Coherent algebras

- The essential properties of centralizer algebras are axiomatized.
- We get definition of coherent algebras.
- Relational language of coherent configurations.
- Origins:
  - B. Weisfeiler & A. Leman (Moscow, 1968): cellular algebras;
Coherent configurations (algebras) provide a basis for combinatorial imitation of permutation groups.

Structure constants aka intersection numbers have a definite combinatorial spirit: numbers of walks in color graph.

How far does this imitation go?
Let $\mathcal{W} \subseteq M_n(\mathbb{C})$ be a matrix algebra over $\mathbb{C}$ which fulfills the following requirements:

**CA1.** Considered as a vector space over $\mathbb{C}$, the algebra $\mathcal{W}$ has a basis $\{A_1, A_2, \ldots, A_r\}$ where each $A_i$ is a $(0, 1)$-matrix, $1 \leq i \leq r$;

**CA2.** $\sum'_{i=1} A_i = J_n$, where $J_n$ is the matrix of order $n$ every entry of which is equal to 1;

**CA3.** For each $i \in \{1, 2, \ldots, r\}$ there is an $i' \in \{1, 2, \ldots, r\}$ such that $A_i^t = A_{i'}$;

**CA4.** The identity matrix $I_n$ belongs to $\mathcal{W}$. 
In this case we call $\mathcal{W}$ a \textit{coherent algebra with standard basis} $\{A_1, A_2, \ldots, A_r\}$.

We indicate this all at once by writing

$$\mathcal{W} = \langle A_1, A_2, \ldots, A_r \rangle.$$
Coherent rings

Given a coherent algebra $\mathcal{W}$, the set $\mathcal{W}' = \mathcal{W} \cap M_n(\mathbb{Z})$ of all integer-valued matrices from $\mathcal{W}$ is both a ring and a $\mathbb{Z}$-module.

A matrix ring with this property is called a coherent ring, denoted by $\mathcal{W}' = \langle A_1, \ldots, A_r \rangle_\mathbb{Z}$. 
If we suppress axiom CA4 in the definition of coherent algebra, we obtain what is called a cellular algebra. The term “cellular algebra” can be traced to the Soviet school of algebraic combinatorics, having been introduced by B.Ju. Weisfeiler and A.A. Leman in 1968.

Thus, a cellular algebra is a matrix algebra over $\mathbb{C}$ which satisfies axioms CA1–CA3.
Now we shall formulate coherent algebras in terms of a relational language.

Let $X = \{1, 2, \ldots, n\}$, and let us consider a collection $\mathcal{R} = \{R_1, R_2, \ldots, R_r\}$ of binary relations on $X$. 
If the following conditions hold:

CC1. $R_i \cap R_j = \emptyset$ for $1 \leq i \neq j \leq r$;

CC2. $\bigcup_{i=1}^{r} R_i = X^2$;

CC3. For each $i \in \{1, 2, \ldots, r\}$ there is an $i' \in \{1, 2, \ldots, r\}$ such that $R_i^{t} = R_{i'}$;

CC4. There exists a subset $I' \subseteq \{1, \ldots, r\}$ such that $\bigcup_{i \in I'} R_i = \Delta$, (here $\Delta = \{(x, x) \mid x \in X\}$);

CC5. For each $i, j, k \in \{1, 2, \ldots, r\}$ the number of elements $z \in X$ for which $(x, z) \in R_i$ and $(z, y) \in R_j$ is constant provided that $(x, y) \in R_k$. We denote this constant by $p_{ij}^k$.

$\mathcal{M} = (X, \mathcal{R})$ is a coherent configuration.
Equivalence of axiomatizations

- Given a coherent configuration $\mathcal{M} = (X, R)$, we consider the graph $\Gamma_i = (X, R_i)$ defined by the relation $R_i$ and we let $A_i = A(\Gamma_i)$ be its adjacency matrix.
- In this case $\mathcal{W} = \langle A_1, A_2, \ldots, A_r \rangle$ is indeed a coherent algebra.
- Note a natural correspondence between axioms CA1-CA4 of a coherent algebra and axioms CC1-CC4 given above.
Fifth axiom

Axiom CC5 ensures that each product $A_i A_j$ is a linear combination of the matrices $A_1, \ldots, A_r$. (More precisely, axiom CC5 is equivalent to the fact that a coherent algebra is, by definition, a matrix algebra.)
Equivalence of axiomatizations

Conversely, given a coherent algebra $\mathcal{W}$ we can easily construct a corresponding coherent configuration $M$ by interpreting each matrix $A_i$ as the adjacency matrix of a graph having arc set $R_i$. 
An important class of examples of coherent configurations is provided by the so-called 2-orbits of permutation groups.

This terminology was introduced by Wielandt and is equivalent to the language of centralizer algebras of permutation groups.
2-orbits

- Let \((G, \Omega)\) be a permutation group. We consider the naturally induced action of \(G\) on \(\Omega^2\) as follows: For \((a, b) \in \Omega^2\) and \(g \in G\) we define \((a, b)^g = (a^g, b^g)\).

- The set of orbits of \((G, \Omega^2)\) will be denoted by \(2\text{-orb}(G, \Omega)\) (following Wielandt).

- We shall refer to the elements of \(2\text{-orb}(G, \Omega)\) as 2-orbits of \((G, \Omega)\).
2-orbits

- Given any permutation group \((G, \Omega)\), the pair \((\Omega, 2\text{-orb}(G, \Omega))\) is clearly a coherent configuration.
- One may establish this fact by checking axioms CC1–CC5 directly, although this is not necessary.
- Indeed, one need only observe that \((\Omega, 2\text{-orb}(G, \Omega))\) corresponds to the coherent algebra \(V(G, \Omega)\).
Example 2.1 ($K_4$)

Consider the complete graph $K_4$ with vertex set $X_1 = \{1, 2, 3, 4\}$. We label the edges of $K_4$ by the elements of $X_2 = \{5, 6, 7, 8, 9, 10\}$ in a fixed but arbitrary manner, and we designate by $e(x)$ the edge of $K_4$ which carries the label $x \in X_2$. Then the symmetric group $S_4 = S(X_1)$ acts intransitively on the set $X := X_1 \cup X_2$ with orbits $X_1$ and $X_2$. 
The 2-orbits of this action:

- \( R_1 = \{ (x, x) \mid x \in X_1 \} \), \( R_2 = \{ (x, x) \mid x \in X_2 \} \),
- \( R_3 = \{ (x, y) \mid x, y \in X_1, x \neq y \} \),
- \( R_4 = \{ (x, y) \mid x, y \in X_2, x \neq y, e(x) \cap e(y) \neq \emptyset \} \),
- \( R_5 = \{ (x, y) \mid x, y \in X_2, e(x) \cap e(y) = \emptyset \} \),
- \( R_6 = \{ (x, y) \mid x \in X_1, y \in X_2, x \in e(y) \} \),
- \( R_7 = \{ (x, y) \mid x \in X_1, y \in X_2, x \notin e(y) \} \),
- \( R_8 = \{ (x, y) \mid x \in X_2, y \in X_1, y \in e(x) \} \),
- \( R_9 = \{ (x, y) \mid x \in X_2, y \in X_1, y \notin e(x) \} \).

Resulting from this action, one obtains the coherent configuration \( \mathcal{M} = (X, \mathcal{R}) \), where \( \mathcal{R} = \{ R_1, R_2, \ldots, R_9 \} \).
A non-standard matrix multiplication called *Schur-Hadamard multiplication* (*SH-multiplication*, for short).

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two square matrices of order $n$, and define

$$c_{ij} = a_{ij}b_{ij}, \quad 1 \leq i, j \leq n.$$ 

The matrix $C = (c_{ij})$ is called the *Schur-Hadamard product* of $A$ and $B$ and it is denoted by $C = A \circ B$. 

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**SH-product**

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Coherent algebras

- Every coherent algebra is closed with respect to SH-multiplication.
- By linearity one need only verify this for Schur-Hadamard products of basis matrices.
- As these are \((0,1)\)-matrices, one clearly has \(A_i \circ A_i = A_i\), and \(A_i \circ A_j = O\) for all \(i \neq j\), where \(O\) denotes the matrix of order \(n\) all of whose entries are equal to 0.
Let $\mathcal{W}$ be a coherent algebra, and let $X = (x_{ij}) \in \mathcal{W}$.

For arbitrary $\nu \in \mathbb{C}$ define $Y_\nu(X) = (y_{ij})$ (cross-section of $X$ by $\nu$) by

$$y_{ij} = \begin{cases} 
\nu & \text{if } x_{ij} = \nu, \\
0 & \text{otherwise.} 
\end{cases}$$

Then $Y_\nu(X) \in \mathcal{W}$ for all $\nu \in \mathbb{C}$.
Proof

- If $x_{ij} \neq \nu$ for all $i, j$ then $Y_\nu(X) = O \in \mathcal{W}$.
- So assume $x_{st} = \nu$ and define the matrix $T_1 = X - X \circ A_{k_1}$, where $A_{k_1}$ is the unique basis matrix for which $(A_{k_1})_{st} = 1$.
- Clearly $T_1$ is in $\mathcal{W}$ and has fewer entries equal to $\nu$ than did $X$. 
Proof

- One repeats the procedure by defining
  \[ T_2 = T_1 - T_1 \circ A_{k_2} ( = X - (X \circ A_{k_1} + X \circ A_{k_2}) ) \]
  and so on, until the matrix \( T_q \) obtained on the \( q \)th iteration is free from entries equal to \( \nu \).

- Then one has
  \[ Y_\nu(X) = X \circ A_{k_1} + X \circ A_{k_2} + \cdots + X \circ A_{k_q} \]
  \[ ( = \nu(A_{k_1} + A_{k_2} + \cdots + A_{k_q}) ) , \]
  which is clearly an element of \( \mathcal{W} \).
Corollary

Assume $\nu \neq 0$.

For any matrix $X \in \mathcal{W}$, there exists a subset $K = \{k_1, k_2, \ldots, k_q\}$ of $\{1, 2, \ldots, r\}$ for which

$$\frac{1}{\nu} Y_\nu(X) = \sum_{k_i \in K} A_{k_i}.$$
Proposition 2.1

A subspace $\mathcal{W}$ of $M_n(\mathbb{C})$ is a coherent algebra if and only if $\mathcal{W}$ contains the matrices $I_n$ and $J_n$ and is closed with respect to the operations of matrix multiplication, $SH$-multiplication, and conjugate-transposition.

- Proof will be discussed in the exercise meeting.
Proposition 2.2

Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be two coherent algebras of order $n$. Then their intersection $\mathcal{W} = \mathcal{W}_1 \cap \mathcal{W}_2$ is also a coherent algebra of order $n$.

Proof: this is an immediate corollary of Proposition 2.1.
Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be coherent algebras with $\mathcal{W}_1 \subseteq \mathcal{W}_2$. Then $\mathcal{W}_1$ is called a **coherent subalgebra** of $\mathcal{W}_2$.

Let $A \in M_n(\mathbb{C})$ be arbitrary. The minimal coherent algebra which contains $A$ is called the **coherent algebra generated by** $A$ and will be denoted by $\langle \langle A \rangle \rangle$. 

**Coherent subalgebras**
In similar fashion, one more generally defines the coherent algebra \( \langle \langle A_1, A_2, \ldots, A_k \rangle \rangle \) generated by the matrices \( A_1, A_2, \ldots, A_k \).

Clearly, if \( \{ A_1, A_2, \ldots, A_r \} \) is the standard basis of a coherent algebra \( \mathcal{W} \), then one has
\[
\langle \langle A_1, A_2, \ldots, A_r \rangle \rangle = \langle A_1, A_2, \ldots, A_r \rangle = \mathcal{W}.
\]
The definitions given above make sense since, by Proposition 2.2, any intersection of coherent algebras is again a coherent algebra.

\{\langle A \rangle \} may be interpreted as the intersection of all coherent subalgebras of $M_n(\mathbb{C})$ which contain $A$. 
Coherent subalgebras

- Let $C(A)$ denote the set of all such subalgebras.
- $C(A) \neq \emptyset$ since it contains the coherent algebra $V_{\mathbb{C}}(\{e\}, \{1, 2, \ldots, n\}) = M_n(\mathbb{C})$.
- Thus

$$\langle\langle A \rangle\rangle = \bigcap_{\mathcal{W} \in C(A)} \mathcal{W}.$$
Computation

- At the moment we do not wish to discuss in evident form, various algorithms for constructing \( \langle \langle A \rangle \rangle \).
- Nonetheless, we mention an extremely effective algorithm which accomplishes this, namely *Weisfeiler-Leman stabilization*.
- The following examples demonstrate how \( \langle \langle A \rangle \rangle \) can be constructed using certain tricks based mainly on the Schur-Wielandt principle.
Let $\Gamma$ be the above graph and let $A = A(\Gamma)$ be its adjacency matrix.

We wish to determine $W = \langle\langle A \rangle\rangle$.

Clearly, $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$
Example 2.2 (cont.)

- All we know from the outset is that $A, I_4, J_4 \in \langle\langle A \rangle\rangle$.

- Since $\langle\langle A \rangle\rangle$ is a linear space, we get $\bar{A} \in \langle\langle A \rangle\rangle$ where

$$\bar{A} = J_4 - I_4 - A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$
Now, since $\langle \langle A \rangle \rangle$ is closed under matrix multiplication, it must contain the two additional matrices:

- $A^2 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ and
- $\bar{A}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$. 
Applying the Schur-Wielandt principle, we further obtain $B_1, B_2 \in \langle \langle A \rangle \rangle$, where

$$B_1 = \frac{1}{3} Y_3(A^2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$B_2 = \frac{1}{2} Y_2(\overline{A}^2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
Example 2.2 (cont.)

- Redefining \( B_3 = A \) and \( B_4 = \bar{A} \), we now have four \((0, 1)\)-matrices \( B_1, B_2, B_3, B_4 \) of \( \langle\langle A \rangle\rangle \) which have mutually disjoint support and sum to \( J_4 \).

- However, these four matrices do not constitute a basis for \( \langle\langle A \rangle\rangle \), since \( \langle\langle A \rangle\rangle \) must additionally contain all products of the form \( B_iB_j \) for \( i, j \in \{1, 2, 3, 4\} \).
Example 2.2 (cont.)

In particular, $B_1 B_3 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $B_3 B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ must be elements of $\langle\langle A \rangle\rangle$, resulting in a “desymmetrization” of the matrix $A$. 
This gives a new set of five matrices:

\[ C_1 = B_1, \quad C_2 = B_2, \quad C_3 = B_1A, \quad C_4 = AB_1, \quad C_5 = B_4, \]

which turns out to provide the desired basis for \( \langle \langle A \rangle \rangle \).

Indeed, all matrix products of the form \( C_iC_j \)
are elements of \( \langle C_1, \ldots, C_5 \rangle \), as is readily verifiable from the following table of products.
Example 2.2 (cont.)

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>$C_1$</td>
<td>$O$</td>
<td>$C_3$</td>
<td>$O$</td>
<td>$O$</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$O$</td>
<td>$C_2$</td>
<td>$O$</td>
<td>$C_4$</td>
<td>$C_5$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$O$</td>
<td>$C_3$</td>
<td>$O$</td>
<td>$3C_1$</td>
<td>$2C_3$</td>
</tr>
<tr>
<td>$C_4$</td>
<td>$C_4$</td>
<td>$O$</td>
<td>$C_2 + C_5$</td>
<td>$O$</td>
<td>$O$</td>
</tr>
<tr>
<td>$C_5$</td>
<td>$O$</td>
<td>$C_5$</td>
<td>$O$</td>
<td>$2C_4$</td>
<td>$2C_2 + C_5$</td>
</tr>
</tbody>
</table>
Example 2.2 (cont.)

- To this point we have shown that \( \langle C_1, \ldots, C_5 \rangle \) is a matrix algebra in the usual sense.

- As the \( C_i \)'s have mutually disjoint support, \( \langle C_1, \ldots, C_5 \rangle \) is closed under SH-multiplication.

- As \( C_1^t = C_1, C_2^t = C_2, C_3^t = C_4, C_4^t = C_3, \) and \( C_5^t = C_5 \), it is closed under conjugate-transposition.
Finally, as $J_4 = C_1 + C_2 + C_3 + C_4 + C_5$ and $I_4 = C_1 + C_2$, we conclude that $\langle C_1, \ldots, C_5 \rangle$ is a coherent algebra.

By construction,

$A \in \langle C_1, \ldots, C_5 \rangle \subseteq \langle \langle A \rangle \rangle$.

Hence, $\langle C_1, \ldots, C_5 \rangle = \langle \langle A \rangle \rangle$. 

Now we will proceed with another example of a coherent algebra, using essentially that it is also a centralizer algebra.

Namely, we start with the octahedron $O$. 
Example 2.3 (cont.)

Let \( g = (2, 3, 4, 5) \), \( h = (1, 2, 3)(4, 5, 6) \).

Check that \( g, h \in \text{Aut}(\mathcal{O}) \).

Check that \( H = \langle g, h \rangle \) is a transitive group of degree 6.

\(|H| = 24\).

We wish to construct \( \mathcal{W} = V(H, [1, 6]) \).
Example 2.3 (cont.)

- Construct $\mathcal{W}_1 = V(\langle g \rangle, [1, 6])$ and $\mathcal{W}_2 = V(\langle h \rangle, [1, 6])$.

- Check that both algebras, by coincidence, have rank 12.

- They are presented by matrices

$$A_1 = \begin{pmatrix}
1 & 4 & 4 & 4 & 4 & 5 \\
6 & 2 & 7 & 8 & 9 & 10 \\
6 & 9 & 2 & 7 & 8 & 10 \\
6 & 8 & 9 & 2 & 7 & 10 \\
6 & 7 & 8 & 9 & 2 & 10 \\
11 & 12 & 12 & 12 & 12 & 3
\end{pmatrix} \quad A_2 = \begin{pmatrix}
13 & 14 & 15 & 16 & 17 & 18 \\
15 & 13 & 14 & 18 & 16 & 17 \\
14 & 15 & 13 & 17 & 18 & 16 \\
19 & 20 & 21 & 24 & 22 & 23 \\
21 & 19 & 20 & 23 & 24 & 22 \\
20 & 21 & 19 & 22 & 23 & 24
\end{pmatrix}$$
Example 2.3 (cont.)

- Now $\mathcal{W} = \mathcal{W}_1 \cap \mathcal{W}_2$ is the intersection of algebras $\mathcal{W}_1$ and $\mathcal{W}_2$.
- To get description of $\mathcal{W}$ consider auxiliary bipartite graph $\Delta$, vertices of which correspond to entries of $A_1$ and $A_2$ ($12 + 12$ vertices).
- Two vertices $x, y$ are adjacent if there exists a cell occupied in $A_1$ and $A_2$ by $x$ and $y$ respectively.
Example 2.3 (cont.)

\[
\begin{array}{ccccccccccccc}
13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\
\end{array}
\]
Example 2.3 (cont.)
Example 2.3 (cont.)

- Connectivity components of the graph \( \Delta \) define matrix for color graph \( \mathcal{W} = \mathcal{W}_1 \cap \mathcal{W}_2 \).
- It has rank 3.
- Different colors correspond to vertices (loops), edges and non-edges of \( \mathcal{O} \).
- \( A = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 3 \\ 2 & 1 & 2 & 3 & 2 & 2 \\ 2 & 2 & 1 & 2 & 3 & 2 \\ 2 & 3 & 2 & 1 & 2 & 2 \\ 2 & 2 & 3 & 2 & 1 & 2 \\ 3 & 2 & 2 & 2 & 2 & 1 \end{pmatrix} \).
Schurian coherent algebras

We call a coherent algebra $\mathcal{W}$ Schurian if it coincides with the centralizer algebra of a suitable permutation group.

In the previous example, $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}$ are all Schurian algebras of ranks 12, 12 and 3 respectively.

Otherwise, $\mathcal{W}$ is called non-Schurian.
The name goes back to Schur, who was considering (1933) Schur rings, a special kind of coherent algebras which are simultaneously group algebras (rings).

Schur believed that all Schur rings are coming from a centralizer algebra of a suitable permutation group.
Non-Schurian

- The smallest counterexamples to the conjecture of Schur in wider context, non-Schurian association schemes, exist on 15, 16, 18 vertices.
- The desired property may be established by group-theoretical or combinatorial arguments.
- In any case this is quite a routine activity (computer is very helpful).
Example 2.4 (DRT on 15 vertices)

\[ \Gamma = \]
Example 2.4 (cont.)

- Doubly regular tournament \( \Gamma \), its opposite graph \( \Gamma^t \) and the reflexive relation form coherent configuration of rank 3.
- The parameters are \((15, 7, 3, 4)\), that is \(A(\Gamma)^2 = 4A(\Gamma) + 3A(\Gamma)^t\)
- It is non-Schurian: \(\text{Aut}(\Gamma)\) has order 21, while \(\Gamma\) has \(\frac{15 \cdot 14}{2} = 105\) arcs.
Example 2.4 (cont.)

- Alternative (combinatorial) proof: count the number of induced subgraphs with 5 vertices of prescribed isomorphism types;
- Distinguish arcs of Γ, using these invariants.
Main references

Klin, Mikhail; Rücker, Christoph; Rücker, Gerta; Tinhofer, Gottfried. Algebraic combinatorics in mathematical chemistry. Methods and algorithms. I. Permutation groups and coherent (cellular) algebras. MATCH No. 40 (1999), 7–138.

Thank You!