, negatively-) selfdual, and whether it is polytopal or polyhedral;
3. The numbers of vertices, edges and faces are respectively written in the form `v = 2 e = 6 f = 2`.

The second line contains information about automorphism group of the map. The line consists of two fields. The first field describes the structure of $\mathrm{Aut}^+(\mathbf{M})$ with respect to Library of Small groups [1, 9]. The second field contains the relations of the presentation of the group $\mathrm{A} = \mathrm{U}/\mathrm{K}$ (see Step 3). This group is isomorphic to $\mathrm{Aut}^+(\mathbf{M})$ in case of families E1 and E3, or it is isomorphic to $\mathrm{Aut}(\mathbf{M})$ otherwise. Let us note that the group presentation of A has generators with names as they are shown in Table 2. The relators were obtained by Magma function `Simplify` [2], with option not to eliminate any generator of the finitely presented group (the argument). We decided to leave this form of presentation to keep track of the generation process.

The last two lines describes the corresponding combinatorial map in terms of the permutations R and L.

Acknowledgements

The authors acknowledge partial support from the grants APVV-0223-10, VEGA 1/0150/14, and from the grant APVV-ESF-EC-0009-10 within the EUROCORES Programme EUROGIGA (project GReGAS) of the European Science Foundation. This work was partially supported by the Agency of the Slovak Ministry of Education for the Structural Funds of the EU, under project ITMS:26110230082 and under project ITMS:26220120007.

References

1. Hans Ulrich Besche, Bettina Eick, and Eamon O'Brien, *The Small Groups library*, `http://www.icm.tu-bs.de/ag_algebra/software/small/`, 2014.
2. Wieb Bosma, John Cannon, Claus Fieker, and Allan Steel, *Handbook of Magma functions*, Edition 2.16, 2010.
3. Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993).
4. Marston D. E. Conder, *All large groups of automorphisms of compact Riemann surfaces of genus 2 to 101*, `http://www.math.auckland.ac.nz/~conder/BigSurfaceActions-Genus2to101-ByGenus.txt`, November 2011.
5. ———, *All chiral (irreflexible) orientably-regular maps on surfaces of genus 2 to 101, up to isomorphism, duality and reflection, with defining relations for their automorphism groups*, `http://www.math.auckland.ac.nz/~conder/ChiralMaps101.txt`, 2014.

6. ______, *All orientable regular maps on surfaces of genus 2 to 101, up to isomorphism and duality, with defining relations for their automorphism groups*, `http://www.math.auckland.ac.nz/~conder/OrientableRegularMaps101.txt`, 2014.
7. Marston D. E. Conder, Jozef Širáň, and Thomas W. Tucker, *The genera, reflexibility and simplicity of regular maps*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 2, 343–364. MR 2608943 (2011i:05092)
8. H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups*, fourth ed., Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 14, Springer-Verlag, Berlin-New York, 1980. MR 562913 (81a:20001)
9. The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.7.5*, 2014.
10. Jack E. Graver and Mark E. Watkins, *Locally finite, planar, edge-transitive graphs*, Mem. Amer. Math. Soc. **126** (1997), no. 601, vi+75. MR 1361828 (97i:05001)
11. Jonathan L. Gross and Thomas W. Tucker, *Topological graph theory*, Dover Publications Inc., Mineola, NY, 2001, Reprint of the 1987 original [Wiley, New York; MR0898434 (88h:05034)] with a new preface and supplementary bibliography.
12. Ján Karabáš, *Actions of finite groups on Riemann surfaces of higher genera*, `http://www.savbb.sk/~karabas/finacts.html`, 2014.
13. ______, *Edge transitive maps on orientable surfaces*, `http://www.savbb.sk/~karabas/science.html#etran`, 2014.
14. Aleksander Malnič, Roman Nedela, and Martin Škoviera, *Lifting graph automorphisms by voltage assignments*, European J. Combin. **21** (2000), no. 7, 927–947.
15. ______, *Regular homomorphisms and regular maps*, European J. Combin. **23** (2002), no. 4, 449–461.
16. Alen Orbanić, Daniel Pellicer, Tomaž Pisanski, and Thomas W. Tucker, *Edge-transitive maps of low genus*, Ars Mathematica Contemporanea **4** (2011), 385–402.
17. Jozef Širáň, *The "walk calculus" of regular lifts of graph and map automorphisms*, Proceedings of the 10th Workshop on Topological Graph Theory (Yokohama, 1998), vol. 47, 1999, pp. 113–128.
18. C. K. Wong, *A uniformization theorem for arbitrary Riemann surfaces with signature.*, Proc. Amer. Math. Soc. **28** (1971), 489–495. MR 0279303 (43 #5026)

E-mail address, J. Karabáš: `karabas@savbb.sk`

E-mail address, R. Nedela: `nedela@savbb.sk`

(J. Karabáš, R. Nedela) Institute of Mathematics and Computer Science, Matej Bel University, Ďumbierska 1, 97411 Banská Bystrica, Slovakia

(J. Karabáš, R. Nedela) Mathematical Institute of Slovak Academy of Sciences, Ďumbierska 1, 97411 Banská Bystrica, Slovakia