Maps, Hypermaps and Related Topics

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Preface

Regular maps and hypermaps are cellular decompositions of closed surfaces exhibiting the highest possible number of symmetries. The five Platonic solids present the most familiar examples of regular maps. The great dodecahedron, a 5-valent pentagonal regular map on the surface of genus 5 discovered by Kepler, is probably the first known non-spherical regular map. Modern history of regular maps goes back at least to Klein (1878) who described in [109] a regular map of type $(3,7)$ on the orientable surface of genus 3. In its early times, the study of regular maps was closely connected with group theory as one can see in Burnside's famous monograph [37], and more recently in Coxeter's and Moser's book [53] (Chapter 8). The present-time interest in regular maps extends to their connection to Dyck's triangle groups, Riemann surfaces, algebraic curves, Galois groups and other areas. Many of these links are nicely surveyed in the papers of Jones [99] and Jones and Singerman [98].

In the present booklet we develop a theory of maps and hypermaps. Symmetries of maps from several points of view are investigated. This booklet can serve as an introductory course to the combinatorial theory of maps and hypermaps. It is equipt with exhaustive, but still incomplete list of literature for further reading on the subject.

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Chapter 1

Maps and hypermaps

1.1 Graphs and group actions

A *graph* is an ordered quadruple $X = (D, V; I, \lambda)$ where D is a set of *darts*, V is a nonempty set of *vertices*, which is required to be disjoint from D, I is a mapping of D onto V, called *the incidence function*, and λ is an involutory permutation of D, called *the dart-reversing involution*. For convenience or if λ is not explicitly specified we sometimes write x^{-1} instead of λx. Intuitively, the mapping I assigns to each dart its *initial vertex*, and the permutation λ interchanges a dart and its *reverse*. The *terminal vertex* of a dart x is the initial vertex of λx. The set of all darts initiated at a given vertex u is denoted by D_u, called the *neighborhood* of u. The cardinality $|D_u|$ of D_u is the *valency* of the vertex u. The orbits of λ are called *edges*; thus each dart determines uniquely its *underlying edge*. An edge is called a *semiedge* if $\lambda x = x$, a *loop* if $\lambda x \neq x$ and $I\lambda x = Ix$, and it is called a *link* otherwise. We represent a graph, as defined above, by a topological space in the usual way as a 1-dimensional CW-complex. Note that from a topological point of view a semiedge is identical with a pendant edge except that its free endpoint is not listed as a vertex.

There is an alternative definition of our graph discovered in context of theoretical physics (see Getzler, E., Kapranov, M.M., Modular operads. Compositio Math. 110(1998), 65–126). They introduce a graph as a triple $(D; \sim, \lambda)$, where D is the set of darts, $\sim$ is an equivalence relation on D and λ is an involutory permutation acting on D. The equivalence relation $\sim$ gives rise to a decomposition of D into equivalence classes $[x]$, $x \in D$. We may set $V = \{[x] \mid x \in D\}$ as the quotient set and $I : x \to [x]$. On the other hand, for a given graph $X = (D, V; I, \lambda)$ the corresponding equivalence relation is defined by the decomposition $D = \cup_{v \in V} I^{-1}(v)$. Thus the two definitions are equivalent.

Our definition of a graph is more convenient for purposes of this book than the classical definition of a graph consisting of vertices V and edges $E \subseteq 2^V$ of cardinality two, since in topological graph theory multiple edges, loops or semiedges naturally arise. In what follows our investigation is restricted to unoriented graphs.

In what follows we introduce basic notions and statements without proofs. For a more detailed explanation the reader is referred to [54, 201].

Let Ω be a set. By $\mathrm{Sym_L}\,\Omega$ and $\mathrm{Sym_R}\,\Omega$ we denote the *left* and the *right symmetric group* on a (nonempty) set Ω, respectively. The right symmetric group on the set $\{0, 1, \ldots, n-1\}$ is commonly denoted by S_n.

A (right) *action* of a group G on a set Ω is defined by a function $\sigma : \Omega \times G \to \Omega$, with the common notation $\sigma(z, g) = z \cdot g$, such that $z \cdot 1_G = z$ and $z \cdot (gh) = (z \cdot g) \cdot h$ for any $z \in \Omega$ and $g, h \in G$. Note that the function $z \mapsto z \cdot g$ is a bijection on Ω for each g in G. Therefore, one may equivalently define a group action of G on a set Ω as a group homomorphism $G \to \mathrm{Sym_R}\,\Omega$. When a group G acts on a set Ω, we call Ω a *G-space.*

The left action is defined in a similar manner, as a function $G \times \Omega \to \Omega$, or a group homomorphism $G \to \mathrm{Sym_L}\,\Omega$.

The subgroup $G_z = \{g \in G \mid z \cdot g = z\}$ is called the *stabilizer* of $z \in \Omega$ under the action of G. The set $\{z \cdot g \in \Omega \mid g \in G\}$ is called the *orbit* of z. An action of a group is called *semi-regular* if all the stabilizers are trivial. It is *transitive* if it has only one orbit, that means for any two $z_2, z_1 \in \Omega$ there is $g \in G$ such that $z_2 = z_1 \cdot g$. The stabilizers in a transitive action are all conjugate, thus the structure of G_z does not depend on the choice of the point z. A transitive and semi-regular action is called *regular*.

Let G act on Ω. The core $\mathrm{Core}_\Omega(G) = \cap_{z \in \Omega} G_z \leq G$ is formed by the elements of G letting all the elements fixed.

If a G-action on Ω is defined by a homomorphism $\alpha : G \to S_\Omega$, then the core $\mathrm{Core}_\Omega(G)$ is nothing but the kernel of α. Hence, the core is a normal subgroup of G. If $\mathrm{Core}_\Omega(G) = 1$ then the action of G on Ω is *faithful*. If G acts faithfully on Ω then the assignment $g \mapsto \alpha_g$, where $\alpha_g(x) = x \cdot g$, is a monomorphism $G \to S_\Omega$, called a *permutation representation* of G on Ω and one can identify G with a subgroup of S_Ω. In a general case, $G/\mathrm{Core}_\Omega(G)$ is isomorphic to a subgroup of S_Ω. The size $|\Omega|$ is called a degree of the permutation representation. A group of permutations G is of degree n if $G \leq S_\Omega$ and $|\Omega| = n$.

Let Ω_i be a G_i-space for $i = 1, 2$. We call a group isomorphism $f : G_1 \to G_2$ *admissible* or *consistent* with the respective actions if there exists a bijection $\phi : \Omega_1 \to \Omega_2$ such that $\phi(z \cdot g) = \phi(z) \cdot f(g)$ for all $z \in \Omega$ and $g \in G$, that is, the diagram

$$\begin{tikzcd}
\Omega_1 \times G_1 \arrow[r] \arrow[d, "\phi \times f"'] & \Omega_1 \arrow[d, "\phi"] \\
\Omega_2 \times G_2 \arrow[r] & \Omega_2
\end{tikzcd}$$

commutes. For a G_i-space Ω_i $(i = 1, 2)$, if there is an admissible isomorphism $f : G_1 \to G_2$, then two actions are called *isomorphic* and the pair (ϕ, f) is called an *isomorphism* of the actions.

As a special case, let $G_1 = G_2 := G$, and $f = id_G$ be the identity. We define a *morphism* from Ω_1 to Ω_2 to be a function $\phi : \Omega_1 \to \Omega_2$ such that $\phi(z \cdot g) = \phi(z) \cdot g$ for all $g \in G$ and all $z \in \Omega_1$. Morphisms of G-spaces are also called *equivariant*. If an equivariant morphism $\phi : \Omega_1 \to \Omega_2$ is bijective then its inverse is also equivariant, and ϕ is an *equivalence* or *G-isomorphism*. The two G-spaces Ω_1 and Ω_2 are isomorphic.

Transitive permutation representations are of crucial importance for investigation of highly transitive combinatorial structures. The following two statements are well-known, see [DiMo, Robinson p.35].

Let $H \leq G$ be a subgroup. For each $g \in G$ define τ_g by $\tau_g : Hx \mapsto Hxg$. Then τ defines a transitive action of G on the right cosets (by right translation) with kernel $H_G = \bigcap\limits_{x \in A} g^{-1}Hg$. It follows that $p : g \mapsto \tau_g$ is a transitive permutation representation of the quotient group G/H_G. In particular, if $H = 1$ then the action is regular, and the statement is known as a Cayley Theorem.

The next theorem shows that all transitive actions can be obtained as above up to equivalence.

Theorem 1.1. *Any transitive action of a finite group G on a set Ω is equivalent to the action of G on the right cosets of some subgroup H of G by right translation.*

Since the kernel of the action of G on the right cosets of H is the core $\mathrm{Core}_G(H) = \bigcap_{g\in G} g^{-1}Hg$ for some $H \le G$, it follows that for an abelian group G, the right translation of G is the only (faithful) transitive representation of G up to equivalence.

Definition 1.1. Let $G \le S_\Omega$, $\Delta \subseteq \Omega$. If for any $g \in G$, either $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$, then we call Δ a *block* of G.

Obviously, Ω, $\emptyset$ and one-element subset $\{\alpha\}$ are blocks of G; they are called the *trivial blocks*.

Definition 1.2. Let $G \le S_\Omega$. If G has a nontrivial block Δ, then G is called an *imprimitive group*, and Δ is called an *imprimitivity set*. Otherwise, we say G a *primitive group*. When $|\Omega| = 2$, we make the following convention: if G is identity, we also call G imprimitive.

In other words, a permutation group G on a set Ω is imprimitive if Ω may be split into nontrivial blocks (of the same size if the action is transitive) such that each $g \in G$ maps a block into a block. For example, all intransitive groups are imprimitive.

Obviously, every primitive group is transitive. Until the end of the section, we assume that G is a transitive permutation group on Ω.

Proposition 1.2. *Let $G \le S_\Omega$. Assume that G is transitive but not primitive. Let Δ be an imprimitivity set of G. Write $H = G_{\{\Delta\}} = \{g \in G \mid \Delta^g = \Delta\}$. Then*

(1) *the subgroup H is transitive on Δ.*

(2) *Assume that $G = \bigcup_{r\in R} Hr$ is a decomposition of G into a union of right cosets of H. Then $\Omega = \bigcup_{r\in R} \Delta^r$, and for any $r \ne r'$ in R, we have $\Delta^r \cap \Delta^{r'} = \emptyset$.*

(3) $|\Delta|$ *divides* $|\Omega|$.

The set $\{\Delta^r \mid r \in R\}$ is called a *complete system of imprimitivity.*

Corollary 1.3. *A transitive group of prime degree is primitive.*

The kernel K of an action of G on Ω is a subgroup formed by elements of G fixing each block of imprimitivity in Δ.

Proposition 1.4. *The kernel K of an action of G with respect to imprimitivity system Δ is a normal subgroup of G*

The following theorem is well-known in a permutation group theory and will be used later.

Theorem 1.5. (see [54, Theorem 4.2A]) *Let G be a transitive subgroup of $Sym(\Omega)$, and let C be the centralizer of G in $Sym(\Omega)$. Then*

(1) *C is semiregular and $C \cong N_G(G_x)/G_x$ for any $x \in \Omega$.*
(2) *C is transitive if and only if G is regular,*
(3) *if C is transitive then it is conjugate to G in $Sym(\Omega)$ and hence C is regular.*

So far we have considered action of a group on its subgroups by left and right translations. There is another important action of a group on its subgroups, namely the action by conjugation. If $H < G$ is a subgroup then the stabilizer $N_G(H) = \{g \in G | H = H^g = g^{-1}Hg\}$ is called the *normalizer* of H in G. Clearly, the normalizer is a normal subgroup of G containing H. Moreover, it is the least subgroup of H with this property. Orbits of the action of G are called *conjugacy classes*. Two groups H and H^g in the same conjugacy class are called *conjugate* subgroups. Counting conjugacy classes of subgroups of finite index in a finitely generated group is closely related to enumeration of isomorphism classes of combinatorial objects.

1.2 Category of maps and ormaps

Recall that a topological map is a (finite) 2-cell decomposition of a compact connected surface. Maps are usually described by 2-cell embeddings of finite connected topological graphs into surfaces. An embedding $i : X \to S$ of a graph X into a surface S is called a *2-cell embedding* if the connectivity components $S - i(X)$ are homeomorphic to open discs (or equivalently, to Euclidean planes). The connectivity components will be called *faces*. Each face is bounded by a closed walk in X. Note that a boundary walk may not be a simple cycle. For instance, if $T \to S$ is an embedding of a tree into the sphere then there is a unique boundary walk traversing each edge of T twice. Recall that compact connected surfaces are characterized by two invariants: orientability and the Euler characteristic (genus). More precisely, the invariants $g \geq 0$, $\tilde{g} \geq 1$ are called the (orientable) genus of S, non-orientable genus of S, respectively. The following statement is well-known.

Theorem 1.6. (Euler-Poincare formula) *Let $\mathcal{M}$ be a map on a surface S with v vertices, e edges and f faces (and no semiedges). Then*

$$v - e + f = \chi(S) = \begin{cases} 2 - 2g & \text{if } S \text{ has genus } g, \\ 2 - \tilde{g} & \text{if } S \text{ has non-orientable genus } \tilde{g}. \end{cases}$$

The embedding is determined by the graph X and a finite collection $\mathcal{F}$ of boundary walks. We say that an edge $\{x, \lambda x\}$ is double-covered by $\mathcal{F}$ if either x, or λx appear twice in walks of $\mathcal{F}$, or both x and λx appear in walks walks of $\mathcal{F}$ exactly ones; and it is one-covered if exactly one of x, λx appears in a walk of $\mathcal{F}$. Given set of closed walks in a graph X we say that two darts x, y form an angle $\alpha = (x, y) = \overrightarrow{xy}$, or an angle $\alpha = (y, x) = \overrightarrow{yx}$, if $I(x) = I(y)$ and there exists W in $\mathcal{F}$ such that either λx, y, or x, λy are consecutive in W. Two darts x, y forming an angle $\overrightarrow{yx}$ or $\overrightarrow{xy}$ will be called elementary $\mathcal{F}$-equivalent. Generally, two darts x and y will be called $\mathcal{F}$-equivalent if there is a sequence of darts $x_0, x_1, \ldots, x_k$ such that $x = x_0$, $y = x_k$ and x_{i-1},x_i are elementary $\mathcal{F}$-equivalent for $i = 1, 2, \ldots, k$.

The following statement gives a simple characterization of sets of walks giving rise to a map.

Lemma 1.7. *Let $i : X \to S$ be a 2-cell embedding of a connected finite graph into a surface S. Let $\mathcal{F}$ be a collection of boundary walks and let $X = (D, V; I, \lambda)$ be the associated combinatorial graph. Then*

(1) *the edges of size two are double-covered by $\mathcal{F}$,*

(2) *semiedges are one-covered by $\mathcal{F}$,*

(3) *x and y are $\mathcal{F}$-equivalent if and only if $I(x) = I(y)$.*

Conversely, for a collection $\mathcal{F}$ satisfying $(1), (2)$ *and* (3) *there exists a 2-cell embedding $i : X \to S$ whose faces are bounded by walks of $\mathcal{F}$.*

Corollary 1.8. *A closed walk of a graph X without semiedges determines a one face 2-cell embedding if and only if each edge is traversed twice (formes a doubly Eulerian trail).*

As far as maps on surfaces are concerned, there are two essentially different approaches to their combinatorial description. The first approach, based on a rotation-involution pair acting on darts, involves the orientation of the supporting surface and so is suitable only for maps on orientable surfaces [73, 94]. The corresponding combinatorial structure is called a combinatorial (or, sometimes, algebraic) oriented map. The other approach, using three involutions acting on mutually incident (vertex, edge, face)-triples called flags, is orientation insensitive and thus allows us to represent maps on non-orientable surfaces as well [95]. The resulting combinatorial structure will be called a combinatorial nonoriented map. Accordingly, we shall usually employ the same notation for a topological map and for the corresponding combinatorial structure on it.

We start with necessary definitions concerning oriented maps. By a (combinatorial) *oriented map* we henceforth mean a triple $(D;\, R, L)$ where $D = D(\mathcal{M})$ is a non-empty finite set of *darts*, and R and L are two permutations of D such that L is an involution and the group $\mathrm{Mon}(\mathcal{M}) = \langle R, L\rangle$ acts transitively on D. The group $\mathrm{Mon}(\mathcal{M})$ is called the oriented *monodromy group* of $\mathcal{M}$. The permutation R is called the *rotation* of $\mathcal{M}$. The orbits of the group $\langle R\rangle$ are the *vertices* of $\mathcal{M}$, and elements of an orbit v of $\langle R\rangle$ are the darts *radiating* (or *emanating*) from v, that is, v is their initial vertex. The cycle of R permuting the darts emanating from v is the *local rotation* R_v at v. The permutation L is the *dart-reversing involution* of $\mathcal{M}$, and the orbits of $\langle L \rangle$ are the *edges* of $\mathcal{M}$. The orbits of $\langle RL \rangle$ define the face-boundaries of $\mathcal{M}$. The incidence between vertices, edges and faces is given by nontrivial set intersection. The vertices, darts and the incidence function define the *underlying graph* $\mathcal{M}$, which is always connected due to the transitive action of the monodromy group. It is easily seen that the cycles of RL satisfy the conditions (1),(2) and (3) in Lemma 1.7, hence with each combinatorial map there is an associated topological map.

An oriented map can be equivalently described as a pair $(\mathcal{G}; R)$ where $\mathcal{G} = (D, V; I, L)$ is a connected graph and R is a permutation of the dart-set of $\mathcal{G}$ cyclically permuting darts with the same initial vertex, that is, $IR(x) = I(x)$ for every dart x of $\mathcal{G}$.

Combinatorial nonoriented maps are built from three involutions acting on a non-empty finite set F of *flags* [95]. Flags can be viewed as the angles of the associated topological map. Given map $\mathcal{M}$ there are three involutions acting as follows: $\rho(\overrightarrow{xy}) = \overrightarrow{yx}$, $\tau(\overrightarrow{xy}) = \overrightarrow{xz}$, $\lambda(\overrightarrow{xy}) = \overrightarrow{x^{-1}z}$. Given dart x, z is the unique dart $z \neq y$ forming an angle with x (x^{-1}). A (combinatorial) *nonoriented map* is a quadruple $(F;\ \lambda, \rho, \tau)$ where λ, ρ and τ are fixed-point free involutory permutations of $F = F(\mathcal{M})$ called the *longitudinal*, the *rotary* and the *transversal involution*, respectively, which satisfy the following conditions:

(i) $\lambda\tau = \tau\lambda$; and

(ii) the group $\langle \lambda, \rho, \tau \rangle$ acts transitively on F.

This group is the nonoriented *monodromy group* $\mathrm{Mon}(\mathcal{M})$ of $\mathcal{M}$.

We define the *vertices* of $\mathcal{M}$ to be the orbits of the subgroup $\langle \rho, \tau \rangle$, the *edges* of $\mathcal{M}$ to be the orbits of $\langle \lambda, \tau \rangle$, and the *face-boundaries* to be the orbits of $\langle \rho, \lambda \rangle$ under the action on F, the incidence being given by nontrivial set intersection. Note that each orbit z of $\langle \lambda, \tau \rangle$ has cardinality 2 or 4 according to whether z is a semiedge or not.

Example 1.1. *The underlying graph of the tetrahedron is the complete graph $K_4 = (D, V; I, L)$ on 4 vertices. We may set $V = \{1, 2, 3, 4\}$,*

$$D = \{12, 21, 13, 31, 14, 41, 23, 32, 24, 42, 34, 43\},$$

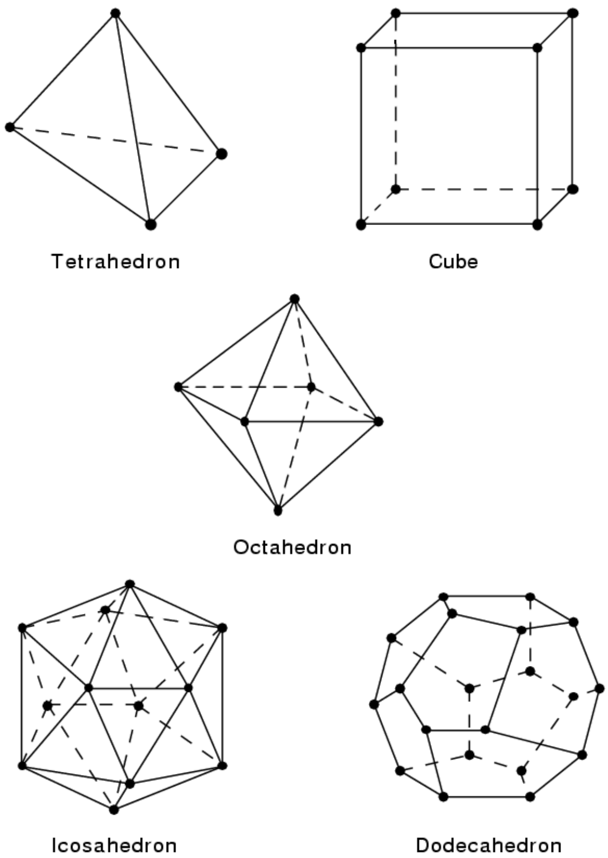


Figure 1.1: The five Platonic solids

$L(ij) = ji$ and $I(ij) = i$, for any $ij \in D$. Then the rotation at vertices compatible with the counterclockwise global orientation in Figure 1.2 is

$$R = (12, 13, 14)(23, 21, 24)(31, 32, 34)(41, 43, 42).$$

Vice-versa, having R we can identify the triangular faces of the map $(D; R, L)$ via cycles of the permutation

$$RL = (12, 24, 41)(21, 13, 32)(31, 14, 43)(23, 34, 42).$$

Figure 1.7 shows the same topological map (the tetrahedron) described *by means of* 3 *involutions acting on* 24 *flags as a map* $(F; \lambda, \rho, \tau)$ *in the category of Maps.*

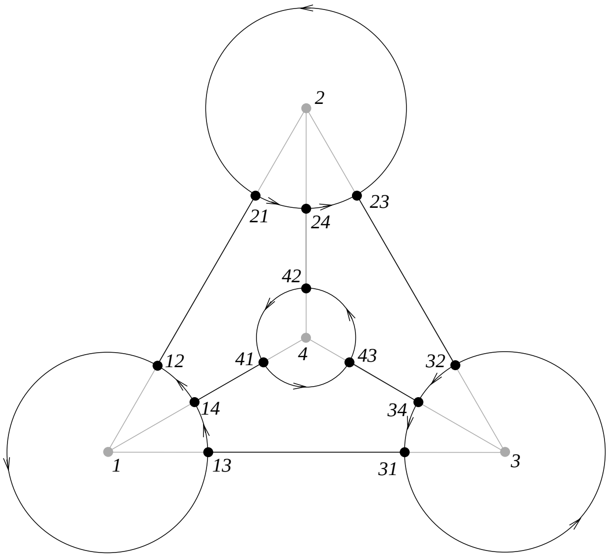


Figure 1.2: The tetrahedron described as an oriented map $(D; R, L)$

The meaning of the condition on λ, ρ, τ requiring these involutions to be fixed-points free becomes clear, if one decides to extend our theory onto *maps on surfaces with boundary*. This was done by Bryant and Singerman in [36]. To do the generalisation we have to allow fixed points of λ,ρ,τ. The underlying surface of a map has a non-empty boundary if and only if at least one of λ,ρ,τ fixes a flag. In fact, the category of nonoriented maps is not complete if we do not consider maps on surfaces with boundary, since a homomorphic image of a map on a closed surface can be a map on a surface with a non-empty boundary. As an example, consider an embedding of a cycle in the sphere and its quotient on the disk given by the reflection interchanging the two faces of the map and fixing the embedded graph point-wise.

Clearly, the even-word subgroup $\langle \rho\tau, \tau\lambda \rangle$ of $\mathrm{Mon}(\mathcal{M})$ has always index at most two. If the index is two, then $\mathcal{M}$ is said to be *orientable*.

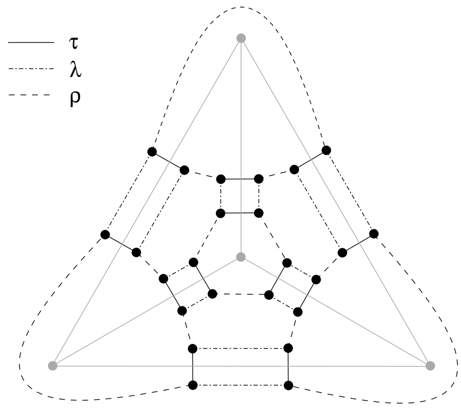


Figure 1.3: The tetrahedron described as a map $(F; \lambda, \rho, \tau)$

With every oriented map $(D;\, R, L)$ we associate the *corresponding nonoriented map* $\mathcal{M}^\natural = (F^\natural;\, \lambda^\natural, \rho^\natural, \tau^\natural)$ by setting $F^\natural = D \times \{1, -1\}$ and defining for a flag $(x, j) \in D \times \{1, -1\}$:

$$\lambda^\natural(x, j) = (L(x), -j), \quad \rho^\natural(x, j) = (R^j(x), -j), \quad \text{and} \quad \tau^\natural(x, j) = (x, -j).$$

Conversely, from an orientable nonoriented map $\mathcal{M} = (F;\, \lambda, \rho, \tau)$ we can construct a pair of oriented maps $\mathcal{M}' = (D;\, R, L)$ and $\mathcal{M}'' = (D;\, R^{-1}, L)$ that are the *mirror image* of each other. We take D to be the set F/τ of orbits of τ on F. Let us denote by $F^+ \subset F$ one of the two orbits induced by the action of the even-word subgroup of $\mathrm{Mon}(\mathcal{M})$. For a dart $\{z, \tau(z)\} = [z]$, where $z \in F^+$, we set $R([z]) = [\rho\tau(z)]$ and $L([z]) = [\lambda\tau(z)]$. Instead of R we could have taken the rotation $R'([z]) = [\tau\rho(z)]$, but since $R' = R^{-1}$ we get nothing but the mirror image – as expected.

Test of orientability

Let $\mathcal{M} = (F;\, \lambda, \rho, \tau)$ be an nonoriented map. We want to determine whether the respective supporting surface S is orientable, or not. The following simple algorithm is well-known, see [73, 151]. Consider the associated 3-valent graph $\mathcal{G} = \mathcal{G}(\lambda, \rho, \tau)$, whose set of vertices is F, each vertex $f \in F$ is incident with darts (f, λ), (f, ρ) and (f, τ) and

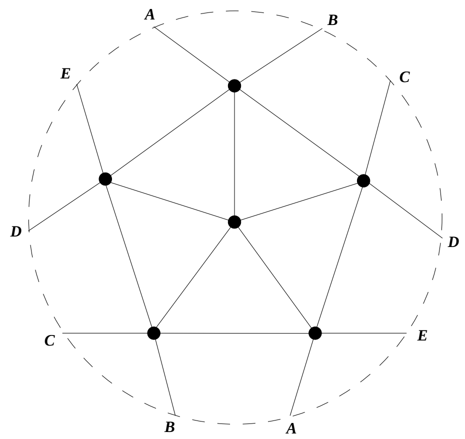


Figure 1.4: K_6 in the Projective plane

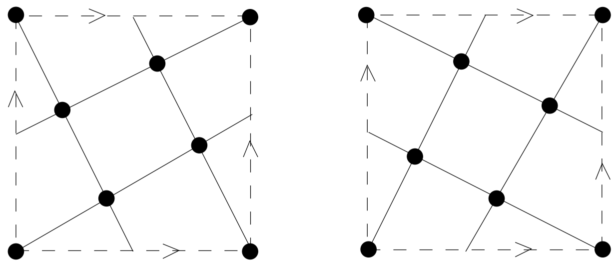


Figure 1.5: Regular embedding of K_5 in the torus appears in two 'enantiomers', in the category of ORMAPS they are not isomorphic.

the dart reversing involution takes $(f, \lambda) \mapsto (\lambda f, \lambda)$, $(f, \rho) \mapsto (\rho f, \rho)$, $(f, \tau) \mapsto (\tau f, \tau)$. Note that $\mathcal{G}$ is nothing but the dual of the barycentric subdivision of $\mathcal{M}$. The result follows.

Proposition 1.9. *A map $\mathcal{M} = (F;\, \lambda, \rho, \tau)$ is orientable if and only if the associated 3-valent graph $\mathcal{G}$ is bipartite.*

Exercises

1.2.1. Prove that two 4-gonal embeddings of K_5 depicted on Figure 11 are not isomorphic in the category of Ormaps, but they are isomorphic in the category of Maps.

1.2.2. How many isomorphism classes of oriented embeddings has K_4? For each class determine a rotation and genus.

1.2.3. There is a pentagonal embedding of the Petersen graph P in the projective plane such that faces are bounded by simple cycles. Is there an embedding of P in torus whose faces are bounded by simple cycles?

1.3 Homomorphisms of maps

Let $\mathcal{M}_1 = (D_1; R_1, L_1)$ and $\mathcal{M}_2 = (D_2; R_2, L_2)$ be two oriented maps. A *homomorphism* $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ of oriented maps is a mapping $\varphi : D_1 \to D_2$ such that

$$\varphi R_1 = R_2\varphi \quad \text{and} \quad \varphi L_1 = L_2\varphi.$$

Analogously, a *homomorphism* $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ of nonoriented maps $\mathcal{M}_1 = (F_1; \lambda_1, \rho_1, \tau_1)$ and $\mathcal{M}_2 = (F_2; \lambda_2, \rho_2, \tau_2)$ is a mapping $\varphi : F_1 \to F_2$ such that

$$\varphi\lambda_1 = \lambda_2\varphi, \quad \varphi\rho_1 = \rho_2\varphi \quad \text{and} \quad \varphi\tau_1 = \tau_2\varphi.$$

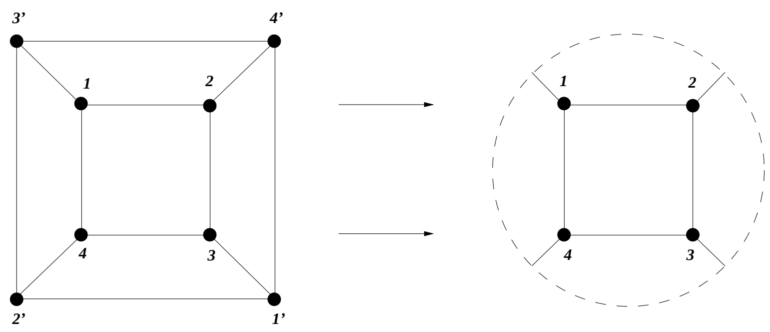


Figure 1.6: Cube smoothly covering its halved quotient in the projective plane.

The properties of homomorphisms of both varieties of maps are similar except that homomorphisms of nonoriented maps ignore orientation. Every map homomorphism induces an epimorphism of the corresponding monodromy groups.

Lemma 1.10. *Let $p\ (F_1;\ \lambda_1, \rho_1, \tau_1) \to (F_2;\ \lambda_2, \rho_2, \tau_2)$ be a map homomorphism. Then the assignment $\lambda_1 \mapsto \lambda_2$, $\rho_1 \mapsto \rho_2$, $\tau_1 \mapsto \tau_2$ extends to an epimorphism p^* called the induced epimorphism of the corresponding monodromy groups.*

Similar statement holds in the category of oriented maps.

PROOF Let w be an element of $\langle \lambda_1, \rho_1, \tau_1 \rangle$ expressed as a word in terms of the generators. If $w = 1$ then for any $x \in F_1$ $w(x) = x$. Then $p^*(w)p(x) = pw(x) = p(x)$. Since p is an epimorphism of maps, every $y \in F_2$ is $p(x)$ for some x and hence $p^*(w) = 1$. Thus p^* takes an identity relation onto an identity relation, it follows that p^* is a homomorphism. By transitivity of $\langle \lambda_2, \rho_2, \tau_2 \rangle$ it is a group epimorphism. □

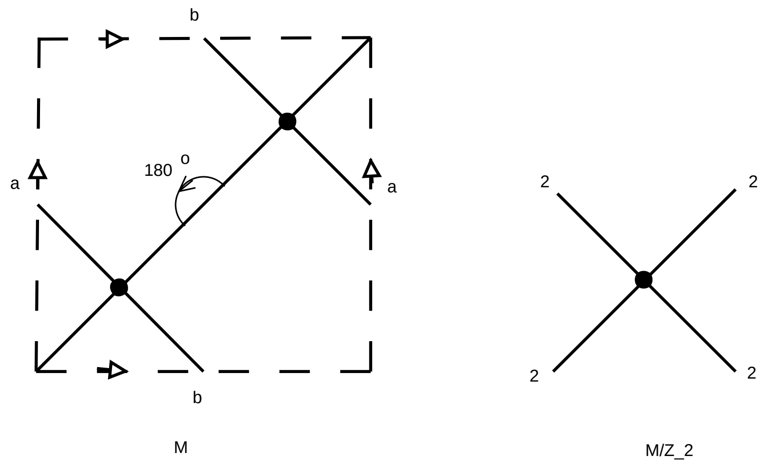


Figure 1.7: Akcion of $\mathbb{Z}_2$ on a toroidal map of type $[0; 2^4]$ and the quotient

The epimorphism p^* will be called the *induced epimorphism* or the *canonical epimorphism* of the corresponding monodromy groups. Furthermore, transitive actions of the monodromy groups ensure that every map homomorphism is surjective and that it also induces an epimorphism of the underlying graphs. Topologically speaking, a map homomorphism is a graph preserving branched covering projection of the supporting surfaces with branch points possibly at vertices, face centers or free ends of semiedges. Therefore we can say that a map $\tilde{\mathcal{M}}$ *covers* $\mathcal{M}$ if there is a homomorphism $\tilde{\mathcal{M}} \to \mathcal{M}$. A map homomorphism is *smooth* if it preserves the valency of vertices, the length of faces and does not send a link or a loop onto a semiedge.

Let us make it more precise. Given oriented map $(D; R, L)$, the *underlying graph* is a graph $X = (D; \cong_R, \lambda)$, where the equivalence relation $\cong_R$ is determined by the decomposition of D into the orbits of R. Similarly, if $(F; \lambda, \rho, \tau)$ is a map, we set D to be the set of orbits of τ. Then the incidence relation is defined by $\{x, \tau x\} \cong \{y, \tau y\}$ if $\{y, \tau y\} = \{(\rho\tau)^i x, \tau(\rho\tau)^i x\}$ for some i, and $L(\{x, \tau x\} = \{\lambda x, \lambda\tau x\}$.

Lemma 1.11. *Let φ be a homomorphism between maps (oriented maps). Then it induces a covering between the respective underlying graphs.*

Proof The proof in the category of oriented maps is easy. We prove the statement for maps. Since $\varphi : F_1 \to F_2$ be a map homomorphism between $\mathcal{M}_1 = (F_1, \lambda_1, \rho_1, \tau_1)$ and $\mathcal{M}_2 = (F_2, \lambda_2, \rho_2, \tau_2)$. Let $\{x, \tau x\} \cong \{y, \tau y\}$ be incident darts. Then $\varphi\{x, \tau_1 x\} = \{\varphi x, \tau_2\varphi(x)\}$ and

$$\varphi\{y, \tau_1 y\} = \varphi\{(\rho_1\tau_1)^i x, \tau_1(\rho_1\tau_1)^i x\} =$$

$$\{\varphi(\rho_1\tau_1)^i x, \varphi\tau_1(\rho_1\tau_1)^i x\} = \{(\rho_2\tau_2)^i\varphi x, \tau_2(\rho_2\tau_2)^i\varphi x\}.$$

Hence the images of incident darts are incident.

Similarly, if $L_1(\{x, \tau_1 x\}) = \{y, \tau_1 y\} = \{\lambda_1 x, \lambda_1\tau_1 x\}$ then

$$L_2\{\varphi x, \varphi\tau_1 x\} = \{\lambda_2\varphi x, \lambda_2\varphi\tau_1 x\} =$$

$$\{\lambda_2\varphi x, \lambda_2\tau_2\varphi x\} = \varphi\{\lambda_1 x, \lambda_1\tau_1 x\} = \varphi L_1\{x, \tau_1 x\}.$$

Hence $\varphi L_1 = L_2\varphi$ and we are done. □

With map homomorphisms we use also isomorphisms and automorphisms. The automorphism group $\mathrm{Aut}(\mathcal{M})$ of an oriented map $\mathcal{M} = (D;\ R, L)$ consists of all permutations in the full symmetry group $S(D)$ of D which commute with both R and L. Similarly, the automorphism group $\mathrm{Aut}(\mathcal{M})$ of an nonoriented map $\mathcal{M} = (F;\ \lambda, \rho, \tau)$ is formed by all permutations in the symmetry group $\mathrm{Sym}(F)$ which commute with each of λ, ρ and τ. Hence, in both cases the automorphism group is nothing but the centralizer of the monodromy group in the full symmetry group of the supporting set of the map (cf. [94, Proposition 3.3(i)]).

Lemma 1.12. *The action of the monodromy group of an (oriented) map M is transitive on the set of (darts) flags of M. The action of the map automorphism group is semiregular on the set of (darts) flags of M.*

Proof Let G be the monodromy group φ be an automorphism. Let $y = \varphi(x)$ for some dart $x \in D$ (or F). By the transitivity of the action of the monodromy group for any dart z there is $w \in \mathrm{Mon}(\mathcal{M})$ such that $z = w(x)$. Then $\varphi(z) = \varphi w(x) = w\varphi(x) = w(y)$. Hence an automorphism is determined by an image of one dart (flag). It follows that a stabilizer of a dart (flag) is trivial. □

Corollary 1.13. *If* $\mathcal{M} = (D; R, L)$ *is an oriented map then* $|\mathrm{Aut}\mathcal{M}| \leq |D| \leq |\mathrm{Mon}(\mathcal{M}|$, *if it is a map* $\mathcal{M} = (F; \lambda, \rho, \tau)$ *then* $|\mathrm{Aut}(\mathcal{M})| \leq |F| \leq |\mathrm{Mon}(\mathcal{M})|$.

Definition 1.3. A map is called *regular* if $|F| = \mathrm{Mon}(\mathcal{M})$, similarly an oriented map is called *regular* if $|D| = \mathrm{Mon}(\mathcal{M})$. A map $(F; \lambda, \rho, \tau)$ on an orientable surface is orientably regular if $|\mathrm{Mon}^+(\mathcal{M})| = |F|/2$.

It follows that the monodromy group of a regular map acts regularly on the supporting set, and $\mathrm{Mon}^+(\mathcal{M})$ acts regularly on the set of white (black) flags. We show the automorphism group of an (oriented) regular map $\mathcal{M}$ acts regularly on (darts) flags of $\mathcal{M}$, and similarly the subgroup of orientation preserving automorphisms $\mathrm{Aut}^+(\mathcal{M})$ of an orientably regular map $\mathcal{M}$ acts regularly on white (black) flags of $\mathcal{M}$, or equivalently, if the associated oriented map is regular.

Theorem 1.5 has the following two corollaries.

Corollary 1.14. *Let* $\mathcal{M}$ *be a map or an oriented map. Set* $G = \mathrm{Mon}(\mathcal{M})$ *and let* G_x *be a stabilizer. Then*

$\mathrm{Aut}(\mathcal{M}) \cong N_G(G_x)/G_x$.

Proof Observe that $\mathrm{Aut}(\mathcal{M})$ is a centralizer of G in $Sym(D)$. The results follows from Theorem 1.5(1).

Corollary 1.15. *Let* $\mathcal{H} = (D; R, L)$ *be an oriented map or a map. Then the following conditions are equivalent:*

(1) *The action of* $\mathrm{Mon}(\mathcal{H})$ *is regular,*

(2) $\mathrm{Mon}(\mathcal{H}) \cong \mathrm{Aut}(\mathcal{H})$,

(3) *the action of* $\mathrm{Aut}(\mathcal{H})$ *is regular.*

PROOF Set $\Omega = D$, $C = \mathrm{Aut}(\mathcal{M})$ and $G = \mathrm{Mon}(\mathcal{M})$ and apply Theorem 1.5(2) and (3). □

It follows that any of the above three conditions can be used to define the regular maps.

Remark Our use of the term *regular map* thus agrees with that of Gardiner et al. [65] and Wilson [207], but is not yet standard. For instance, Jones and Thornton [95] uses the term "reflexible", and White [198] calls such maps "reflexible symmetrical". On the other hand, our oriented regular maps are called "regular" in Coxeter and Moser [53], "symmetrical" in [21] and [198], and "rotary" in Wilson [207].

For each homomorphism $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ of oriented maps there is the corresponding homomorphism of maps $\varphi^\natural : \mathcal{M}_1^\natural \to \mathcal{M}_2^\natural$ defined by $\varphi^\natural(x, i) = (\varphi(x), i)$. If $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$, that is, φ is an automorphism, then this definition and the assignment $\varphi \mapsto \varphi^\natural$ yield the isomorphic embedding of $\mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(\mathcal{M}^\natural)$. This allows us to treat $\mathrm{Aut}(\mathcal{M})$ as a subgroup of $\mathrm{Aut}(\mathcal{M}^\natural)$ and, consequently, speak that every orientable regular map is orientably-regular (but not necessarily vice versa). It is easy to see that the index $|\mathrm{Aut}(\mathcal{M}^\natural) : \mathrm{Aut}(\mathcal{M})|$ is at most two. If it is two, then the map $\mathcal{M}$ is said to be *reflexible*, otherwise it is *chiral*. In the former case, there is an isomorphism ψ of the map $\mathcal{M} = (D;\, R, L)$ with its mirror image $(D;\, R^{-1}, L)$ called a *reflection* of $\mathcal{M}$. Clearly, $\psi^\natural$ is an automorphism that extends $\mathrm{Aut}(\mathcal{M})$ to $\mathrm{Aut}(\mathcal{M}^\natural)$. Topologically speaking, oriented map automorphisms preserve the chosen orientation of the supporting surface whereas reflections reverse it.

Transitivity of the action of the automorphism group of a regular (orientably regular) map forces all the vertices to have the same valency and all the faces to have same size (covalency). We say that a map $\mathcal{M}$ has a *type* (p, q) if the covalency of every face is p and the valency of every vertex of $\mathcal{M}$ is q for some integers p, q. Generally, we can define the *type* of a map to be the couple (p, q) of integers, where p (q) is the least common multiple of covalencies (valencies) of the covalencies (valencies) of faces (vertices) of $\mathcal{M}$.

Homomorphisms between regular maps

Given homomorphism $\mathcal{M} \to \mathcal{N}$ between oriented maps more can be said if one of $\mathcal{M}$, $\mathcal{N}$, is a regular map. Let $G \le \mathrm{Aut}(\mathcal{M}) = (D; R, L)$ be a subgroup of the automorphism group of an oriented map. For any $x \in D$ denote $[x]$ the orbit in the action of G containing x. Set $\bar{R}[x] = [Rx]$, $\bar{L}[x] = [Lx]$ for any $x \in D$ and denote $\bar{D} = \{[x]; x \in D\}$. Then $\bar{\mathcal{M}} = (\bar{D}; \bar{R}, \bar{L})$ is a well defined oriented map and the natural assignment $x \mapsto [x]$ defines a map homomorphism $\mathcal{M} \to \bar{\mathcal{M}}$. Homomorphisms arising in the above way are called *regular*, the group $G \le \mathrm{Aut}(\mathcal{M})$ is called the *group of covering transformations.* Above definitions can be modified for maps as well.

Lemma 1.16. *Let $p : \mathcal{M} \to \mathcal{N}$ be a regular covering between maps with $A = CT(p) \le \mathrm{Aut}(\mathcal{M}) = G$. Then the normalizer $N_G(A)$ projects. In particular, the group of lifts coincides with $N_G(A)$.*

Proof The covering p takes $x \mapsto [x]_A$. If $\tilde{f} \in N_G(A)$ the projection $f : [x]_A \to [\tilde{f}x]_A$ is well-defined. It follows that $N_G(A) \le Lft(\mathrm{Aut}(\mathcal{N})$. To see that the equality holds, let f has a lift $\tilde{f}$, that means $p\tilde{f} = fp$. It follows that $\tilde{f}$ preserves the fibres, but the latter implies $\tilde{f}^{-1}\alpha\tilde{f} \in A$ for any $\alpha \in A$. □

The following statement comes from [145].

Proposition 1.17. *Let $p : \mathcal{M} \to \mathcal{N}$ be a map homomorphism. Then*

(1) *If $\mathcal{M}$ is regular then p is regular and every automorphism of $\mathcal{N}$ lifts.*

(2) *If $\mathcal{N}$ is regular and p is regular then any automorphism projects.*

In particular, a homomorphism between regular maps is regular.

Proof Since $\mathcal{M}$ is regular for every $x, y \in \mathrm{fib}_b$, over a dart (or flag) there exists $\varphi \in \mathrm{Aut}(\mathcal{M})$ taking x onto y. We claim that $p\varphi = p$ which means that $CT(p)$ acts regularly on each fibre. Indeed, if z is any element of the supporting set of the map, then there is $w \in \mathrm{Mon}(\mathcal{M})$ such that $w(x) = z$.

We have

$$p\varphi(z) = p\varphi(w(x)) = pw\varphi(x) = w^* p\varphi(x) = w^*(b) = w^* p(x) = pw(x) = p(z),$$

where $w^* = p^*(w)$.

Let $f \in \mathrm{Aut}(\mathcal{N})$ be an automorphism taking $x \mapsto y$. Choose any $\tilde{x} \in p^{-1}(x)$ and any $\tilde{y} \in p^{-1}(y)$. Since $\mathcal{M}$ is regular, there is $\tilde{f} \in \mathrm{Aut}(\mathcal{M})$ taking $\tilde{x} \mapsto \tilde{y}$. We prove that $\tilde{f}$ is a lift of f. Let $\tilde{z} = \tilde{w}(\tilde{x})$ for some $\tilde{w} \in \mathrm{Mon}(\mathcal{M})$ and let $w = p^*(\tilde{w})$. We have

$$p\tilde{f}(\tilde{z}) = p\tilde{f}\tilde{w}(\tilde{x})wp\tilde{f}(\tilde{x}) = w(y) = wfp(\tilde{x}) = fp\tilde{w}(\tilde{x}) = fp(\tilde{z}).$$

□

Proposition 1.18. *Let $p : \mathcal{M} \to \mathcal{N}$ be a map homomorphism. Then the following two conditions are equivalent:*

(1) *$\mathcal{M}$ is regular, p is regular and $CT(p) \lhd \mathrm{Aut}(\mathcal{M})$,*

(2) *both $\mathcal{M}$ and $\mathcal{N}$ are regular.*

Proof Assume first that both $\mathcal{M}$ and $\mathcal{N}$ are regular. By Proposition 1.17 p is regular. By Lemma 1.16 the group of lifts coincides with the normaliser $N_G(CT(p))$, where $G = \mathrm{Aut}(\mathcal{M})$. If $\mathcal{N}$ is regular that means that every automorphism of $\mathcal{M}$ is a lift and so $N_G(CT(p)) = \mathrm{Aut}(\mathcal{M})$ implying $CT(p) \lhd \mathrm{Aut}(\mathcal{M})$.

Conversely, if $\mathcal{M}$ is regular, p is regular and $CT(p) \lhd \mathrm{Aut}(\mathcal{M})$, then $\mathcal{N}$ is isomorphic to a quotient map $\bar{\mathcal{M}}$. We set $\bar{\psi}[x] = [\psi x]$ for any $\psi \in \mathrm{Aut}(\mathcal{M})$. This mapping is an automorphism of $\bar{\mathcal{M}}$ provided it is well-defined. Let $z \in [x]$. Then $\bar{\psi}[z] = [\psi\alpha x] = [\beta\psi x] = [\psi x] = \bar{\psi}[x]$, for some $\alpha, \beta \in CT(p)$. Since for any two darts $[x]$ and $[y]$ there is $\psi \in \mathrm{Aut}(\mathcal{M})$ such that $y = \psi(x)$ the map $\bar{\mathcal{M}}$ is regular, and so is $\mathcal{N}$.
□

Similar statements hold for homomorphisms between regular maps in the category of MAPS.

Example 1.2. Maps of type $(4, 4)$.

Let consider an infinite rectangular grid on the Euclidean plane which can be viewed as an oriented map $\mathcal{M}$ of type $(4, 4)$. Let introduce a Cartesian coordinate system such that vertices of the map are points with integer coordinates. Then given any integer vector (b, c) the translation $(x, y) \mapsto (x, y) + (b, c)$ induces a map isomorphism $\tau_{b,c}$ acting freely on darts, vertices and edges of $\tilde{\mathcal{M}}$. Let H be the group of all such translations, clearly $H \cong \langle \tau_{1,0}, \tau_{0,1} \rangle \cong \mathbb{Z} \times \mathbb{Z}$. Let ρ be the 90 degree counter-clockwise rotation around the vertex $(0, 0)$. Then the group $\langle \rho, H \rangle \leq \mathrm{Aut}(\tilde{\mathcal{M}})$ is transitive on darts of $\mathcal{M}$, and consequently, $\mathrm{Aut}(\tilde{\mathcal{M}}) = \langle \rho, H \rangle$, and the map is regular. Take any $\tau_{b,c}$ and let $K = \langle \tau_{b,c}, \tau_{b,c}^{\rho} \rangle$. Since $\tau_{b,c}^{\rho} = \tau_{-c,b}$ is a translation, it commutes with every element of H. Moreover,

$$K^{\rho} = \langle \tau_{b,c}^{\rho}, \tau_{b,c}^{\rho^2} \rangle = \langle \tau_{b,c}^{\rho}, \tau_{-b,-c} \rangle = K.$$

It follows that K is a normal subgroup of $\mathrm{Aut}(\mathcal{M})$. By Proposition 1.18 there is an oriented regular quotient map $\{4, 4\}_{b,c} = \bar{\mathcal{M}} = (\bar{D}, \bar{R}, \bar{L})$ determined by the action of K (see Figure). The vertex-set $\bar{V}$ of the quotient map consists of orbits of K, it is straightforward to see that each orbit has a representative in the quadrilateral whose corners have coordinates $[0, 0], [b, c], [-c, b], [b - c, b + c]$, taking just the internal vertices and the vertices on the two sides $([0, 0], [b, c])$, $([0, 0], [-c, b])$ we get a transversal of the set of all orbits. It follows that $\bar{\mathcal{M}}$ is a finite oriented map of type $(4, 4)$ of Euler characteristic 0. Hence the supporting surface is torus. This way an infinite family of regular oriented maps on the torus is constructed. Later we prove that in fact every regular map of type $(4, 4)$ arises in this way. The maps $\{4, 4\}_{b,c}$ were investigated by Coxeter and Mozer in [53, Chapter 4].

Exercises

1.3.1. Prove that an automorphism group of an oriented regular map can be generated by an automorphism cyclically permuting darts based at a vertex v, and by an automorphism taking a dart incident to v into its inverse.

1.3.2. Considering quotients of the hexagonal grid in the Euclidean construct an infinite family of oriented regular maps of type $(6, 3)$.

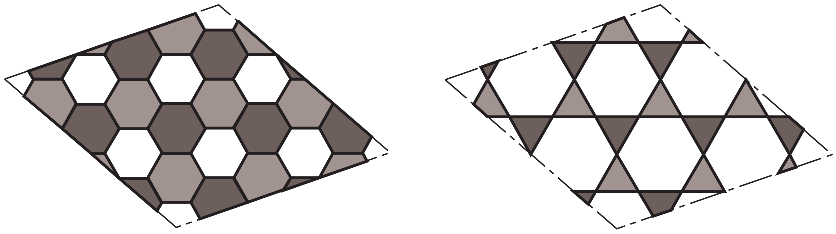
1.3.3. Classify those oriented regular map which are isomorphic to its mirror image $\{4,4\}_{b,-c}$.
1.3.4. Show that the maps that correspond to the oriented maps $\{4,4\}_{b,c} \cong \{4,4\}_{b,-c}$ are regular in the category of MAPS.
1.3.5. Prove that the monodromy group, as well as the automorphisms group of the maps $\{4,4\}_{b,c}$ has presentation $G = \langle r, \ell | r^4 = \ell^2 = (r\ell)^4 = (r\ell r)^b (r^2 \ell)^c = 1\rangle$.
1.3.6. Prove that $G = \langle x; y | x^3 = y^2 = (xy)^6 = (x^{-1}yxy)^b (xyx^{-1}y)^c = 1\rangle$ is of finite index, for any choice of integers a, b. (Hint: Try to identify it with a group of isometries of a regular trivalent hexagonal grid in the Euclidean plane) Draw a figure of the map $A(G/1; x, y)$ for $b = 3$ and $c = 3$.

1.4 Generalization to hypermaps, Walsh map of a hypermap

A *topological hypermap* $\mathcal{H}$ is a cellular embedding of a connected trivalent graph X into a closed surface S such that the cells are 3-colored (say by black, grey and white colours) with adjacent cells having different colours. Numbering the colours 0, 1 and 2, and labeling the edges of X with the missing adjacent cell number, we can define 3 fixed points free involutory permutations r_i, $i = 0$, 1, 2, on the set F of vertices of X; each r_i switches the pairs of vertices connected by i-edges (edges labeled i). The elements of F are called *flags* and the group G generated by r_0, r_1 and r_2 will be called the *monodromy group*[1] $\mathrm{Mon}(\mathcal{H})$ of the hypermap $\mathcal{H}$. The cells of $\mathcal{H}$ colored 0, 1 and 2 are called the *hypervertices*, *hyperedges* and *hyperfaces*, respectively. Since the graph X is connected, the monodromy group acts transitively on F and orbits of $\langle r_0, r_1\rangle$, $\langle r_1, r_2\rangle$ or $\langle r_0, r_2\rangle$ on F determine hyperfaces, hypervertices and hyperedges, respectively. The order of the element $k = ord(r_0 r_1)$, $m = ord(r_1 r_2)$ and $n = ord(r_2 r_0)$ is called the *valency* of a hyperface, hypervertex and hyperedge, respectively. The triple (k, m, n) is called *type* of the hypermap.

Maps correspond to hypermaps satisfying condition $(r_0 r_2)^2 = 1$, or

[1]This group has been called the monodromy group of $\mathcal{H}$ [99, 155], the connection group of $\mathcal{H}$ [207] and the Ω-group of $\mathcal{H}$ [26].



(a) (b)

Figure 1.8: Regular embedding of the Fano plane in the torus gives rise to a regular hypermap of type $(3, 3, 3)$.

in other words, maps are hypermaps of type $(p, q, 2)$ or of type $(p, p, 1)$. Thus we can view the category of Maps as a subcategory of the category *of Hypermaps* which is formed by 4-tuples $(F;\ r_0, r_1, r_2)$, where r_i $(i = 0, 1, 2)$ are (fixed points free) involutory permutations generating the monodromy group $\mathrm{Mon}(\mathcal{H})$ acting transitively on F. Similarly, the category of Oriented Hypermaps arises by relaxing the condition $L^2 = 1$ in the definition of an oriented map. More precisely, an *oriented hypermap* is a 3-tuple $(D;\ R, L)$, where R and L are permutations acting on D such that the oriented monodromy group is transitive on D. The notions defined in the previous section extend from maps to hypermaps in an obvious way. For more information on hypermaps the reader is referred to [50].

The modified Euler formula for hypermaps reads as follows.

Theorem 1.19. (Euler formula for hypermaps) *Let* $\mathcal{H} = (F;\ r_0, r_1, r_2)$ *be a hypermap on a closed surface* S *of genus* g *having* v *hypervertices,* e *hyperedges and* f *hyperfaces. Then*

(1) $v + e + f - |F|/2 = 2 - 2g$*, if* S *is orientable,*

(2) $v + e + f - |F|/2 = 2 - g$*, if* S *is nonorientable.*

Proof Let $X \to S$ be a 2-cell embedding of the associated 3-edge-coloured graph. It has $|F|$ vertices, $3|F|/2$ edges and $v + e + f$ faces. Consequently, the Euler characteristic $\chi = |F| - 3|F|/2 + v + e + f$ and we are done. $\square$

Similarly as for maps, a hypermap $(F; r_0, r_1, r_2)$ is *orientable* if and only if the associated 3-edge-coloured graph is bipartite.

Remark. In order not to exclude factorisations of hypermaps by subgroups containing reflections, we need to relax the condition that the three involutions r_0, r_1 and r_2 acts on flags without fixed points. Topological maps corresponding to such objects are maps on compact surfaces with non-empty boundary. Later we will discuss them in a detail. Each hypermap covers a trivial one-flag hypermap embedded in the disc.

Walsh representation

An important and convenient way to visualize hypermaps was introduced by Walsh in [195]. Topologically, a map can be seen as a cellular embedding of a graph in a closed surface and a hypermap as a cellular embedding of a hypergraph in a closed surface. Since hypergraphs are in a sense bipartite graphs (with one monochromatic set of vertices representing the hypervertices and the other monochromatic set of vertices representing the hyperedges) a hypermap can be viewed as a bipartite map, as well. In fact, given any topological hypermap $\mathcal{H}$ we can construct a topological bipartite map $W(\mathcal{H})$, called the Walsh bipartite map associated to $\mathcal{H}$ by taking first the dual of the underlying 3-valent map and then deleting the vertices (together with the edges attached to them) lying inside the hyperfaces of $\mathcal{H}$. The resulting map is bipartite with one monochromatic set of vertices lying on the faces colored black, representing the hypervertices of $\mathcal{H}$, and the other monochromatic set lying on the faces colored grey, representing the hyperedges.

This construction can be reversed: given any topological bipartite map $\mathcal{B}$, where the vertices are bipartitioned in black and grey, we construct an associated topological hypermap $W^{-1}(\mathcal{B}) = \mathrm{Tr}(\mathcal{B}^*)$ by truncating the dual map $\mathcal{B}^*$; the faces of the resulting 3-valent map $\mathrm{Tr}(\mathcal{B}^*)$ contains the vertices and the face-centres of the original map and are henceforth 3-colorable black, grey and white, with all these colours meeting at each vertex of $\mathrm{Tr}(\mathcal{M}^*)$. If $\mathcal{B} = W(\mathcal{H})$ is the Walsh bipartite map of an oriented hypermap $\mathcal{H} = (D; R, L)$ then $\bar{R}$ and $\bar{L}$ are the respective rotations on the two bipartition sets of the dart set of $\mathcal{B}$, so the rotation of $\mathcal{B}$ is $\bar{R}\bar{L} = \bar{L}\bar{R}$. Note that the darts of the hypermap

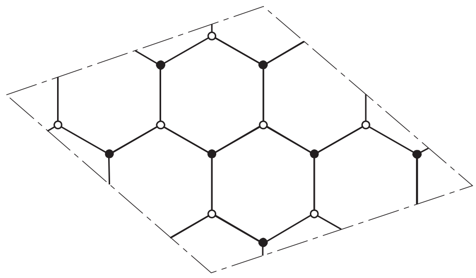


Figure 1.9: The Walsh map of the Fano plane embedding.

correspond to edges of the Walsh map, hence R, L and $\bar{R}$, $\bar{L}$ act on different sets of objects.

Chapter 2

Groups and hypermaps

2.1 Maps, hypermaps and groups

Schreier representations

In the previous sections we have seen that maps and hypermaps can be represented by means of two or three permutations satisfying some conditions. The aim of this section is to show that one can study hypermaps as purely group theoretical objects. The idea emerges from the fact that every transitive permutation group is equivalent to a group acting on cosets by translation. Following [189, 190], we call these representations Schreier representations.

Schreier representations of oriented maps appear implicitly in Jones and Singerman [94], and explicitly in [155]. Vince [189] developed a theory of Schreier representations of (hyper)maps on closed surfaces described by three involutions. Here we introduce Schreier representations of oriented hypermaps. Between Schreier representations the representations employing the actions of monodromy groups, triangle groups and the universal triangle group are the most important. Thus we shall investigate them in detail.

Let G be a finite group generated by two elements r and ℓ. In other words, G is a finite quotient of some triangle group $\Delta^+(k, m, n) = \langle r, \ell;\ \ell^n = r^m = (r\ell)^k = 1\rangle$. k,m and n being positive integers. Further let S be a subgroup of G. Using the action of G on the set $C = G/S$ of left cosets of S in G by the left translation, we construct a hypermap

$A(G/S;\, r, \ell)$ whose monodromy group is a homomorphic image of G and the local monodromy group is a homomorphic image of S. We take the cosets as darts of the hypermap and define the rotation R and the dart reversing involution L by setting

$$R(hS) = rhS, \tag{2.1}$$
$$L(hS) = \ell hS, \tag{2.2}$$

respectively, hS being an arbitrary element of C. For the resulting hypermap $(C;\, R, L) = A(G/S;\, r, \ell)$ we easily check that the assignment $r \mapsto R$, $\ell \mapsto L$ extends to a homomorphism $\Delta^+(k, m, n) \to \mathrm{Mon}(A(G/S;\, r, \ell))$.

Let $\mathcal{H}$ be a hypermap. A *Schreier representation* of $\mathcal{H}$ is an isomorphism $\mathcal{H} \to A(G/S;\, r, \ell)$ for an appropriate group $G = \langle r, \ell \rangle$ and a subgroup $S \leq G$, or simply the hypermap $A(G/S;\, r, \ell)$ itself. Given any hypermap $\mathcal{H} = (D;\, R, L)$, it is not difficult to find a Schreier representation for $\mathcal{H}$. Indeed, we first fix any dart a of $\mathcal{H}$ and set $G = \mathrm{Mon}(\mathcal{H}) = \langle R, L \rangle$ and $S = \mathrm{Mon}(\mathcal{H}, a)$, to be the stabilizer of a. Then, for an arbitrary dart x we take any element $h \in \mathcal{H}$ with $h(a) = x$ and label x by the coset $hS \in C$, thereby obtaining a labeling $\alpha(x) = hS$. Observe that α is well-defined since for any two elements h and h' of $\mathrm{Mon}(\mathcal{M})$ with $h(a) = x = h'(a)$ we have $hS = h'S$. In fact, α is a bijection of $D(\mathcal{H})$ onto C. Clearly, $\alpha(Rx) = R\alpha(x)$ and $\alpha(Lx) = L\alpha(x)$ which means that $\alpha : \mathcal{H} \to A(G/S;\, R, L)$ is the required isomorphism.

We summarise the above discussion in the following proposition.

Proposition 2.1. *Every oriented hypermap has Schreier representation $A(G/S;\, r, \ell)$, where $G = \mathrm{Mon}(\mathcal{H})$ and S is the stabiliser of a dart.*

If we start from a given hypermap $\mathcal{H}$, the Schreier representation we have just described is in some sense best possible because the monodromy group $\mathrm{Mon}(\mathcal{H})$ is not merely a homomorphic image of G but is actually isomorphic to it. In this case we say that the Schreier representation is *effective*. In general, a Schreier representation $A(G/S;\, r, \ell)$ is effective if and only if G acts faithfully on C, i.e., the translation by every non-identity element of G is a non-identity permutation of C. Elementary theory of group actions or straightforward computations yield that the latter occurs precisely when the subgroup $\bigcap_{h\in G} hSh^{-1}$, the *core* of S in G, is trivial (cf. Rotman [166]). Given Schreier representation of a hypermap its automorphism group can be computed as follows.

Lemma 2.2. *Let $A(G/S; r, \ell)$ be an arbitrary Schreier representation of a hypermap $\mathcal{H}$. Then $\mathrm{Aut}(\mathcal{H}) \cong N_G(S)/S$.*

Proof The statement is a consequence of Theorem 1.5. However, to give the reader more insight we prove it directly. For every element $c \in N_G(S)$ we define the permutation ξ_c by setting $hS \mapsto hcS = hSc$. Note that ξ_c is a permutation if and only if $c \in N_G(S)$. By the definition ξ_c commutes with both the rotation and the dart reversing involution of the map. Moreover, two automorphisms ξ_c and ξ_d are equal if and only if $cd^{-1} \in S$. Hence, S is the kernel of a morphism: $\Phi : N_G(S) \to \mathrm{Aut}(\mathcal{H})$ and so there is a monomorphism $N_G(S)/S \to \mathrm{Aut}(\mathcal{H})$.

We end the proof by showing that every automorphism is ξ_c for some $c \in N_G(S)$. Assume that $\varphi \in \mathrm{Aut}(\mathcal{H})$ is an automorphism taking $S \mapsto gS$. Let $h \in S$. Then

$$hgS = h(\varphi S) = \varphi(hS) = \varphi S = gS.$$

It follows that $g^{-1}hgH = H$ and thus $g \in N_g(S)$. □

Similarly as for maps Theorem 1.5 implies:

Theorem 2.3. *Let be an oriented hypermap or a hypermap. Then $\mathrm{Aut}(\mathcal{H})$ acts semiregulary on the supporting set and the following conditions are equivalent:*

(1) *The action of $\mathrm{Mon}(\mathcal{H})$ is regular,*

(2) $\mathrm{Mon}(\mathcal{H}) \cong \mathrm{Aut}(\mathcal{H})$,

(3) *the action of $\mathrm{Aut}(\mathcal{H})$ is regular.*

Let us remark that the above isomorphism assigns the left translation by an element $h \in G$ (representing a monodromy of $\mathcal{H}$) to the right translation ξ_h (representing an automorphism of $\mathcal{H}$).

Schreier representations provide a convenient tool to deal not only with automorphisms but also with homomorphisms between hypermaps. If

$$G = \langle r, \ell;\ \ell^n = r^m = (r\ell)^k = 1, \ldots \rangle$$

is a finite quotient of the triangle group $\Delta^+(k, m, n)$ and $S \leq S' \leq G$ are two subgroups then the natural projection $\pi : G/S \to G/S', \quad hS \mapsto hS'$ $(h \in G)$, is a homomorphism $A(G/S;\ r, \ell) \to A(G/S';\ r, \ell)$. In fact, every hypermap homomorphism $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$, where $\mathcal{H}_i = (D_i;\ R_i, L_i)$, is in the following sense equivalent to an appropriate natural projection.

Proposition 2.4. *Let* $\mathcal{H}_i = (D_i; R_i, L_i)$ *be two oriented hypermaps and* $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$. *Then there are subgroups* $S_1 \leq S_2 \leq H = \mathrm{Mon}(\mathcal{M}_1)$, *isomorphisms* κ_i *and a homomorphism* $\alpha : xS_1 \mapsto xS_2$ *such that the following diagram commutes:*

$$\mathcal{M}_1 \to \varphi\mathcal{M}_2$$

$$\downarrow \kappa_1 \downarrow \kappa_2$$

$$A(H/S_1;\ R_1, L_1) \to \alpha A(H/S_2;\ R_1, L_1)$$

Proof Let $a_2 = \varphi(a_1)$, $a_i \in D_i$, be two darts. We set $S_1 = \mathrm{Mon}(M_1, a_1) = H_{a_1}$. Then H acts on D_2 by $hx = \varphi^*(h)x$. Set S_2 to be the stabilizer of a_2 in this action. Clearly $S_1 \leq S_2 \leq H$. Set $\kappa_i(x) = \kappa_i(ga_i) = gS_i$, where $g \in H$ such that $x = ga_i \in D_i$. We need to verify several things. First,

$$\kappa_1(R_1(x)) = \kappa_1(R_1(ga_1)) = \kappa_1(R_1 g)a_1 = R_1 g S_1 = R_1^* \kappa(x),$$

where R_1^* is the rotation in $A(H/S_1;\ R_1, L_1)$. Secondly,

$$\kappa_2(R_2(x)) = \kappa_2(R_2(\varphi^*(g)a_2)) = \kappa_2(R_1 g)a_2 = R_1 g S_2 = R_2^* \kappa(x),$$

where R_2^* is the rotation in $A(H/S_2;\ R_1, L_1)$. Similar computations give $\kappa_i(L_i(x)) = L_i^* \kappa_i$, for $i = 1, 2$. Hence κ_i are coverings. Since $|D_i| = |\{xS_i | x \in H\}|$ the mappings κ_i are bijective.

Now α is a homomorphism, because

$$R_2^* \alpha(xS_1) = R_1 x S_2 = \alpha(R_1 x S_1) = \alpha R_1(xS_1) = \alpha R_1^*(xS_1)$$

and by a similar argument $L_2^*\alpha = \alpha R_1^*$.

Finally we show that the diagram commutes. Indeed,

$$\alpha\kappa_1(x) = \alpha\kappa_1(ga_1) = \alpha(gS_1) = gS_2 = \kappa_2(ga_2)$$

$$= \kappa_2 g(\varphi a_1) = \kappa_2\varphi(ga_1) = \kappa_2\varphi(x).$$

$\square$

Generic hypermap

One consequence of these considerations is that every oriented hypermap is a quotient of a (finite) oriented regular hypermap. In fact, for every oriented hypermap $\mathcal{H} = (D;\, R, L)$ there exists a regular hypermap $\mathcal{H}^{\#}$ and a homomorphism $\alpha : \mathcal{H}^{\#} \to \mathcal{H}$ with the following universal property: for every regular hypermap $\tilde{\mathcal{H}}$ and a homomorphism $\varphi : \tilde{\mathcal{H}} \to \mathcal{H}$ there is a homomorphism $\varphi' : \tilde{\mathcal{H}} \to \mathcal{H}^{\#}$ such that $\varphi = \alpha\varphi'$. In terms of Schreier representations, the homomorphism α is equivalent to the natural projection $A(G/1;\, R, L) \to A(G/S; R, L) \cong \mathcal{H}$ where $G = \mathrm{Mon}(\mathcal{H})$ and $S = \mathrm{Mon}(\mathcal{H}, a)$, is the stabilizer of some dart $a \in D(\mathcal{H})$. We shall call the hypermap $\mathcal{H}^{\#}$ the *generic regular hypermap* over $\mathcal{H}$ and $\alpha : \mathcal{H}^{\#} \to \mathcal{H}$ the *generic homomorphism*. It is obvious that the induced homomorphism $\alpha^* : \mathrm{Mon}(\mathcal{H}^{\#}) \to \mathrm{Mon}(\mathcal{H})$ is an isomorphism and that $\mathcal{H}^{\#}$ and $\mathcal{H}$ have the same type.

To construct the generic hypermap $\mathcal{H}^{\#} = (D^{\#};\, R^{\#}, L^{\#})$ for an oriented hypermap $\mathcal{H} = (D;\, R, L)$ it is sufficient to set $D^{\#} = \mathrm{Mon}(\mathcal{H})$, $R^{\#}(x) = Rx$, and $L^{\#}(x) = Lx$ for any $x \in D^{\#}$. Observe that the automorphisms of $\mathcal{H}^{\#}$ are just the right translations of $D^{\#} = \mathrm{Mon}(\mathcal{H})$ by the elements of $\mathrm{Mon}(\mathcal{H})$, and so $\mathcal{H}^{\#}$ is indeed an oriented regular hypermap. Similarly, if $\mathcal{H} = (F;\, \lambda, \rho, \tau)$ is an nonoriented hypermap then the generic regular hypermap $\mathcal{H}^{+} = (F^{+};\, \lambda^{+}, \rho^{+}, \tau^{+})$ over $\mathcal{H}$ can be constructed by setting $F^{+} = \mathrm{Mon}(\mathcal{H})$, $\lambda^{+}(x) = \lambda x$, $\rho^{+}(x) = \rho x$ and $\tau^{+}(x) = \tau x$, for any $x \in F^{+}$. Again, the hypermap automorphisms are given by the right translations of F^{+} by the elements of $\mathrm{Mon}(\mathcal{H}) = F^{+}$.

It is obvious that if $\mathcal{H}$ is orientable, then so is $\mathcal{H}^{+}$. Moreover, the hypermap $\mathcal{H}^{+}$, as a topological hypermap, smoothly covers $(\mathcal{H}^{\#})^{\natural}$.

In the following sections we shall see that maps or Walsh bipartite hypermap representations combined with the generic hypermap construction provide a convenient tool for construction regular hypermaps satisfying certain constrains (see for instance [93, 156]). Let us note that there is an interesting relationship between hypermaps and algebraic curves (see Section 5). Via this relationship, actions of Galois groups of algebraic number fields on maps on surfaces are investigated (see [74, 99, 98, 171]). In this context the base maps are (following Grothendieck) called *dessigns d'enfants.*

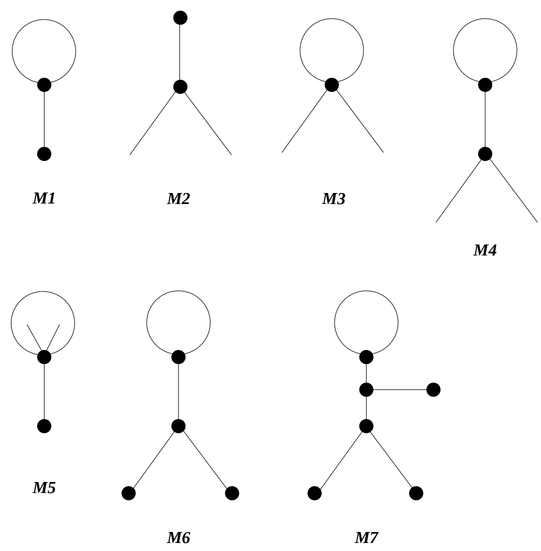


Figure 2.1: Dessign d'enfants

Example 2.1. Figure 2.1 shows spherical maps $\mathcal{M}_1, \ldots \mathcal{M}_5$ which (oriented) generic maps are the five Platonic solids. The respective (oriented) monodromy groups are $\mathbb{A}_4$, $\mathbb{S}_4$, $\mathbb{S}_4$, $\mathbb{A}_5$ and $\mathbb{A}_5$. Dessign d'enfants $\mathcal{M}_6$ and $\mathcal{M}_7$ represent (oriented) regular maps on surfaces with higher genera. The associated monodromy groups are the projective linear group $PSL(2,7)$ and Mathieu group $\mathbb{M}_{12}$, respectively. The generic map $M_6^{\#}$ is known as the dual of the Klein's triangulation of the surface of genus 3 and this map is the smallest Hurwitz map (see the following Section).

A nice application of the construction of the generic map over a map follows. In 1978 Grünbaum asked the following question:

Problem 2.1. First Grünbaum's problem: Is there for any two positive integers p and q, $\frac{1}{p}+\frac{1}{q}<\frac{1}{2}$, a map without semiedges of type (p,q)?

There is a long story related with this problem (see the historical notes). Perhaps the most elementary affirmative solution of the Grunbaum's problem is obtained in [93] giving the dessing's d'enfants (a planar map) for each hyperbolic type (p,q) and deriving the respective generic covering maps, see Figure 2.2. We establish the result explicitly.

Proposition 2.5. *For any two positive integers p and q, $\frac{1}{p}+\frac{1}{q}<\frac{1}{2}$ there is an (oriented) regular map of type (p,q).*

Proof By duality we may assume $p \geq q > 2$. Then $p = qt + r$, for some $0 \leq r < q$ and $t \geq 1$.

We distinguish three cases: Case (a) $p = q$, Case (b) $2q > p > q$ and Case (c) $p \geq 2q$. For each of the three cases a spherical map $\mathcal{M}$ of type (p,q) is drawn on Figure 2.2. If $p > q$ the map is path-like and it contains t vertices of degree q and r vertices of degree one. Since there is just one face, its length is $qt + r = p$. Taking the generic map over $\mathcal{M}$, we get an (oriented) regular map $(\mathcal{M}^{\#})$ $\mathcal{M}^{+}$ of type (p,q). Since the only (oriented) regular maps with semiedges are the semistars st_n of type (n,n), the maps we have constructed are without semiedges. $\square$

Maps and hypermaps from triangle groups

The above theory of Schreier representations can be applied to infinite hypermaps as well. It follows that oriented maps and hypermaps of given type (k,m,n) can be described as quotients of the *universal oriented hypermap* of type (k,m,n) which (oriented) monodromy group is $\Delta^{+}(k,m,n)$. This is the even-word subgroup of the extended triangle group

$$\Delta(k,m,n) = \langle r_0, r_1, r_2;\, r_0^2 = r_1^2 = r_2^2 = (r_1r_2)^k = (r_0r_1)^m = (r_2r_0)^n = 1\rangle,$$

which is the monodromy group of the *universal hypermap*

$$A(\Delta(k,m,n); r_0, r_1, r_2)$$

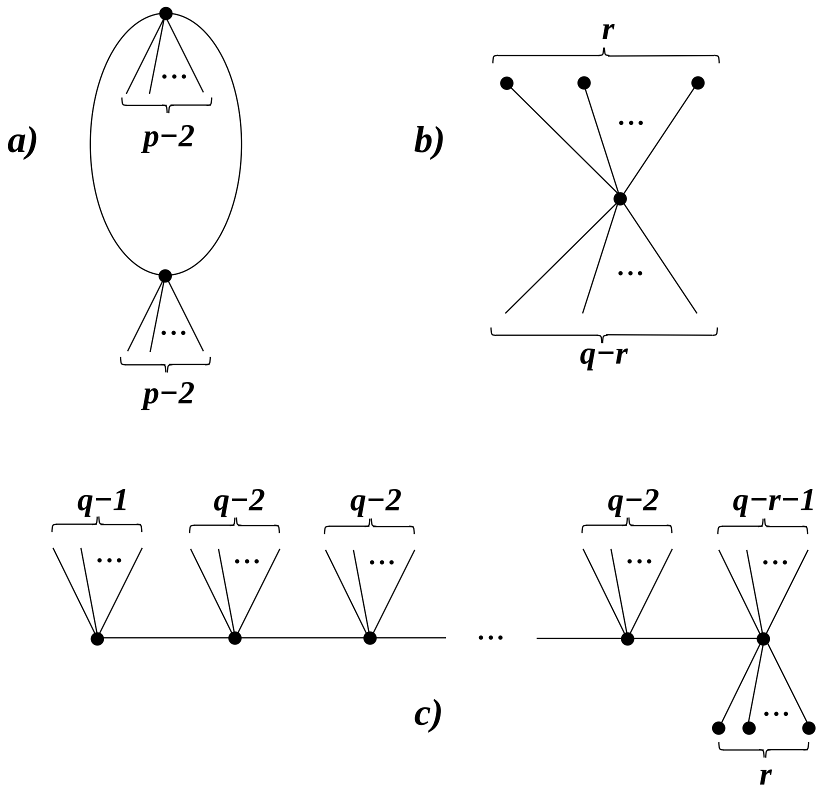


Figure 2.2: Dessigns d'enfants of hyperbolic types

for the category of (nonoriented) hypermaps of type (k, m, n). It follows that
$$\Delta^+(k, m, n) = \langle r, \ell; (r\ell)^k = \ell^n = r^m = 1\rangle.$$
Note that the universal maps of type (k, m) with the monodromy group $\Delta(k, m, 2)$ are the well known tessellations of the sphere, plane or hyperbolic plane by k-gons (m of them meeting at each vertex) provided the expression $\frac{1}{k} + \frac{1}{m}$ is greater, equal or less than $\frac{1}{2}$, respectively.

A subgroup S of a group G is called *torsion-free* if it contains no elements of finite order.

Proposition 2.6. *$\mathcal{H}$ is a regular hypermap of type (k, m, n) if and only if there is a normal torsion-free subgroup N of the (extended) triangle group $\Delta(k, m, n) = \langle r_0, r_1, r_2 | (r_0r_1)^k = (r_1r_2)^m = (r_2r_0)^n = 1\rangle$ such* *that* $\mathcal{H} \cong A(\Delta(k,m,n)/N; r_0, r_1, r_2)$.

Proof. Elements of finite order in $\Delta(k,m,n)$ for $\frac{1}{k}+\frac{1}{m}+\frac{1}{n} \leq 1$ belong to one of the following subgroups $\langle r_i, r_{i+1}\rangle^g$, for $i = 0,1,2$ and $g \in G$. Hence if $N \lhd \Delta(k,m,n)$ contains a non-trivial element of finite order, the algebraic hypermap $A(\Delta(k,m,n)/N; r_0, r_1, r_2)$ has a different type than (k,m,n). Conversely, if the type of $A(\Delta(k,m,n)/N; r_0, r_1, r_2)$ is not (k,m,n) it means that N contains a non-trivial element of finite order belonging to a stabilizer of a hyper-vertex, hyper-face or hyper-edge. □

Similarly for the category of oriented maps we have.

Proposition 2.7. *$\mathcal{H}$ is a regular oriented hypermap of type (k,m,n) if and only if there is a normal torsion-free subgroup N of the oriented triangle group $\Delta^+(k,m,n) = \langle r, \ell | (r\ell)^k = r^m = \ell^n = 1\rangle$ such that $\mathcal{H} \cong A(\Delta(k,m,n)/N;)$.*

We employ Proposition 2.7 to prove the following.

Proposition 2.8. *Every oriented regular map of type $(4,4)$ is isomorphic to a map $\{4,4\}_{b,c}$.*

Proof. Let $\mathcal{M}$ be an oriented map of type $(4,4)$.

□

It is well-known that a free 2-generator group can be represented as a matrix-group [165, pages 48-49]. Denote by $\nu = 2cos(\pi/k)$, $\eta = 2cos(\pi/m)$ and $\xi = 2cos(\pi/n)$. Set

$$r = \begin{pmatrix} 1 & \xi & \nu\xi+\eta \\ 0 & -1 & -\nu \\ 0 & \nu & \nu^2-1 \end{pmatrix} \tag{2.3}$$

$$\ell = \begin{pmatrix} -1 & -\xi & 0 \\ \xi & \xi^2-1 & 0 \\ \eta & \eta\xi+\nu & 1 \end{pmatrix}. \tag{2.4}$$

Proposition 2.9. *The assignment $r \mapsto R$ and $\ell \mapsto L$ extends to a group monomorphism.*

Hence triangle groups are matrix groups, see [177, 178, 1] for more details. It follows that (hyper)maps can viewed as quotients of certain matrix groups, while regular (hyper)maps correspond to factor groups of the matrix groups representing the respective triangle groups. This approach to maps and hypermaps is used to study regular maps and hypermaps of large planar width. Here we apply this approach to solve

Problem 2.2. Second Grünbaum's problem: Are there for any two positive integers p,q and r $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$, infinitely many maps without semiedges of type (p,q)?

A group G is called *residually finite* if for each element $g \in G$ there exists a finite quotient H of G such that the epimorphism $G \to H$ does not send g onto the identity.

The following deep statement was proved by Mal'cev (see, e.g., Kaplansky [108]).

Theorem 2.10. (Mal'cev) *Every finitely generated matrix group is residually finite.*

Now we are ready to prove.

Proposition 2.11. *For any two positive integers p and q, $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\le 1$ there are infinitely many (oriented) regular map of type (p,q).*

Proof Let $G=\Delta(p,q,r)=\langle r_0,r_1,r_2;\ r_0^2=r_1^2=r_2^2=(r_1r_2)^p=(r_0r_1)^q=(r_2r_0)^r=1\rangle$, be the triangle group. For $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\le 1$ the group is the monodromy group of a hypermap whose associated topological map $\mathcal{M}$ is a 3-valent tesselation of a hyperbolic plane or plane, by p-gons, r-gons and q-gons. Hence G is infinite. For each $w \in G$ denote by $|w|$ the minimum number of generators to express w. Let $S_d=\{w\in\Delta(p,q,r)|\ |w|\le d\}$. Since G is residually finite there is a normal subgroup $N_d \lhd G$ of finite index I_d such that $xy^{-1}\notin N_d$ for any two distinct elements $x,y\in S_d$. It follows that the epimorphism $G\to G/N$ maps S_d injectively.

If $d > max\{2p,2q,2r\}$ then the quotient G/N_d is torsion-free, meaning that p, q and r are true orders of the images of generators. Moreover, for $d > max\{2p,2q,2r\}$ we may find d_1 such that $|S_{d_1}|>|G/N_d|$ and repeat the construction thus getting N_{d_1} such that $|G/N_{d_1}|>|G/N_d|$.

Repeating this procedure we construct infinitely many algebraic hypermaps $A(G/N_d; r_0, r_1, r_2)$ of type (p, q, r). $\square$

Fundamental group of a hypermap

We can go even one step further. Let us denote by

$$\Delta = \Delta(\infty, \infty, \infty) = \langle r_0, r_1, r_2;\ r_0^2 = r_1^2 = r_2^2 = 1\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2,$$

the free product of three two-element groups. Since the monodromy group of any hypermap $\mathcal{H}$ is a finite quotient of Δ we can identify every hypermap with the algebraic hypermap $A(\Delta/S; r_0, r_1, r_2)$ for some $S \leq \Delta$ of finite index. The subgroup S will be called the *fundamental group of a hypermap.* Given monodromy epimorphism $p^* : G \to \mathrm{Mon}(\mathcal{H})$ from a group G generated by 3-involutions we define a *relative fundamental group* $\pi_G(\mathcal{H}, x)$ at a flag x of $\mathcal{H}$ to be the stabilizer of a flag in the action of G on flags via the monodromy epimorphism defined by: $gx = g^*x$, where $g^* = p^*(g)$.

One can study hypermaps via the torsion-free subgroups of Δ of finite index. In what follows we generalize the concepts of maps and hypermaps by relaxing the condition that the monodromy action is torsion-free. The topological counterparts of such hypermaps will be discussed later. Hypermaps satisfying the original requirement (the monodromy action on flags is torsion free) will be called *closed*.

The facts listed in the following statement are well-known between map- and hypermap experts (see [50, 52]).

Theorem 2.12. *Let $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ be hypermaps, and let $G \cong \Delta/K$ and let* $\mathrm{Mon}(\mathcal{H})$, $\mathrm{Mon}(\mathcal{H}_1)$, $\mathrm{Mon}(\mathcal{H}_2)$ *are monodromy images of G. Then the following hold:*

(a) *$\mathcal{H}_1$ covers $\mathcal{H}_2$ if and only if there are $S_1 \leq S_2 \leq G$ such that $\mathcal{H}_1 \cong A(G/S_1;\ r_0, r_1, r_2)$ and $\mathcal{H}_2 \cong A(G/S_2;\ r_0, r_1, r_2)$,*

(b) *$\mathcal{H}_1 \cong \mathcal{H}_2$ if and only if the corresponding relative fundamental groups are conjugate in G,*

(c) *Let $[G : G^+] = 2$. Then $\mathcal{H}$ is orientable if and only if its relative fundamental group is contained in the even-word subgroup $G^+ \leq G$,*

(d) *the fundamental group of the generic hypermap for a an algebraic hypermap given by $S \leq G$ is the largest normal subgroup contained in S. In particular, regular hypermaps covered by $A(G/1; r_0, r_1, r_2)$ correspond to normal subgroups of G of finite index.*

PROOF The proof of (a) is similar to the proof of Proposition 2.4. Let $a_2 = \varphi(a_1)$, $a_i \in F_i$, be two flags. Let κ_i^* be the monodromy epimorphisms $\kappa_i^* : G \to \mathrm{Mon}(\mathcal{H}_i)$ taking r_j onto the respective involutory generators of $\mathrm{Mon}(\mathcal{H}_i)$, for $j = 0, 1, 2$ and $i = 1, 2$. Then G acts on F_i by $gx = \kappa_i^*(g)x$. Let S_i be the stabilizers of a_i for $i = 1, 2$ in the action of G on F_i. This gives Schreier representations $A(G/S_i; r_0, r_1, r_2)$ of $\mathcal{H}_i$ for $i = 1, 2$. Then the covering φ induces an epimorphism φ^* of the monodromy groups defined by $\varphi^*(gS_1) = gS_2$. In particular if $s \in S_1$ then $sS_1 = S_1$ and so $S_2 = \varphi^* s S_1 = s\varphi^* S_1 = sS_2$. Hence $S_1 \leq S_2$. If $S_1 \leq S_2$ then the mapping $gS_1 \mapsto gS_2$ is covering.

(b) Let $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ be an isomorphism between hypermaps. Let $\mathcal{H}_1 \cong A(G/S_1;\, r_0, r_1, r_2)$ and $\mathcal{H}_2 \cong A(G/S_2;\, r_0, r_1, r_2)$ be some Schreier representations of the hypermaps, where S_i are the stabilizers of some flags a_i, $i = 1, 2$. By part (a) if $a_2 = \varphi a_1$ then $S_1 = S_2$, otherwise S_1 is a stabilizer of a flag in $\mathcal{H}_2$ different from φa_1. Since the action of G is transitive, $S_2 = S_1^g$ for some $g \in G$. On the other hand, if $S_2 = S_1^g$ then S_1 is the stabilizer of some flag in $\mathcal{H}_2$, thus $\mathcal{H}_1 \cong A(G/S_1;\, r_0, r_1, r_2) \cong \mathcal{H}_2$.

(c) $\mathcal{H}$ is orientable if and only if the even word subgroup $H^+ < H = \mathrm{Mon}(\mathcal{H})$ is a proper subgroup of index two the monodromy group. Since the monodromy epimorphism $\alpha : G \to H$ takes $G^+ \to H^+$, we are done.

(d) By Proposition 1.17 the generic hypermap $\mathcal{H}^+ \cong A(G/N; r_0, r_1, r_2)$ for some normal subgroup $N \lhd G$. By part (a) there is $S \leq N \lhd G$ such that $\mathcal{H} \cong A(G/S; r_0, r_1, r_2)$. The subgroup N is a maximal normal subgroup of a hypermap subgroup S of $\mathcal{H}$, since if $N < K \leq S$, then by part (a) we get $\mathcal{H}^+ \to A(G/K; r_0, r_1, r_2) \to \mathcal{H}$ contradicting the minimality of $\mathcal{H}^+ \to \mathcal{H}$. □

Note that the group $N \lhd G$ in part (d) of the proof is nothing but the kernel of the monodromy epimorphism $G \to \mathrm{Mon}(\mathcal{H})$ and $\mathrm{Mon}(\mathcal{H}) \cong G/N$.

It follows that fundamental groups of hypermaps play in the theory of hypermaps a similar role as fundamental groups of graphs for the theory of graph coverings. Using the algebraic representation via fundamental groups one can handle many problems. An (oriented) hypermap is called *rooted* if a flag (dart), called *root*, is distinguished. Morphisms between rooted hypermaps take a root onto a root. It is assumed that in an algebraic hypermap $A(\Delta/S; r_0, r_1, r_2)$ the flag S is the root. Given two rooted hypermaps $\mathcal{H}_1$, $\mathcal{H}_2$ with the respective hypermap subgroups S_1, S_2, the intersection $S_1 \cap S_2$ defines the *least common cover* for both $\mathcal{H}_1$ and $\mathcal{H}_2$ and $\langle S_1, S_2 \rangle$ determines the largest rooted hypermap covered my both $\mathcal{H}_1$, $\mathcal{H}_2$ called *the largest common quotient*. The following is straightforward.

Proposition 2.13. *There is an anti-isomorphism between the lattice of subgroups of the free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ and the lattice of rooted (possibly infinite) hypermaps ordered by coverings.*

The sublattice formed by normal subgroups is in the context of this book particularly important, because the corresponding hypermaps are regular hypermaps. So regularity of a hypermap is a synonymum for the normality of the respective fundamental group.

Exercises

2.1.1. Prove that (oriented) regular maps with semiedges are precisely the spherical embeddings of semistars st_n.

2.1.2. Classify regular and oriented regular maps of elliptic types (p, q), $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$. For each map N in the list find a map $\mathcal{M}$ with the least number of flags (darts) such that $N \cong \mathcal{M}^+$ ($N \cong \mathcal{M}^\#$).

2.1.3. Using the construction of a generic map find orientably regular 3-valent maps which faces are bounded by walks of lengths 7,8,9 and 10.

2.1.4. Prove that any bridgeless cubic graph is a truncation of an oriented map. Does the statement hold if we replace 'oriented map' by 'map'?

2.1.5. Find all non-isomorphic oriented maps whose is isomorphic to the Petersen graph. Determine the genera of the generic regular covers of them.

2.1.6. Prove Proposition 2.9.

2.1.7. Determine all maps (up to isomorphism) covered by the cube Q_3.

2.1.8. Determine the regular hypermaps and maps whose automorphism group is $\Delta(k, m, m)$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. Draw the corresponding figures.

2.1.9. Prove that a group G is residually finite if and only if for any finite set S of elements of G there is an epimorphism on G into a finite group which does not take $g \in S$ onto an identity.

2.2 Join and intersection of regular hypermaps

The smallest common cover of given rooted hypermaps $\mathcal{H}_1 = (F_1;\ r_0, r_1, r_2)$ and $\mathcal{H}_2 = (F_2;\ q_0, q_1, q_2)$ can be viewed as a *join of two hypermaps.* The question arises whether we can construct the product explicitly, or in other words, can we derive the monodromy group of the product in terms of the monodromy groups of factors? The most natural approach is to set $F = F_1 \times F_2$ and $p_i(x, y) = (r_i x, q_i y)$ for any $(x, y) \in F$ and $i = 0, 1, 2$. Unfortunately, the hypermap $\mathcal{H} = (F; p_0, p_1, p_2)$ is, in general, not correctly defined since the action of $\mathrm{Mon}(\mathcal{H}_1) \times \mathrm{Mon}(\mathcal{H}_2)$ may not be transitive on F. This and other related questions will be investigated in this section.

Within this section Δ will denote the universal triangle group, $\Delta = \langle r_0, r_1, r_2 | r_0^2 = r_1^2 = r_2^2 = 1\rangle$. In order to shorten explanation, we use the symbol Δ/H to denote an algebraic hypermap $(\Delta/H; r_0, r_1, r_2)$. In case $H \lhd \Delta$ thus Δ/H will denote both the factor group and the respective algebraic hypermap which we hope will make the reader no trouble.

Let $\mathcal{H} = \Delta/H$ and $\mathcal{K} = \Delta/K$ be algebraic hypermaps. Set $\mathcal{H} \vee \mathcal{K} = \Delta/(H \cap K)$ and $\mathcal{H} \wedge \mathcal{K} = \Delta/\langle H, K\rangle$. The hypermaps $\mathcal{H} \vee \mathcal{K}$, $\mathcal{H} \wedge \mathcal{K}$ will be called *join* and *intersection* of $\mathcal{H}$ and $\mathcal{K}$, respectively.

The following two propositions are direct consequences of definitions.

Proposition 2.14. *Let $\mathcal{H} = \Delta/H$ and $\mathcal{K} = \Delta/K$ be algebraic hypermaps. Then*

(1) *if both $\mathcal{H}$ and $\mathcal{K}$ are finite then $\mathcal{H} \vee \mathcal{K}$ and $\mathcal{H} \wedge \mathcal{K}$ are finite,*

(2) *if both $\mathcal{H}$ and $\mathcal{K}$ are regular then $\mathcal{H} \vee \mathcal{K}$ and $\mathcal{H} \wedge \mathcal{K}$ are regular as well,*

(3) *if $\mathcal{H} \to \mathcal{K}$ is a covering then $\mathcal{H} \vee \mathcal{K} = \mathcal{H}$ and $\mathcal{H} \wedge \mathcal{K} = \mathcal{K}$.*

(4) *if a hypermap* $\mathcal{X} = \Delta/X$ *covers both* $\mathcal{H}$ *and* $\mathcal{K}$ *then it covers* $\mathcal{H} \vee \mathcal{K}$*,*

(5) *if a hypermap* $\mathcal{X} = \Delta/X$ *is covered by both* $\mathcal{H}$ *and* $\mathcal{K}$ *then it is covered by* $\mathcal{H} \vee \mathcal{K}$*,*

Proposition 2.15. *If* $\mathcal{H}$ *and* $\mathcal{K}$ *are regular hypermaps then* $\mathcal{H} \vee \mathcal{K}$ *and* $\mathcal{H} \wedge \mathcal{K}$ *are well-defined binary operations on isomorphism classes of hypermaps.*

Proof By normality the fundamental groups are unique. □

It follows from the above propositions that for any two regular hypermaps $\mathcal{H}$ and $\mathcal{K}$ there is a unique regular hypermap $\mathcal{Y} = \mathcal{H} \vee \mathcal{K}$, called the *least common cover*, satisfying the following property: if $\mathcal{X} \to \mathcal{H}$ and $\mathcal{X} \to \mathcal{K}$ then $\mathcal{X}$ covers the join $\mathcal{Y}$. Similarly, any regular hypermap covered by two regular hypermaps is covered by $\mathcal{H} \wedge \mathcal{K}$ and so we can view the intersection as the *largest common quotient* of $\mathcal{H}$ and $\mathcal{K}$.

The following lemma lists the properties of the join and intersection which are trivial consequences of the definitions. In particular, regular hypermaps form a lattice anti-isomorphic to the lattice of all normal subgroups of Δ. The ordering on regular hypermaps is given by hypermap coverings.

Lemma 2.16. *Let* $\mathcal{X}$*,* $\mathcal{Y}$ *and* $\mathcal{Z}$ *be algebraic hypermaps (regular hypermaps). Let* Δ *and* 1 *be the universal and trivial hypermaps. Then*

(1) $\mathcal{X} \vee (\mathcal{Y} \vee \mathcal{Z}) = (\mathcal{X} \vee \mathcal{Y}) \vee \mathcal{Z}$*,*

(2) $\mathcal{X} \vee \mathcal{Y} = \mathcal{Y} \vee \mathcal{X}$*,*

(3) $\mathcal{X} \vee \Delta = \Delta$ *and* $\mathcal{X} \vee 1 = \mathcal{X}$*,*

(4) $\mathcal{X} \wedge (\mathcal{Y} \vee \mathcal{Z}) \to (\mathcal{X} \wedge \mathcal{Y}) \vee (\mathcal{X} \wedge \mathcal{Z})$.

Interchanging joins and intersections in the above statements we get a dual version of the above lemma. In particular, we have

$$\mathcal{X} \vee (\mathcal{Y} \wedge \mathcal{Z}) \leftarrow (\mathcal{X} \vee \mathcal{Y}) \wedge (\mathcal{X} \vee \mathcal{Z}).$$

Denote by $|\mathcal{H}|$ the number of flags of a hypermap $\mathcal{H}$. Of course if $\mathcal{H}$ is a regular hypermap we have $|\mathcal{H}| = |\mathrm{Mon}(\mathcal{H})| = |\Delta/H|$. The following statement relates the monodromy groups of the join and intersection of hypermaps with the monodromy groups of the original hypermaps.

Proposition 2.17. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular hypermaps. Then the monodromy group of $\mathcal{H} \vee \mathcal{K}$ is a subgroup of the direct product* $\mathrm{Mon}(\mathcal{H}) \times \mathrm{Mon}(\mathcal{K})$ *and we have*

$$\mathrm{Mon}(\mathcal{H} \wedge \mathcal{K}) = \mathrm{Mon}(\mathcal{H} \vee \mathcal{K})/\big(H/H \cap K \times K/H \cap K\big),$$

where $H/H \cap K \times K/H \cap K$ is an internal direct product. Moreover,

$$|\mathcal{H} \vee \mathcal{K}| \cdot |\mathcal{H} \wedge \mathcal{K}| = |\mathcal{H}| \cdot |\mathcal{K}|.$$

Proof We show that the mapping $\psi : g(H \cap K) \mapsto (gH, gK)$ is a monomorphism $\Delta/(H \cap K) \to \Delta/H \times \Delta/K$. Indeed, for any $x, y \in \Delta$

$$\psi((xH \cap K)(yH \cap K)) = \psi(xyH \cap K) = (xyH, xyK) =$$

$$(xH, xK)(yH, yK) = \psi(xH \cap K)\psi(yH \cap K).$$

Now let $\psi(xH \cap K) = 1 = (H, K)$ for some $x \in \Delta$. Then $(xH, xK) = (H, K)$, and consequently $x = 1$. Hence, ψ is a monomorphism.

By the third isomorphism theorem

$$HK/H \cap K = H/H \cap K \times K/H \cap K \cong HK/K \times HK/H.$$

Using this we get

$$|\mathcal{H} \vee \mathcal{K}||\mathcal{H} \wedge \mathcal{K}| = |\Delta/H \cap K||\Delta/HK| = |\Delta/HK||HK/K \times HK/H||\Delta/HK| =$$

$$|\Delta/HK||HK/K||\Delta/HK||HK/H| = |\Delta/K||\Delta/H| = |\mathcal{K}||\mathcal{H}|.$$

By the second isomorphism theorem we obtain

$$\mathrm{Mon}(\mathcal{H} \wedge \mathcal{K}) = \Delta/HK \cong (\Delta/H \cap K)/(HK/H \cap K) =$$

$$\mathrm{Mon}(\mathcal{H} \vee \mathcal{K})/\big(H/H \cap K \times K/H \cap K\big).$$

$\square$

The equality $|\mathcal{H} \vee \mathcal{K}| \cdot |\mathcal{H} \wedge \mathcal{K}| = |\mathcal{H}| \cdot |\mathcal{K}|$, combined with the well-known statement in elementary number theory establishing

$$|\mathcal{H}| \cdot |\mathcal{K}| = gcd(|\mathcal{H}|, |\mathcal{K}|) \cdot lcm(|\mathcal{H}|, |\mathcal{K}|),$$

may suggest that $|\mathcal{H} \vee \mathcal{K}| = lcm(|\mathcal{H}|, |\mathcal{K}|)$, or equivalently $|\mathcal{H} \wedge \mathcal{K}| = gcd(|\mathcal{H}|, |\mathcal{K}|)$. However, this is not true in general. In general, we can only claim that $lcm(|\mathcal{H}|, |\mathcal{K}|)$ divides $|\mathcal{H} \vee \mathcal{K}|$, and $|\mathcal{H} \wedge \mathcal{K}|$ divides $gcd(|\mathcal{H}|, |\mathcal{K}|)$. The above two equalities imply

$$\frac{|\mathcal{H} \vee \mathcal{K}|}{lcm(|\mathcal{H}|, |\mathcal{K}|)} = \frac{gcd(|\mathcal{H}|, |\mathcal{K}|)}{|\mathcal{H} \wedge \mathcal{K}|}.$$

This observation led us to a new concept allowing us to relate two hypermaps. Given two regular hypermaps $\mathcal{H}$ and $\mathcal{K}$ the integer

$$s(\mathcal{H}, \mathcal{K}) = \frac{|\mathcal{H} \vee \mathcal{K}|}{lcm(|\mathcal{H}|, |\mathcal{K}|)} = \frac{gcd(|\mathcal{H}|, |\mathcal{K}|)}{|\mathcal{H} \wedge \mathcal{K}|}$$

will be called the *shared cover index* of $\mathcal{H}$ and $\mathcal{K}$. Clearly, if one of $\mathcal{H}$, $\mathcal{K}$ covers the other then $s(\mathcal{H}, \mathcal{K}) = 1$. Generally, it can be equal to any divisor of $gcd(|\mathcal{H}|, |\mathcal{K}|)$.

Replacing hypermaps by oriented hypermaps one can see that the concept of the shared cover index applies in the category of oriented regular maps as well.

Monodromy groups of the join and intersection

Throughout this section all the considered hypermaps will be regular. In the above section we have derived some information on the structure of the monodromy groups of $\mathcal{H} \vee \mathcal{K}$ and $\mathcal{H} \wedge \mathcal{K}$. In what follows we shall consider the problem how to calculate the above monodromy groups by using the action of monodromy groups of $\mathcal{H}$ and $\mathcal{K}$. Let $A = \langle t_0, \ldots, t_k \rangle$ and $B = \langle s_0, \ldots, s_k \rangle$ be two k-generated groups. Let us define their monodromy product $A \times_m B$ to be the subgroup of the direct product generated by (t_i, s_i), where $i = 0, 1, \ldots, k$. Note that S. Wilson calls it the parallel product in [**?**]. Further, denote by $\pi_1 : A \times_m B \to A$, $\pi_2 : A \times_m B \to B$ the natural projections erasing the second and first coordinate, respectively.

Theorem 2.18. *Let $\mathcal{H} = (A; t_0, t_1, t_2)$ and $\mathcal{K} = (B; s_0, s_1, s_2)$ be regular hypermaps, and let $\Delta = \langle r_0, r_1, r_2 | r_0^2 = r_1^2 = r_2^2 = 1\rangle$. Then* $\mathrm{Mon}(\mathcal{H} \vee \mathcal{K}) = \mathrm{Mon}(\mathcal{H}) \times_m \mathrm{Mon}(\mathcal{K})$ *and* $\mathrm{Mon}(\mathcal{H} \wedge \mathcal{K}) = \mathrm{Mon}(\mathcal{H}) \times_m \mathrm{Mon}(\mathcal{K})/Ker\,\pi_2 Ker\,\pi_1$.

Proof Recall that the fundamental group of $\mathcal{H}$ can be reconstructed as a stabiliser of a flag x_0 in the action of $H = \Delta_{x_0}$ and similarly for $\mathcal{K}$, $K = \Delta_{y_0}$. Denote by $\psi_1 : \mathcal{H} \to A(\Delta/H; r_0H, r_1H, r_2H)$ the isomorphism of hypermaps and by $\psi_1^* : \mathrm{Mon}\mathcal{H} \to \Delta/H$ the induced group epimorphism sending $t_i \mapsto r_iH$, for $i = 0, 1, 2$. Similarly, denote by ψ_2 the isomorphism $\mathcal{K} \to A(\Delta/K; r_0K, r_1K, r_2K)$ of hypermaps and by ψ_2^* the respective group epimorphism taking $s_i \mapsto r_iK$. Then we have an isomorphism $\Psi : \Delta/H \times_m \Delta/K \to \mathrm{Mon}(\mathcal{H}) \times_m \mathrm{Mon}(\mathcal{K})$ taking

$$(t_iH, r_iK) \to ((\psi_1^*)^{-1}(r_iH), (\psi_2^*)^{-1}(r_iK)) = (r_i, s_i).$$

In the proof of Proposition 2.17 we have already verified that the mapping $\Phi : \Delta/H \cap K \to \Delta/H \times_m \Delta/K$, taking $g(H \cap K)$ onto (gH, gK), is an isomorphism of groups. Now the composition $\Psi\Phi$ establishes an isomorphism $\mathrm{Mon}(\mathcal{H} \vee \mathcal{K}) \to \mathrm{Mon}(\mathcal{H}) \times_m \mathrm{Mon}(\mathcal{K})$.

Regarding the intersection of $\mathcal{H}$ and $\mathcal{K}$, by Proposition 2.17 we have

$$\mathrm{Mon}(\mathcal{H} \wedge \mathcal{K}) = \mathrm{Mon}(\mathcal{H} \vee \mathcal{K})/(H/H \cap K \times K/H \cap K).$$

In view of what we have proved it is enough to see that $\Psi\Phi$ sends $K/H \cap K$ onto Ker π_2, and $H/H \cap K$ onto Ker π_1. Indeed,

$$\Psi\Phi(K/H \cap K) = \Psi(\{(gH, K) | g \in K\}) =$$

$$\{(w, 1) | (w, 1) \in \mathrm{Mon}(\mathcal{H}) \times_m \mathrm{Mon}(\mathcal{K})\} = \mathrm{Ker}\,\pi_2.$$

Similar calculation verifies the statement $\Psi\Phi(H/H \cap K) = \mathrm{Ker}\,\pi_1$. □

Chirality of hypermaps

Let $\Delta = \langle r_0, r_1, r_2 | r_0^2 = r_1^2 = r_2^2 = 1\rangle$ and $H \leq \Delta^+$, where $\Delta^+ = \langle r_0r_1, r_1r_2\rangle = \langle L, R\rangle$. Since $H \lhd \Delta^+$ then $H^{r_0} = H^{r_1} = H^{r_2}$ and we may denote this common conjugate by H^r. Then $\mathcal{H} = A(\Delta^+/H; R, L)$ is a regular oriented hypermap and $\mathcal{H}^r = A(\Delta/H^r; r_0, r_1, r_2)$ is the *mirror image* of $\mathcal{H}$. We call $\mathcal{H}$ *mirror symmetric* or *reflexible* if $\mathcal{H} \cong \mathcal{H}^r$. Mirror asymmetric hypermaps are called chiral.

Topologically speaking, a reflexible oriented hypermap admits an orientation reversing self-homeomorphism of the supporting surface preserving the embedded graph and colours of faces. All spherical regular oriented hypermaps are reflexible. Figure 2.2 shows the least chiral orientably regular map determining a 2-cell embedding of K_5 into the torus.

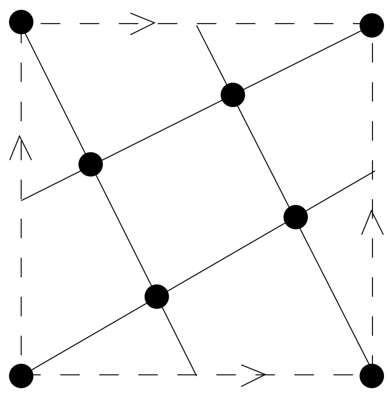


Figure 2.3: The smallest chiral orientably regular map is toroidal $\{4,4\}_{2,1}$

From the first point of view the "chirality" seems to be a binary invariant. Surprisingly, it turns out [32] that one can measure it by a group, called the *chirality group* $X(\mathcal{H})$ of an orientably regular hypermap $\mathcal{H}$, measuring of how much given oriented hypermap deviates from being mirror symmetric. The chirality group is the group $H/(H \cap H^r) \cong HH^r/H$ (by the isomorphism theorems).

The integer $\kappa(\mathcal{H}) = |H/(H \cap H^r)|$ is called the *chirality index* of $\mathcal{H}$.

Proposition 2.19. *Let $\mathcal{H}$ be an oriented regular hypermaps and $\mathcal{H}^r$ is the mirror image of it. Then $s(\mathcal{H}, \mathcal{H}^r) = \kappa(\mathcal{H})$, where $\kappa(\mathcal{H})$ is the chirality index of $\mathcal{H}$.*

PROOF

$$s(\mathcal{H},\mathcal{H}^r) = \frac{|\mathcal{H}\vee\mathcal{H}^r|}{lcm(|\mathcal{H}|,|\mathcal{H}^r|)} = \frac{|\mathcal{H}\vee\mathcal{H}^r|}{|\mathcal{H}|} = \frac{|\Delta/H\cap H^r|}{|\Delta/H|} =$$

$$|H/H\cap H^r| = \kappa(\mathcal{H}).$$

It follows that $X(\mathcal{H})$ is trivial if and only if $\mathcal{H}$ is reflexible. Structure of chirality groups is studied in [32] in a more detail. It is proved there that every abelian group is the chirality group of an oriented regular hypermap. On the other hand, many non-abelian groups including symmetric groups and dihedral groups cannot serve as chirality groups. If the presentation of the monodromy group of a hypermap is known, we can compute the chirality group employing the following statement.

Theorem 2.20. *Let $\mathcal{H}$ be a regular oriented hypermap with the monodromy group:*

$$G = \mathrm{Mon}(\mathcal{H}) = \langle r,\ell|\mathcal{R}(r,\ell)\rangle,$$

where $\mathcal{R}(r,\ell)$ is a finite set of relators.

Then the chirality group of $\mathcal{H}$ is isomorphic to the normal closure of $\mathcal{R}(r^{-1},\ell^{-1})$ in G.

PROOF Since $G=\Delta^+/H$ we have

$$G/X(\mathcal{H}) \cong (\Delta^+/H)/(HH^r/H) \cong \Delta^+/HH^r = \langle r,\ell|\mathcal{R}(r,\ell),\mathcal{R}(r^{-1},\ell^{-1})\rangle.$$

By Von Dyck theorem [107, page 28], $X(\mathcal{H}) = \langle\mathcal{R}(r^{-1},\ell^{-1})\rangle^G$. □

Example 2.2. Consider the metacyclic group

$$G = \langle\, a,b \mid a^n=1,\ b^m=a^s,\ bab^{-1}=a^r\,\rangle$$

of order mn, where $rs\equiv s \pmod n$ and $r^m\equiv 1 \pmod n$ for some $m>2$. If we regard a and b as the monodromy generators for an orientably regular hypermap $\mathcal{H}=A(G/1;a,b)$ with monodromy group G, then the chirality group $X(\mathcal{H})$ is the least normal subgroup N of G such that the assignment $a\mapsto a^{-1}$ and $b\mapsto b^{-1}$ induces an automorphism of G/N. We obtain this quotient from G by adding the extra relations formed from those of G by replacing a and b with their inverses. In this case, it is sufficient to add $b^{-1}a^{-1}b=a^{-r}$, or equivalently, $b^{-1}ab=a^r$, so that $a = b(b^{-1}ab)b^{-1} = (a^r)^r = a^{r^2}$ in G/N. Thus $a^{r^2-1} = 1$ in G/N, so in G it follows that $K = \langle a^{r^2-1}\rangle$ is a subgroup of N. On the other hand, it is easy to see that K is a normal subgroup of G such that G/K is invariant under replacement of the generators with their inverses. By minimality $N = K$, so $X(\mathcal{H}) \cong \langle a^{r^2-1}\rangle$ and $\kappa(\mathcal{H}) = n/\gcd(n, r^2 - 1)$; since $m > 2$, this can be arbitrarily large.

Recall that the oriented map $\{4,4\}_{b,c}$ (see Exercises) has monodromy group $G = \langle r, \ell | r^4 = \ell^2 = (r\ell)^4 = (r\ell r)^b(r^2\ell)^c = 1\rangle$. Employing Theorem 2.20 the chirality groups of toroidal regular maps are computed in [35]. In particular, we have

Theorem 2.21. *The chirality group of* $\{4,4\}_{b,c}$ *is cyclic. The chirality index is given by*

$$\kappa = \frac{n}{(n, 2d^2)}$$

where $n = b^2 + c^2$ *and* $d = \gcd(b, c)$.

Exercises

2.2.1. Find all non-isomorphic regular oriented hypermaps whose monodromy group is either A_n for $n \leq 6$ or S_n for $n \leq 5$. Prove that all of the hypermaps are reflexible.

2.2.2. Prove that for each $n \geq 7$ there are regular oriented hypermaps whose monodromy group is A_n and chirality index is $n!/2$.

2.2.3. For which of the maps $\{4,4\}_{b,c}$ the chirality index is $b^2 + c^2$ and when it is 1?

2.2.4. A regular oriented map $\mathcal{M}$ oriented map is indecomposable, if $\mathcal{M} = \mathcal{K} \vee \mathcal{N}$ implies that either $\mathcal{K} \cong \mathcal{M}$ or $\mathcal{N} \cong \mathcal{M}$. Classify indecomposable maps between the maps $\{4,4\}_{a,b}$.

2.2.5. Prove that a regular hypermap $\mathcal{H}$ is indecomposable if and only if $G = \mathrm{Mon}(\mathcal{H})$ contains exactly one maximal normal subgroup $N < G$, $N \neq G$.

2.2.6. Prove that for every integer $p > 3$ there infinitely many self-dual maps of type (p, p).

2.2.7. Prove that there are infinitely many self-Petrie-dual maps.

2.3 Orthogonal hypermaps and direct products

Two regular hypermaps $\mathcal{H}$, $\mathcal{K}$ will be called *orthogonal* if $HK = \Delta$. We shall use $\mathcal{H} \perp \mathcal{K}$ to denote the orthogonality of $\mathcal{H}$ and $\mathcal{K}$. Let G, H be two groups. A *common epimorphic* image of G and H is a group Q such that there are epimorphisms $G \to Q$ and $H \to Q$. Let $H = \langle r_0, r_1, r_2 \rangle$, $K = \langle s_0, s_1, s_2 \rangle$ and $Q = \langle t_0, t_1, t_2 \rangle$ be groups. We say that Q is a *monodromic common epimorphic image* of H and K if both the assignments $r_i \mapsto t_i$ and $s_i \mapsto t_i$ (for $i = 0, 1, 2$) extend to group epimorphisms $H \to Q$ and $K \to Q$.

The following theorem gives several characterisations of the orthogonality.

Theorem 2.22. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular hypermaps. Then the following conditions are equivalent:*

(1) $\mathcal{H} \perp \mathcal{K}$,

(2) $\mathcal{H} \wedge \mathcal{K}$ *is a trivial hypermap,*

(3) $\mathcal{H}$ *and* $\mathcal{K}$ *have no nontrivial common quotients,*

(4) *the monodromy groups* $\mathrm{Mon}(\mathcal{H})$ *and* $\mathrm{Mon}(\mathcal{K})$ *have no common monodromic epimorphic images,*

(5) $\mathrm{Mon}(\mathcal{H} \vee \mathcal{K}) = \mathrm{Mon}(\mathcal{H}) \times \mathrm{Mon}(\mathcal{K})$.

Proof *(i)* $\Leftrightarrow$ *(ii)* Since the flags of the intersection are the elements of Δ/HK, the intersection is a trivial hypermap if and only if $HK = \Delta$.

(ii) $\Leftrightarrow$ *(iii)*. If $\mathcal{H} \wedge \mathcal{K}$ is nontrivial then it forms a non-trivial common quotient.

Vice-versa if there is a non-trivial common (possibly irregular) quotient $\mathcal{Q}$ then there are $g, h \in \Delta$ such that $\Delta > Q^g \geq K$ and $\Delta > Q^h \geq H$. By normality of both H and K we get $\Delta > Q^x \geq K$, $\Delta > Q^x \geq H$ for any $x \in \Delta$. Hence $\mathcal{Q}_\Delta = \Delta / \bigcap_{x \in \Delta} Q^x \to \mathcal{Q}$ is a non-trivial regular common quotient. However, since $\mathcal{Q}_\Delta$ is covered by $\mathcal{H} \wedge \mathcal{K}$, thus the intersection is a non-trivial hypermap.

$(ii) \Leftrightarrow (v)$. By Proposition 2.17 $\mathrm{Mon}(\mathcal{H} \vee \mathcal{K}) \leq \mathrm{Mon}(\mathcal{H}) \times \mathrm{Mon}(\mathcal{K})$. The second part of Proposition 2.17 implies that the equality holds if and only if $\mathcal{H} \perp \mathcal{K}$.

$(iii) \Leftrightarrow (iv)$ If there is a common quotient $\mathcal{Q}$ for $\mathcal{H}$ and $\mathcal{K}$ then the coverings $\mathcal{H} \to \mathcal{Q}$ and $\mathcal{K} \to \mathcal{Q}$ induce, respectively, monodromy epimorphisms $\mathrm{Mon}(\mathcal{H}) \to \mathrm{Mon}(\mathcal{Q})$ and $\mathrm{Mon}(\mathcal{K}) \to \mathrm{Mon}(\mathcal{Q})$. Vice-versa, if Q is a monodromic common epimorphic image, then representing the hypermaps via hypermap subgroups we get that the assignments $gH \mapsto gQ$, $gK \mapsto gQ$, where g ranges in Δ extend to group epimorphisms. However, the same mappings establish coverings $\Delta/H \to \Delta/Q$ and $\Delta/K \to \Delta/Q$. The statement follows. □

Denote by $\mathcal{O} = (F; r_0, r_1, r_2)$ the two-flag hypermap with $r_0 = r_1 = r_2$ being equal to the non-trivial involution interchanging the two flags. It is easy to see that the hypermap subgroup of $\mathcal{O}$ is Δ^+.

Proposition 2.23. *Let $\mathcal{H}$ and $\mathcal{K}$ be (regular) hypermaps. If $\mathcal{H}$ and $\mathcal{K}$ are orthogonal then at least one of the hypermaps $\mathcal{H}$ and $\mathcal{K}$ is nonorientable.*

Proof Assume both $\mathcal{H}$ and $\mathcal{K}$ are orientable. The orientability implies that both $H \leq \Delta^+$, $K \leq \Delta^+$ are subgroups of the even-word subgroup Δ^+ of Δ. Then $\mathcal{O} \cong \Delta/\Delta^+$ is a common non-trivial quotient, a contradiction. □

In general, it can be difficult to see the orthogonality of hypermaps. In what follows we give some sufficient conditions implying the orthogonality of hypermaps. The following proposition is a straightforward consequence of Theorem 2.22.

Proposition 2.24. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular hypermaps. If the monodromy groups of $\mathcal{H}$ and $\mathcal{K}$ have no nontrivial common epimorphic images then the hypermaps $\mathcal{H}$ and $\mathcal{K}$ are orthogonal.*

Thus a regular hypermap with a non-abelian simple monodromy group is orthogonal to any other hypermap.

Numerical conditions implying the orthogonality may be useful in constructions. We shall present a sample of them.

Proposition 2.25. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular hypermaps of types (m_0, m_1, m_2) and (n_0, n_1, n_2). Let one of them, say $\mathcal{H}$, be non-orientable.*

If for any two $i, j \in \{0, 1, 2\}$ the integers m_i, n_i and m_j, n_j are respectively coprimes then the hypermaps $\mathcal{H}$ and $\mathcal{K}$ are orthogonal.

Proof Let the monodromy groups be generated by the triples of involutions $\mathrm{Mon}(\mathcal{H}) = \langle r_0, r_1, r_2\rangle$ and $\mathrm{Mon}(\mathcal{K}) = \langle s_0, s_1, s_2\rangle$. Denote by $R_i = r_i r_{i+1}$ and $S_i = s_i s_{i+1}$, $i = 0, 1, 2$. By the assumption, two of $gcd(R_i, S_i)$, $i = 0, 1, 2$ are equal to 1. Without loss of generality we assume $gcd(R_0, S_0) = 1$ and $gcd(R_2, S_2) = 1$. Then the following equality for the even word subgroup of the monodromy product holds true:

$$(\mathrm{Mon}(\mathcal{H})\times_m \mathrm{Mon}(\mathcal{K}))^+ = \langle R_0, R_2\rangle \times_m \langle S_0, S_2\rangle = \mathrm{Mon}^+(\mathcal{H})\times_m \mathrm{Mon}^+(\mathcal{K}).$$

To prove the orthogonality of $\mathcal{H}$ and $\mathcal{K}$ we show that the projections of the latter group into the coordinate factors contain isomorphic copies of the even-word subgroups of the original hypermaps. Since the orders of R_0 and S_0 are coprime, $(R_0, 1)$ and $(1, S_0)$ are elements of the cyclic group $\langle (R_0, S_0)\rangle$. For the same reason we see that $(R_2, 1)$ and $(1, S_2)$ belong to $\langle (R_2, S_2)\rangle$. Now observe $\mathrm{Mon}^+(\mathcal{H}) \cong \langle (R_0, 1), (R_2, 1)\rangle$, and similarly we get $\mathrm{Mon}^+(\mathcal{K}) \cong \langle (1, S_0), (1, S_2)\rangle$. Hence we have that $\mathrm{Mon}(\mathcal{H} \vee \mathcal{K})$ contains a subgroup $G = \mathrm{Mon}^+(\mathcal{H}) \times \mathrm{Mon}^+(\mathcal{K})$. Since $\mathcal{H}$ is non-orientable, $\mathrm{Mon}^+(\mathcal{H}) = \mathrm{Mon}(\mathcal{H})$. By Theorem 2.18 the monodromy group of the intersection

$$\mathrm{Mon}(\mathcal{H} \wedge \mathcal{K}) = \mathrm{Mon}(\mathcal{H}) \times_m \mathrm{Mon}(\mathcal{K})/Ker\,\pi_2 Ker\,\pi_1.$$

Since $\mathrm{Mon}(\mathcal{H}) \times \mathrm{Mon}^+(\mathcal{K}) \leq Ker\,\pi_2 Ker\,\pi_1$ the intersection is either trivial or a 2-flag hypermap. However, the only (regular) 2-flag hypermap is $\mathcal{O}$ which is obviously not covered by $\mathcal{H}$. Hence, the intersection is the trivial hypermap and we are done. □

A cellular embedding of a graph into a surface is called a *regular embedding* if the corresponding map is regular.

Proposition 2.26. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular maps determined by regular embeddings of two non-bipartite graphs with coprime valency. Then $\mathcal{H} \perp \mathcal{K}$ if and only if at least one of $\mathcal{H}$, $\mathcal{K}$ is non-orientable.*

Proof If both embeddings define orientable maps then they both cover $\mathcal{O}$, and consequently, they are not orthogonal.

Let one of the maps associated with the embeddings of graphs is non-orientable. With the same notation as above we have $R_2^2 = 1 = S_2^2$, because the hypermaps are maps now. Since the valences of the maps are coprime we have that $(R_1, 1)$ and $(1, S_1)$ belong to the monodromy product of the even word subgroups. Since the graphs are non-bipartite there are identities of the form $\prod_{i=1}^{k} R_1^{m_i} R_2 = 1$, $\prod_{i=1}^{n} S_1^{p_i} S_2 = 1$, where k and n are some odd integers. Replacing above R_1 by $(R_1, 1)$, R_2 by (R_2, S_2), S_1 by $(1, S_1)$ and S_2 by (R_2, S_2) we get that the involutions $(R_2, 1)$ and $(1, S_2)$ are elements of the monodromy product. Hence the even-word subgroup of the monodromy product is the direct product of the even-word subgroups of the original maps. Now we can complete the proof as above. □

There are oriented versions of the above propositions. We shall state them without proofs.

Proposition 2.27. *Let $\mathcal{H}$ and $\mathcal{K}$ be oriented regular hypermaps of types (m_0, m_1, m_2) and (n_0, n_1, n_2).*

If for any two $i, j \in \{0, 1, 2\}$ the integers m_i, n_i and m_j, n_j are respectively coprimes then the hypermaps $\mathcal{H}$ and $\mathcal{K}$ are orthogonal.

A cellular embedding of a graph into an orientable surface is called *orientably regular* if the corresponding oriented map is regular.

Proposition 2.28. *Orientably regular embeddings of non-bipartite graphs with coprime valency determine a couple of orthogonal oriented maps.*

2.4 Operations on maps and hypermaps, external symmetries

Generally, an operation Φ is a function associating a given (hyper)map $\mathcal{M}$ another (hyper)map $\Phi(\mathcal{M})$. Typically, we require that Φ preserve some important properties of maps such as the underlying surface, or the underlying graph, or the monodromy group, or the hypermap subgroup etc. With each operation Φ and a fixed map $\mathcal{M}$ a family of external symmetries, defined by the relation $\mathcal{M} \cong \Phi(\mathcal{M})$, is associated. Depending on the class of operations we consider, the external symmetries, called exomorphisms in [155], form a group $\mathrm{Exo}(\mathcal{M})$ containing the automorphism group $\mathrm{Aut}(\mathcal{M})$ as a (normal) subgroup. The factor group $\mathrm{Ex}(\mathcal{M}) = \mathrm{Exo}(\mathcal{M})/\mathrm{Aut}(\mathcal{M})$ then can be interpreted as a group of outer symmetries leaving $\mathcal{M}$ invariant. A systematic study of actions of operations on regular maps was done by S. Wilson in his thesis.

Functors

Important class of operations is formed by *functors* between categories of hypermaps, i.e., it is required that morphisms between hypermaps are preserved. Perhaps the most familiar functors in the category of nonoriented maps is the *duality* operation defined by $(F;\, \lambda, \rho, \tau) \mapsto (F;\, \tau, \rho, \lambda)$, and the *Petrie duality* operation defined by $(F;\, \lambda, \rho, \tau) \mapsto (F;\, \tau, \rho, \lambda)$. These two functors have a nice geometric description as well.

Let $N \lhd \Delta$ and $G = \Delta/N$. Consider a category $\mathcal{C} = \mathcal{C}(G)$ of hypermaps formed by the set of hypermaps covered by $\mathcal{U} = A(\Delta/N; r_0, r_1, r_2)$. The outer automorphism group $Out(G) = \mathrm{Aut}(G)/Inn(G)$ of $G = \Delta/N$ acts on isomorphism classes of hypermaps, as follows. If $\mathcal{H} \cong A(G/S; r_0, r_1, r_2)$ and $[g] \in Out(G)$ then $[\mathcal{H}] \mapsto [A(G/S^g; r_0, r_1, r_2)]$, where S^g is the image of S under $g \in \mathrm{Aut}(G)$. Since $S_1 \leq S_2 \leq G$ is mapped by g onto $S_1^g \leq S_2^g \leq G$, in view of Theorem 2.12 $[g]$ fixes $\mathcal{C}$, $[g]$ permutes the isomorphism classes and coverings between hypermaps in $\mathcal{C}$ are preserved. Hence each element of $Out(G)$ determines a functor for the category $\mathcal{C}$. We summarize the above discussion as follows.

Proposition 2.29. *Let $\mathcal{C}(G)$ be a category of hypermaps with the universal monodromy group $G = \Delta/N$, for some $N \lhd \Delta$. Then each element of $Out(G)$ determines a functor for $\mathcal{C}(G)$ and the automorphism group of $\mathcal{C}(G)$ is isomorphic to $Out(G)$.*

Lemma 2.30. *Given category of hypermaps $\mathcal{C}(G)$ the group of exomorphisms of $\mathcal{H}$ is isomorphic to stabilizer* $\mathrm{Exo}(\mathcal{H}) \cong STAB_{\mathrm{Aut}(G)}(S)/S$. *In particular,* $\mathrm{Aut}(\mathcal{H}) \lhd \mathrm{Exo}(\mathcal{H})$.

In what follows we shall discuss distinguished categories of hypermaps induced by triangle groups $G = \Delta(k, m, n)$, $G = \Delta^+(k, m, n)$ separately. We write $\mathcal{C}(k, m, n)$ instead $\mathcal{C}(\Delta(k, m, n))$.

ORIENTED HYPERMAPS: $\mathcal{C}^+(\infty, \infty, \infty)$.

In this case the universal monodromy group is a free group F_2 of rank two. Nielsen [160] (see [131, page 25]) proved the following theorem.

Theorem 2.31. (Nielsen) *Let* $\Phi : \mathrm{Aut}(F_2) \to GL(2, \mathbb{Z})$ *be the homomorphism induced by the abelianization of* F_2. *Then* $\mathrm{Ker}(\Phi) \cong \mathrm{Inn}(F_2)$ *and consequently,* $\mathrm{Out}(F_2) \cong GL(2, \mathbb{Z})$.

Let $F_2 = \langle R, L\rangle$. Define some automorphisms of F_2 by setting:

1. $\rho : R \mapsto R^{-1}L^{-1},\ L \mapsto L$, vertex-face duality,
2. $\sigma : R \mapsto L,\ L \mapsto R$, vertex-edge duality,
3. $\tau : R \mapsto R,\ L \mapsto L^{-1}$, oriented Petrie duality,
4. $\beta : R \mapsto R^{-1},\ L \mapsto L^{-1}$, reflection.

One can prove that the respective operators $\bar\rho$, $\bar\sigma$ and $\bar\tau$ on oriented hypermaps $(D; R, L)$ generate the group of functors. One can check, that the group of functors is an amalgam of two dihedral groups $\langle\bar\tau, \bar\sigma\rangle \cong D_4$ and $\langle\bar\sigma, \bar\rho\rangle \cong D_6$, whose intersection is $\langle\bar\sigma, \bar\beta\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The group of dualities $\langle\bar\sigma, \bar\rho\rangle$ has size 12. The geometric explanation follows: it permutes hyper-vertices, -edges and -faces, and preserves or switches the orientation. The reflection $\bar\sigma$ is a central involution of $\mathrm{Out}(F_2)$.

Proposition 2.32. $\mathrm{Out}(F_2) \cong GL(2, \mathbb{Z}) \cong \langle\bar\rho, \bar\sigma, \bar\tau\rangle$.

HYPERMAPS: $\mathcal{C}(\infty, \infty, \infty)$.

Given a hypermap $\mathcal{H} = (F; r_0, r_1, r_2)$ we can derive virtually six hypermaps by permuting the three colours 0, 1, 2 of its cells; in fact, for each permutation $\sigma \in S_3 = S_{\{0,1,2\}}$ we define the σ-dual $\mathcal{H}^\sigma$ to be the hypermap $\mathcal{H} = (F; r_{\sigma 0}, r_{\sigma 1}, r_{\sigma 2})$. As expected this six operations form a group (see Machi [134]). L. James [92] proved that the outer automorphism group of $\Delta(\infty,\infty,\infty)$ is isomorphic to $PGL(2,\mathbb{Z})$. It is easy to see that $\tau : r_0 \mapsto r_2r_0r_2, r_1 \mapsto r_1, r_2 \mapsto r_2$ extends to an automorphism of Δ. Moreover, it cannot be an inner automorphism because it takes the toroidal embedding of the Fano plane $\mathcal{F}$ (see Figure ...) which is a hypermap of type $(3,3,3)$ onto a hypermap of type $(7,3,3)$. Recall that an inner automorphism by Theorem 2.12 preserve isomorphism classes. L. James [92] proved that the outer automorphism group of $\Delta(\infty,\infty,\infty)$ is generated by the six permutations of the three generators and by τ. More precisely, we have the following theorem.

Theorem 2.33. L. James [92] *The outer automorphism group* $Out\Delta$ *is isomorphic to* $PGL(2,\mathbb{Z})$. *It induces a set of functors generated by* σ-*duals and one other operator defined by* $(F; r_0, r_1, r_2) \mapsto (F; r_2r_0r_2, r_1, r_2)$.

A hypermap isomorphic with all its five nontrivial σ-duals will be called *totally self-dual*.

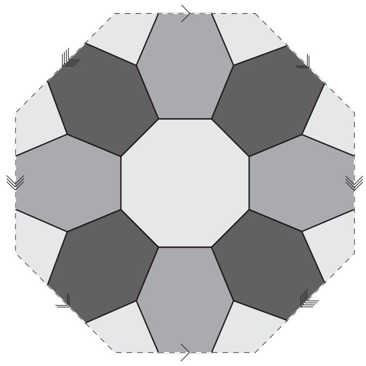


Figure 2.4: Totally self-dual regular hypermap of type $(4,4,4)$ of genus 2 arises as a quotient of the triangle $\Delta(4,4,4)$ by a characteristic subgroup .

Corollary 2.34. *Let* $H \lhd \Delta$ *and* $\mathcal{H} \cong A(\Delta/H; r_0, r_1, r_2)$. *Then* H *is characteristic if and only if* $\mathcal{H}$ *is totally self-dual and* $\mathcal{H} \cong \mathcal{H}^\tau$.

MAPS: $\mathcal{C}(\infty, \infty, 2)$.

The map duality and Petrie duality generate a group of functors isomorphic to S_3 (see Wilson [204] and Lins [122]). Jones and Thornton showed that the above group of six operations is induced by the action of the outer automorphism group on conjugacy classes of subgroups of the group $\Delta(\infty, \infty, 2) = \langle \lambda, \rho, \tau;\ \rho^2 = \tau^2 = \lambda^2 = (\tau\lambda)^2 = 1\rangle$, which is the automorphism group of the universal map for the category of nonoriented maps.

Theorem 2.35. [95] *The set of operations acting on the family of (nonoriented) maps induced by the action of outer automorphism group of* $\Delta(\infty, \infty, 2)$ *consists of six operations forming a group* G *generated by the duality and Petrie operations, and* G *is isomorphic to the symmetric group* $\mathbb{S}_3$*.*

Moreover, Léger and Terrasson [118] proved that the outer automorphism group of the category of nonoriented maps is $\mathbb{S}_3$, that is, they proved that the quotient of the group of invertible elements from the category of maps onto itself, modulo the normal subgroup induced by inner automorphisms of $\Delta(\infty, \infty, 2)$ is $\mathbb{S}_3$. As it was noted by Jones (personal communication), the above result is nothing but a restatement of Theorem 2.35 in the language of categories. Further generalisation of the above result can be found in [3].

K-valent ORMAPS: $\mathcal{C}(\Delta^+(\infty, k, 2))$.

We assume that $k \geq 3$. Call two oriented maps $\mathcal{M}_1 = (D_1; R_1, L_1)$, $\mathcal{M}_2 = (D_2; R_2, L_2)$ in $\mathcal{C}(\Delta^+(\infty, k, 2))$ *congruent* if $\mathcal{M}_2 \cong (D_1; R_1^e, L_1)$ for some e coprime to k. Clearly, the mapping $R \mapsto R^e$, $L \mapsto L$ extends to an automorphism of $\Delta^+(\infty, k, 2)$ and this automorphism is not an inner automorphism since $g^{-1}Lg = L$ implies $g = 1$. The underlying graph of the universal map in this category is the infinite k-valent tree.

Theorem 2.36. *The outer automorphism group* $Out\Delta^+(\infty, k, 2)$*,* $k > 2$*, is isomorphic to* $\mathbb{Z}_k^*$*. The induced functors of* $\mathcal{C}(\Delta^+(\infty, k, 2))$ *are the congruences.*

Proof Let σ be an outer automorphism of $G = \Delta^+(\infty, k, 2) = \langle R, L | R^k = L^2 = 1\rangle$. Consider the images of the generators $R \mapsto R^\sigma$ and $L \mapsto L^\sigma$.

G can be viewed as the automorphism group of the map $A(G/1; R, L)$, and consequently, it acts as a group of automorphisms of the underlying graph which is the infinite k-valent tree T. But any automorphism of T of finite order fixes either a vertex or an edge. In particular R^σ has order k and so it fixes a vertex, consequently we may assume $R^\sigma = R^e$, for some $e \in \mathbb{Z}_k^*$. By a similar argument, we get that L^σ fixes an edge, hence $L^\sigma = L^h$ for some $h \in G$. Assume $h \notin \langle R \rangle$. Then $L \notin \langle R^\sigma, L^\sigma \rangle$, a contradiction. Thus $L^\sigma = L^{R^f}$. Then composing σ with the conjugation by R^{-f} we get an automorphism taking $R \mapsto R^e$ and $L \mapsto L$. It follows that $\sigma \in \tau_e \mathrm{Inn}(G)$, where τ_e is the congruence automorphism associated with $e \in \mathbb{Z}_k^*$. ■

This suggests the following definition: an integer e will be called an exponent of $\mathcal{M} = (D;\ R, L)$ if $\mathcal{M} \cong \mathcal{M}_e = (D;\ R^e, L)$. Note that the oriented map $\mathcal{M}$ is reflexible (mirror symmetric) if and only if $\mathcal{M} \cong (D;\ R^{-1}, L)$, hence the concept of an exponent generalizes the concept of reflexibility. The mapping which realizes this isomorphism is an exomorphism of $\mathcal{M}$. In general, the map $\mathcal{M}_e$ is correctly defined provided e is coprime with the valency n of a map, i.e., with the least common multiple of lengths of cycles of R. If this is the case, the underlying graphs as well as the monodromy groups of both $\mathcal{M}$ and $\mathcal{M}_e$ are identical. This is the most important property of the *congruence operation* associated with an exponent $e \in \mathbb{Z}_n^*$. For regular maps, the converse statement is proved in [155].

Theorem 2.37. [155] *If $\mathcal{M}$ and $\mathcal{M}'$ are two oriented regular maps with the same underlying graph of valency n and with identical monodromy groups. Then they are congruent, i.e., $\mathcal{M}' = \mathcal{M}_e$ for some $e \in \mathbb{Z}_n^*$.*

The exponents of a map $\mathcal{M}$ reduced modulo its valency form an Abelian group, the exponent group $\mathrm{Ex}(\mathcal{M})$ of $\mathcal{M}$. Similarly, the exomorphisms of $\mathcal{M}$ form a group $\mathrm{Exo}(\mathcal{M})$, and this group is an extension of the automorphism group of $\mathcal{M}$ by the exponent group.

The exponent group of a map $\mathcal{M}$ is naturally embedded in the multiplicative group $\mathbb{Z}_n^*$ of invertible elements of the ring $\mathbb{Z}_n$ of integers modulo n, n being the valency of $\mathcal{M}$. Thus the order of $\mathrm{Ex}(\mathcal{M})$ divides $\phi(n)$ (ϕ is Euler's function).

Another interesting property of exomorphisms follows

In [155] the reader can find more information on the exponents of maps.

k-valent MAPS $\mathcal{C}(\Delta(\infty, k, 2))$.

Theorem 2.38. *The outer automorphism group $Out\Delta(\infty, k, 2)$, $k > 2$, is isomorphic to $\mathbb{Z}_k^* \times \mathbb{Z}_2$. The induced functors of $\mathcal{C}(\Delta^+(\infty, k, 2))$ are the congruences and Petrie duality.*

Inner exponents and antipodality

Among the exponents of a map $\mathcal{M}$, so called inner exponents play a special role. The corresponding exomorphisms act on darts as inner automorphisms of the monodromy group. These give rise to a subgroup of the exponent group, the inner exponent group $\mathrm{IEx}(\mathcal{M})$ of $\mathcal{M}$. In contrast to the general exponent groups, the inner exponents are functorial, which means that a map homomorphism induces a homomorphism between the inner exponent groups of the corresponding maps. The importance of inner exponents is emphasized by the fact that, apart from a short list of exceptions, inner exponent -1 implies that a regular map covers a map on a non-orientable surface. On the other hand, the fact that a map antipodally covers a map on a non-orientable surface forces -1 to be inner exponent. More details about antipodality and exponents can be find in [155].

Theorem 2.39. *If an oriented map $\mathcal{M} = (D; R, L)$ antipodaly double covers a map on a non-orientable surface, then -1 is an inner exponent of $\mathcal{M}$.*

Converesely, if $\mathcal{M}$ has an inner exponent -1 and $\mathcal{M}$ is neither a semistar, nor an odd cycle or its dual, then $\mathcal{M}$ covers a map in a non-orientable surface.

Note that given map on a non-orientable surface with Euler characteristic χ, its antipodal double cover on an orientable surface of characteristic 2χ is uniquely determined. However, the same map on an orientable surface of Euler characteristic χ may have more than one halved quotients on the non-orientable surface with Euler characteristic $\chi/2$.

Generalised Petrie duality on bipartite maps

In general the Petrie duality is not an operator for the category of oriented maps, since the topological Petrie dual of a map on an orientable surface may be a map on a non-orientable surface. However, for bipartite maps we have the following.

Proposition 2.40. *[157] Let $\mathcal{M}$ be a bipartite oriented regular map. Then the following statements are equivalent:*

(1) M is reflexible,

(2) the oriented Petrie dual of $\mathcal{M}$ is regular.

For bipartite maps $\mathcal{M} = (D; R, L)$ the set of darts is a disjoint union $D = D_0 \cup D_1$, where D_0, D_1 is formed by darts based at 'black', 'white' vertices, respectively. Consequently, $R = R_0 R_1$, where R_i is the restriction of R onto D_i, $i = 0, 1$. The oriented Petrie dual of $\mathcal{M}$ is a map $\mathcal{M}^{-1}_{1/2} = (D; R_0 R_1^{-1}, L)$. Replacing -1 by another involutory exponent e of the bipartite map $\mathcal{M}$ we get an involutory operator acting on the set of bipartite maps. Properties of this switch operator are studied in [157]. The following statement is proved.

Proposition 2.41. *Let $\mathcal{M} = (D; R, L)$ be a bipartite n-valent regular map and let $e^2 \equiv 1$ mod n. Then the following statements are equivalent:*

(1) $e \in \mathrm{Ex}(\mathcal{M})$ is an exponent,

(2) $\mathcal{M}^e_{1/2} = (D, R_0 R_1^e, L)$ is regular.

Similar results can be derived for other categories of hypermaps [101]. As concerns functors between distinguished categories of hypermaps, Singerman's list of triangle group inclusions [173] gives rise to a set of functors between different categories of hypermaps [101]. Let us note that the representation of a hypermap by the 3-valent 3-coloured map as well as the Walsh bipartite representation are examples of such functors. In the context of the correspondence between oriented maps and Riemann surfaces we would like to mention the following result.

Functors between categories of hypermaps

The following Singerman list [173] of inclusions between triangle groups determine a set of injective functors between different categories of hypermaps. Assume, say we have an oriented map $\mathcal{M}$ of virtual type $(n, 2n)$. Then $\mathcal{M} = A(G/S; R, L)$

non-normal inclusion	index	condition
$\Delta^+(7,7,7) \subset \Delta^+(2,3,7)$	24	
$\Delta^+(2,7,7) \subset \Delta^+(2,3,7)$	9	
$\Delta^+(3,3,7) \subset \Delta^+(2,3,7)$	8	
$\Delta^+(4,8,8) \subset \Delta^+(2,3,8)$	12	
$\Delta^+(3,8,8) \subset \Delta^+(2,3,8)$	10	
$\Delta^+(9,9,9) \subset \Delta^+(2,3,9)$	12	
$\Delta^+(4,4,5) \subset \Delta^+(2,4,5)$	6	
$\Delta^+(n,4n,4n) \subset \Delta^+(2,3,4n)$	6	$n \geq 2$
$\Delta^+(n,2n,2n) \subset \Delta^+(2,4,2n)$	4	$n \geq 3$
$\Delta^+(3,n,3n) \subset \Delta^+(2,3,3n)$	4	$n \geq 3$
$\Delta^+(2,n,2n) \subset \Delta^+(2,3,2n)$	3	$n \geq 4$

normal inclusion	index	condition
$\Delta^+(n,n,n) \subset \Delta^+(3,3,n)$	3	$n \geq 4$
$\Delta^+(n,n,n) \subset \Delta^+(2,3,2n)$	6	$n \geq 4$
$\Delta^+(m,n,n) \subset \Delta^+(2,n,2m)$	2	$n \geq 3$,$n+m \geq 7$

Remark that most of the above inclusions have a geometric explanation. For instance, the inclusion $\Delta^+(2,n,2n) \subset \Delta^+(2,3,2n)$ of index three is geometrically realized as a truncation of the $2n$-valent tesselation of the hyperbolic plane by n-gons.

The monodromy group of an oriented regular map $\mathcal{M}$ of type (p, q) is an epimorphic image of $\Delta^+(p, q, 2)$. Let K be the kernel of this epimorphism. Then U/K is a Riemann surface which we denote by $S(\mathcal{M})$, here K is considered to be a group of isometries of U. In [174, 175] Singerman and Syddall consider the problem whether the assignment $\mathcal{M} \mapsto S(\mathcal{M})$ is injective, or in other words, whether the same Riemann surface can underlie different regular maps. The late situation certainly happens for genus 0 regular maps, since the only Riemann surface of genus 0 is the Riemann sphere. As concerns genus

1, up to duality, there are two infinite families of regular maps (maps of type $(3,6)$ and of type $(4,4)$) and there are two Riemann surfaces associated with these two families. To continue our discussion we need to define the following two functorial operations. By the *truncation* of $\mathcal{M}$ we mean the cubic map whose vertices are darts of $\mathcal{M}$ and two are joined by an edge if they form an angle of $\mathcal{M}$, or they underlie the same edge. By the *medial map* we mean the map whose vertices are the middle points of edges of $\mathcal{M}$, two being adjacent if the respective edges form an angle of $\mathcal{M}$. Finally, let $e(\mathcal{M})$ denotes the number of edges of $\mathcal{M}$. Using the above defined operations one can describe regular maps sharing the same Riemann surface in almost all cases.

Theorem 2.42. [174, 175] *Let $\mathcal{M}$ and $\mathcal{N}$ be two orientably regular maps of genus > 1 with $S(\mathcal{M}) = S(\mathcal{N})$ and $e(\mathcal{M}) \leq e(\mathcal{N})$. Then one of the following statements holds:*

(a) $\mathcal{N}$ is the medial map of $\mathcal{M}$,

(b) $\mathcal{N}$ is the truncation of $\mathcal{M}$,

(c) $\mathcal{N}$ has type $(7,3)$ and $\mathcal{M}$ has type $(7,7)$,

(d) $\mathcal{N}$ has type $(8,3)$ and $\mathcal{M}$ has type $(8,8)$.

(e) $\mathcal{N} \cong \mathcal{M}$ or to its dual map.

Exercises

2.4.1. Determine terms of rotations an orientably regular self-dual map of type $\{5,5\}$.

2.4.2. Find a regular map which is in the same time self-Petrie and self-dual.

2.4.3. Is there a cubic regular toroidal and self-Petrie map?

Chapter 3

Lifts and quotients

3.1 Lifting automorphisms of maps

Let $\pi : \tilde{X} \to X$ be a covering of topological spaces. Assume we have a homeomorphism $\psi : X \to X$. A question whether there is homeomorphism $\tilde{\psi} : \tilde{X} \to \tilde{X}$ such that $\psi\pi = \pi\tilde{\psi}$ is called the lifting automorphism problem. This problem is studied in [7, 22, 142, 215]. Particularly, the lifting automorphism problem for graphs is investigated in [19, 55, 21, 83, 111, 112, 143].

We show how a fruitful concept of voltage assignment (see [71, 72, 73]), used to describe coverings of graphs, can be modified in order to describe homomorphisms between oriented regular maps. Voltage assignments can be used to build up regular maps from their regular quotients. The definitions and results presented in this section are taken from [145] (see also [76, 5, 6, 153, 154, 181, 100]).

Lifting condition

Let $\mathcal{M} = (D; R, L)$ be an oriented map. An (oriented) *angle* of $\mathcal{M}$ is an ordered pair $\alpha = (x, y) = \overrightarrow{xy}$, where x and y are darts of $\mathcal{M}$ such that $y \in \{R(x), R^{-1}(x), L(x)\}$. The angle $\alpha^{-1} = (y, x)$ is the inverse of $\alpha = (x, y)$. Let us denote by $A(\mathcal{M})$ the set of all angles of $\mathcal{M}$. An *angle walk* is a sequence $\alpha_1\alpha_2 \ldots \alpha_n$ of angles such that the initial dart of α_i coincides with the terminal dart of α_{i-1} for $i = 2, 3, \ldots, n$.

Let G be a finite group. A voltage assignment on $\mathcal{M}$ valued in G is a function $\alpha : A(\mathcal{M}) \to G$ such that for any angle $\alpha \in A(\mathcal{M})$ one has $\alpha(\alpha^{-1}) = \alpha^{-1}(\alpha)$. A voltage assignment extends from angles to walks in an obvious way by setting $\alpha(W) = \alpha(\alpha_1)\alpha(\alpha_2)\dots\alpha(\alpha_n)$. The group generated by voltages of closed walks based at a fixed dart x is called the *local voltage group* G^x. Since $\mathcal{M}$ is connected all the local groups are conjugate subgroups of G.

Given a voltage assignment α on $\mathcal{M} = (D; R, L)$ valued in G set $D^\alpha = D \times G$ and define permutations on D^α by

$$R^\alpha(x, h) = (R(x), h\alpha(\overrightarrow{xRx})),$$

$$L^\alpha(x, h) = (L(x), h\alpha(\overrightarrow{xLx})).$$

One can prove that $\langle R^\alpha, L^\alpha \rangle$ is transitive on D^α if and only if $G^x = G$. If this is the case the map $\mathcal{M}^\alpha = (D^\alpha; R^\alpha, L^\alpha)$ is a correctly defined map called the *derived map*. In what follows we shall always assume that a considered voltage assignment on $\mathcal{M}$ valued in G satisfies the condition $G^x = G$

As one can expect, a regular homomorphism $\tilde{\mathcal{M}} \to \mathcal{M}$ is equivalent to a natural projection $\mathcal{M}^\alpha \to \mathcal{M}$ erasing the second coordinates (see [145]). It is important to stress that using angle voltage assignments we can handle also coverings of maps (map homomorphisms) which may not induce coverings between the respective underlying graphs (a disadvantage of the classical approach based on associating ordinary voltages to darts of the quotient map [72, 73]). Maybe a bit surprising is the fact that an arbitrary homomorphism defined on an orientably regular map is necessarily regular see Proposition 1.17 or [145]. It follows that angle voltage assignments provide a convenient tool for study of orientably regular maps. A natural question arises: Under what condition a voltage assignment α defined on an orientably regular map $\mathcal{M}$ determines an orientably regular derived map $\mathcal{M}^\alpha$?

We say that a voltage assignment $\alpha : A(\mathcal{M}) \to G$ is *locally invariant* if for any $\tau \in \mathrm{Aut}(\mathcal{M})$ and any closed walk W based at a dart x

$$\alpha(W) = 1 \Rightarrow \alpha(\tau W) = 1.$$

The above lifting condition appears in an explicit form firstly in [76], where map homomorphisms induced by graph coverings of the underlying graphs are investigated. The following characterization of homomorphisms between regular maps is proved in [145].

Theorem 3.1. [145] *Let $\mathcal{M}$ and $\tilde{\mathcal{M}}$ be (oriented) regular maps. Then*

(1) *if α is a locally invariant assignment on $\mathcal{M}$, then the derived map $\mathcal{M}^\alpha$ is (oriented) regular.*

(2) *if $\varphi : \tilde{\mathcal{M}} \to \mathcal{M}$ is a homomorphism, then there exists a locally invariant voltage assignment α on $\mathcal{M}$ such that the natural projection $\mathcal{M}^\alpha \to \mathcal{M}$ is equivalent to φ.*

Let us note also that lifting part of Theorem 3.1 is, in a weaker form, proved in [76]. Weaker form of the above theorem can be found also in [143]. Voltage assignments on flags of hypermaps and lifting conditions are studied by Surowski in [183].

By Theorem 3.1 coverings between regular maps are always regular with the group of covering transformations isomorphic to the voltage group. With some effort one can show that every oriented regular map with a non-solvable monodromy group covers regularly either a regular map with a non-abelian simple group monodromy group, or it covers regularly a bipartite oriented regular map which partition stabilizer is either simple non-abelian, or a direct product of two isomorphic simple non-abelian groups (see [158]).

Abelian voltage group

A global version of the invariance condition was used a long time ago in the proof that there are infinitely many 5-arc-transitive cubic graphs (see [19]). We say that a voltage assignment $\alpha : A(\mathcal{M}) \to G$ is *invariant* if for any $\tau \in \mathrm{Aut}(\mathcal{M})$ and any walk W

$$\alpha(W) = 1 \Rightarrow \alpha(\tau W) = 1.$$

Of course, an invariant voltage assignment is locally invariant as well. If the assignment $\alpha : A(\mathcal{M}) \to G$ is invariant, then $\mathrm{Aut}(\mathcal{M}^\alpha)$ is a split extension of G by $\mathrm{Aut}(\mathcal{M})$ (see [145]). Each map $\mathcal{M}$ admits at least one invariant voltage assignment which can be constructed as follows: Let $G = Z_2^n$, where n is $|A(\mathcal{M})|/2$. Take a set $X = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ of generators of G and let τ be a bijection $A(\mathcal{M}) \to X$. Then τ is an invariant voltage assignment. Since the local voltage group is not equal to G, the derived map splits into isomorphic connected parts.

A restriction of the covering onto any connectivity component yields a map homomorphism such that $\mathrm{Aut}(\mathcal{M})$ lifts. In particular, if $\mathcal{M}$ is regular then so is the lifted map. Another example of an invariant voltage assignment can be constructed by assigning the voltage $1 \in (Z_2, +)$ to each angle of $\mathcal{M}$. The double cover is connected if and only if the truncation of $\mathcal{M}$ is not bipartite.

The lifting condition significantly simplifies if the voltage group is abelian. Since a map homomorphism between regular maps with a solvable automorphism groups decomposes into a sequence of coverings, each with an elementary abelian group of covering transformations, elementary abelian covers of maps such that the map automorphism group lifts are of special importance.

Voltage assignments in elementary abelian groups were studied by Malnič, Marušič and Potočnik in [146] and by Du, Kwak and Xu in [57]. Using some ideas of linear algebra an algorithm to check the lifting condition is derived. Invariant voltage assignments defined on graphs and valued in products of cyclic groups are used in [19, 21, 181, 143] as well. Regular cyclic covers over Platonic solids are described in [182] and [100], regular cyclic covers over toroidal regular maps are classified in [172].

Homology and cohomology theories of hypermaps are built in works of Machi [135], Surowski and Schroeder [184]. The following construction shows that every regular map of genus g admits a locally invariant 'homology voltage assignment' of rank $2g$.

Theorem 3.2. *(*Theorem 7.1. in [184]*) Let M be a regular orientable map of Euler characteristic $2-2g$ and genus g. Then for each positive integer n, there is a regular (connected) map of Euler characteristic $2gn(2-2g)$ covering $\mathcal{M}$.*

3.2 Maps on orbifolds

Maps on oriented orbifolds

Given regular covering $\psi : \mathcal{M} \to \mathcal{N}$ between two oriented maps, let $x \in V(\mathcal{N}) \cup F(\mathcal{N}) \cup E(\mathcal{N})$ be a vertex, face or edge of $\mathcal{N}$. The ratio of degrees $b(x) = deg(\tilde{x})/deg(x)$, where $\tilde{x} \in \psi^{-1}(x)$ is a lift of x along ψ, will be called a *branch index* of x. By regularity of the action of $\mathrm{CT}(\psi)$ on fibres branch index is a well-defined positive integer not depending on the choice of the lift $\tilde{x}$. In some considerations, it is important to save information about branch indexes coming from some regular covering defined over a map N. This can be done by introducing a signature σ on $\mathcal{N}$. A *signature* is a function $\sigma : x \in V(\mathcal{N}) \cup F(\mathcal{N}) \cup E(\mathcal{N}) \to Z^+$ assigning a positive integer to each vertex, edge and face, with the only restriction: if x is an edge of degree 2 then $\sigma(x) = 1$, if it is of degree 1 then $\sigma(x) \in \{1, 2\}$. We say that a signature σ on $\mathcal{N}$ is *induced* by a covering $\psi : \mathcal{M} \to \mathcal{N}$ if it assigns to vertices, faces and edges of $\mathcal{N}$ their branch indexes with respect to ψ.

Given a couple $(\mathcal{M}, \sigma)$, where $\mathcal{M}$ is a finite oriented map of genus g and σ is a signature we define an *orbifold type* of $(\mathcal{M}, \sigma)$ to be an $(r+1)$-tuple of the form $[g; m_1, m_2, \ldots, m_r]$, where $1 < m_1 \leq m_2 \leq \cdots \leq m_r$ are integers, and m_i appears in the sequence $s_i > 0$ times if and only if σ takes the value m_i exactly s_i times. Denote by $\bar{\mathcal{M}}$ the respective topological map on S_g.

A topological counterpart of a (combinatorial) map $\mathcal{M}$ with a signature σ can be established as follows. A *topological orbifold* associated with $(\mathcal{M}, \sigma)$ is a surface $S_g \setminus \mathcal{B}$ where $\mathcal{B}$ is a finite set of *branch points* satisfying the following rules:

1. at most one point of an edge or of a face is in $\mathcal{B}$,
2. $x \in \mathcal{B}$ is a vertex $\bar{x}$ of if and only if $\sigma(x) > 1$,
3. $x \in \mathcal{B}$ belongs to an edge $\bar{e}$ of $\bar{\mathcal{M}}$ if and only if $\sigma(e) > 1$ and e is not a semi-edge,
4. $x \in \mathcal{B}$ is a free-end of a semi-edge $\bar{e}$ if $\sigma(e) = 2$,
5. $x \in \mathcal{B}$ belongs to a face $\bar{f}$ of $\bar{\mathcal{M}}$ if and only if $\sigma(f) > 1$.

It follows that $|\mathcal{B}| = r$.

The *orbifold fundamental group* $\pi_1(\mathcal{M}, \sigma)$ of $(\mathcal{M}, \sigma)$ is a F-group

$$\pi_1(\mathcal{M}, \sigma) = F[g; m_1, m_2, \ldots, m_r] =$$

$$\langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g, e_1, \ldots, e_r | \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{r} e_j = 1, e_1^{m_1} = \ldots e_r^{m_r} = 1 \rangle.$$

In particular, the oriented triangle group $\Delta^+(k,m,n)$ is the F-group $F[0;k,m,n]$. Let $\psi : \mathcal{M} \to \mathcal{N}$ be a *regular* covering and σ be a signature defined on $\mathcal{N}$. We say that ψ is *σ-compatible* if for each element $x \in V(\mathcal{N}) \cup E(\mathcal{N}) \cup F(\mathcal{N})$ the branch index of $b(x)$ of x is a divisor of $\sigma(x)$. Signature σ defined on $\mathcal{N}$ lifts along a σ-compatible regular covering $\psi : \mathcal{M} \to \mathcal{N}$ to a derived signature σ_ψ on $\mathcal{M}$ defined by the following rule: $\sigma_\psi(\tilde{x}) = \sigma(x)/b(x)$ for each $\tilde{x} \in \psi^{-1}(x)$ and each $x \in V(\mathcal{N}) \cup E(\mathcal{N}) \cup F(\mathcal{N})$. Let us remark that if $\sigma(x) = 1$ for each $x \in V(\mathcal{N}) \cup E(\mathcal{N}) \cup F(\mathcal{N})$ then σ-compatible covers over $\mathcal{M}$ are just smooth regular covers over $\mathcal{M}$. Such a signature will be called *trivial*. In this terminology (closed) surfaces are orbifolds with signature $[g;\emptyset]$. In general if O is an orbifold we write $O = O[g;m_1,m_2,\ldots,m_r]$. The fundamental group $\pi_1(O)$ of an orbifold O is an F-group defined above. A (topological) map on an orbifold O is a map on the underlying surface S_g of genus g satisfying the following properties:

(P1) if $x \in B$ then x is either an internal point of a face, or a vertex, or an end-point of a semiedge which is not a vertex,

(P2) each face contains at most one branch point,

(P3) the branch index of x lying at the free end of a semiedge is two.

A mapping $\psi : \tilde{O} \to O$ is *a covering* if it is a branched covering between the underlying surfaces mapping the set of branch points $\tilde{B}$ of $\tilde{O}$ onto the set B of branch points of O and each $\tilde{x}_i \in \psi^{-1}(x_i)$ is mapped uniformly with the same branch index d dividing the prescribed index r_i of $x_i \in B$. It follows that there is a correspondence between compatible coverings of maps with signatures and the respective topological orbifolds.

The following well-known statement relates the signature induced by a regular covering between two oriented maps with the genera of the respective surfaces.

Theorem 3.3. (Riemann-Hurwitz formula) *Let $\mathcal{M} \to \mathcal{M}/G$ be a regular covering between maps on orientantable surfaces with a covering transformation group G, let $\mathcal{M}$ be finite. Let the respective orbifold type of $\mathcal{N} = \mathcal{M}/G$ be $[\gamma;m_1,m_2,\ldots,m_r]$. Then the Euler characteristic of the underlying surface of $\mathcal{M}$ is given by the*

$$\chi = |G|\big(2 - 2\gamma - \sum_{i=1}^{r}(1 - \frac{1}{m_i})\big).$$

Proof. Consider the hypermaps $\mathcal{H}$ on S and $\mathcal{H}' = \mathcal{H}/G$ on S_g associated to to $\mathcal{M} \to \mathcal{M}/G = \mathcal{N}$. As usually, denote by v, e, f and by v', e', f' the respective numbers of hyper-vertices, hyper-edges, hyper-faces. By F and F' we denote the respective sets flags. By Euler-Poincare formula $\chi = v+e+f-|F|/2$, where χ is the Euler characteristic of S. Note that the hyper-vertices, hyper-edges, hyper-faces, correspond to faces of the associated 3-valent topological maps. Now the numbers m_i are orders of non-trivial cyclic stabilisers of faces of the associated 3-valent map on S in the action of G. If m is the order of a face-stabiliser then the projected hyper-element lifts to $|G|/m$ hyper-elements covering it. In particular $v+e+f = |G|(v'+e'+f'-r+\sum_{i=1}^{r}\frac{1}{m_i})$ and $|F| = |G|\cdot|F'|$. Inserting in the E-P formula and employing $2-2\gamma = v'+e'+f'-|F'|/2$ we obtain

$$\chi = |G|(v'+e'+f'-r+\sum_{i=1}^{r}\frac{1}{m_i}-|F'|/2) = |G|\big(2-2\gamma-\sum_{i=1}^{r}(1-\frac{1}{m_i})\big),$$

as was required.

□

There is an analogous Riemann-Hurwitz formula for the category of maps which is more complicated. The problem is that quotients of maps on surfaces may be maps on surfaces with a non-empty boundary, hence we first need to derive an Euler-Poincare formula for such maps.

Corollary 3.4. *Let G be a group of automorphisms of an oriented map of genus $g > 1$. Then $|G| \le 84(g-1)$.*

Setting $\chi = 2 - 2g$ in Theorem 3.3 we get

$$|G| = \frac{2g-2}{2\gamma - 2 + \sum_{i=1}^{r}(1-\frac{1}{m_i})}.$$

Denote

$$f(\gamma; m_1, m_2, \ldots, m_r) = 2\gamma - 2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i}).$$

To maximize $|G|$ is equivalent to minimize the function $f(\gamma; m_1, m_2, \ldots, m_r)$ on the set of $r+1$-tuples of integers satisfying $r \geq 0$, $\gamma \geq 0$, $m_i \geq 2$ for every $i = 1, \ldots r$ and $2\gamma - 2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) > 0$. First observe that $f(0; 2, 3, 7) = 1/42$ which gives $|G| = 84(g-1)$. We want to show that this is a minimum value of f. Assume, on the contrary, that there is $(\gamma; m_1, \ldots, m_r)$ such that $f(\gamma; m_1, m_2, \ldots, m_r) < 1/42$.

Claim 1. If $0 < f(\gamma; m_1, m_2, \ldots, m_r) < 1/42$ then $\gamma = 0$ and $r = 3$.

Assuming $\gamma \geq 2$ we get $f(\gamma; \emptyset) \geq 2$, a contradiction. If $\gamma = 1$ then $r \geq 1$ implying $f(1; m_1, m_2, \ldots, m_r) \geq 1/2$. Thus $\gamma = 0$. The condition $-2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) > 0$ implies $r \geq 3$.

If $r \geq 5$ then $f(0; m_1, m_2, \ldots, m_r) \geq f(0; 2, 2, 2, 2, 2) \geq 1/2$. If $r = 4$ then $f(0; m_1, m_2, m_3, m_4) \geq f(0; 2, 2, 2, 3) = 1/6$. It follows that $\gamma = 0$ and $r = 3$.

Claim 2. Let $m_1 \leq m_2 \leq m_3$.
Then $0 < f(0; m_1, m_2, m_3) < f(0; m_1, m_2, m_3 + 1)$.

Claim 3. $f(0; m_1, m_2, m_3) \geq min\{f(0; 3, 3, 4), f(0; 2, 4, 5), f(0; 2, 3, 7)\}$.

Claim 3 can be proved as follows. List all triples (m_1, m_2, m_3) such that $1 < m_1 \leq m_2 \leq m_3$ and $0 \leq f(0; m_1, m_2, m_3)$. Check that for each triple from the (finite) list either $f(0; m_1, m_2, m_3 + 1) \leq 0$ or $(m_1, m_2, m_3 + 1)$ is one of the three triples $(3, 3, 4)$, $(2, 4, 5)$, $(2, 3, 7)$. Now starting from any triple (m_1, m_2, m_3) such that $0 < f(0; m_1, m_2, m_3)$ we may subtract the largest term by 1. By Claim 2 we decrease the value of f. Repeating this procedure we end in one of the above three triples.

Computing the critical values of f we get

$$f(0; 3, 3, 4) = 1/12, \ f(0; 2, 4, 5) = 1/20$$

and $f(0; 2, 3, 7) = 1/42$, a contradiction. $\square$

Corollary 3.5. *Let G be an automorphism group of a regular oriented hypermap of genus $g > 1$. Then $|G| \leq 84(g-1)$.*

In particular, such a hypermap cannot have more than $84(g-1)$ darts, andonsequently, given orientable surface supports only finitely many regular hypermaps.

Note that there is no upper bound for the size of a discrete group of automorphisms acting on the two orientable surfaces of Euler characteristic 0 and 2. On the other hand, the orientation preserving groups of automorphisms of the sphere are well-known (crystallographic groups).

Restricting ourselves to spherical oriented regular maps the respective groups are the orientation preserving automorphisms groups of the five Platonic solids and all the cyclic groups of finite order. The actions on the torus Klein bottle are closely related with the Euclidean crystallographic groups.

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