

# Surfaces and maps on surfaces

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# 1 Classification of compact connected surfaces

By a *surface* we mean a connected Hausdorff space such that every point has a neighbourhood homeomorphic to the Euclidean plane  $\mathbb{E}^2$ . Sphere, torus, double-torus and projective plane are examples of compact surfaces. Cutting two disjoint open discs from a given surface and identifying the boundary cycles with the two boundary cycles of a cylinder following a couple of boundary homeomorphisms we form a new surface. Similarly, cutting a single hole in a surface and gluing the boundary cycle with the boundary cycle of a Möbius strip along a homeomorphism we get a new surface as well. Main aim of this section is to prove that every surface can be obtained from the sphere using repetitively the above operations.

A *topological map* is a (finite) 2-cell decomposition of a surface. Maps are usually described by *2-cell embeddings* of finite connected topological graphs into surfaces. An embedding  $i : X \rightarrow S$  of a graph  $X$  into a surface  $S$  is called a *2-cell embedding* if the connectivity components  $S - i(X)$  are homeomorphic to open discs (or equivalently, to Euclidean planes). The connectivity components will be called *faces*. Each face is bounded by a closed walk in  $X$ . If all the faces are bounded by closed walks of length three, the respective map is called a *triangulation*. If the underlying graph of a triangulation is simple and all the vertices have degrees at least 3, the map is called a *simplicial triangulation* and the surface with a fixed triangulation is called a triangulated surface. The following statement is well-known.

**Theorem 1.1** (see [?, Theorem 3.1.1]) *Every compact surface admits a finite triangulation.*

Given two surfaces  $S_1$  and  $S_2$  we may form a new surface  $S_1 \# S_2$  first by making holes  $D_1, D_2$  in  $S_1$  and  $S_2$ , respectively, such that the boundary walks  $\partial D_i$ ,  $i = 1, 2$ , are simple cycles. Then  $S$  is a quotient space given by a homeomorphism  $\partial D_1 \rightarrow \partial D_2$ . It is clear that  $S = S_1 \# S_2$  is a surface and it is compact if both  $S_1$  and  $S_2$  were compact. Till the end of this section by a surface we shall mean always compact and connected surface.

**Definition.** The most simple compact connected surfaces are the sphere, the projective plane, the torus and Klein bottle. Although they can be defined analytically (for instance, the sphere as a subspace of the Euclidean 3-dimensional space formed by points  $[x, y, z]$  satisfying  $x^2 + y^2 + z^2 = 1$ ), for the purposes of the classification it is more convenient to introduce them as quotients of a closed disc, see Figure 1.

As a step towards the classification we first prove the following.

**Theorem 1.2** *Every compact surface is either a sphere, or a connected sum of tori, or a connected sum projective planes.*

The proof is divided into several steps. A labelled polygon with a set of sides  $D$  is a regular polygon with  $2n$  sides each endowed with one of the two orientations and a labelling  $f : D \rightarrow E$ , where  $|E| = n$  and  $|f^{-1}(x)| = 2$ , for each  $x \in E$ .

We say that a surface has a *polygonal representation* if it is homeomorphic to a regular  $2n$ -gon whose sides are identified following the prescribed labelling and orientation.

**Lemma 1.3** *Every surface has a polygonal representation.*

PROOF By Theorem 1.1 a surface  $S$  can be triangulated. Let  $T_1, T_2, \dots, T_m$  be the (closed) triangles of the triangulation. Each  $T_i$  is homeomorphic to a triangle  $T'_i$  in the Euclidean plane and we may assume that the triangles  $T'_1, T'_2, \dots, T'_m$  are mutually disjoint. There is an associated collection of homeomorphisms  $\Phi_i : T'_i \rightarrow T_i$ . For every side  $f_1$  in  $T_i$  there exists  $j \neq i$  such that  $f_2 = \Phi_j^{-1} \Phi_i(f_1)$  is a side of  $T'_j$ . Moreover, we may introduce an orientation at  $f_1$  at random and

choose the orientation on  $f_2$  such that  $f_2 = \Phi_j^{-1}\Phi_i(f_1)$  is orientation preserving. Each such two sides  $f_1 \approx f_2$  with the prescribed orientation will be called equivalent. It follows that  $S$  can be obtained as a quotient space  $S = \cup_{i=1}^m T'_i / \approx$ . The sides of the triangles  $T'_i$ ,  $i = 1, \dots, m$ , are preimages of the edges of the graph  $K$  underlying the triangulation, and the equivalence classes of  $\approx$  are in correspondence with the edges of  $K$ . Hence we have a labelling of the sides of  $\cup_{i=1}^m T'_i$  by  $E(K)$ . Since  $S$  is connected, we can construct a sequence of edges  $e_1, e_2, \dots, e_{m-1}$  such that for every  $1 \leq j \leq m-1$  the quotient space  $\mathcal{P}_j = \cup_{i=1}^j T'_i / \{e_1, e_2, \dots, e_{j-1}\}$  is homeomorphic to a closed disc and exactly one edge on the boundary  $\partial\mathcal{P}_j$  is labelled by  $e_j$ . Clearly,  $\mathcal{P}_m$  gives the required polygonal representation of  $S$ . ■

Given a polygonal representation  $\mathcal{P}$  of a surface  $S$ , the underlying embedded graph  $X$  is determined as follows. The set of edges is the set of labels of sides of  $\mathcal{P}$ . The vertices of  $X$  are the classes of an equivalence relation defined on the set of vertices of  $\mathcal{P}$ . Two vertices  $u$  and  $v$  are called *elementary equivalent* if there is  $e \in E$  such that they are heads, or tails, of the two sides of  $\mathcal{P}$  labelled by  $e$ . We say that  $u \equiv v$  are *equivalent* if there is a sequence of elementary equivalences such that  $u = v_1 \equiv v_2 \equiv \dots \equiv v_k = v$ . A vertex  $v$  is incident with an edge  $e$  if there is a vertex  $u \in v$  of  $\mathcal{P}$  incident to a side of  $\mathcal{P}$  labelled  $e$ . Let us remark that the size of the equivalence class of a vertex  $v$  is equal to the degree of  $v$  in the underlying embedded graph. It follows that  $S$  has a polygonal representation if and only if it admits a cellular one-face embedding of a graph  $X$ , where the unique face is bounded by a closed walk  $W = x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_{2n}^{\epsilon_{2n}}$ ,  $\epsilon_i \in \{-1, 1\}$ . An edge  $e = \{x, x^{-1}\}$  is said to be of the *first kind* if both  $x$  and  $x^{-1}$  appear in  $W$ , and it is called an edge of the *second kind* otherwise.

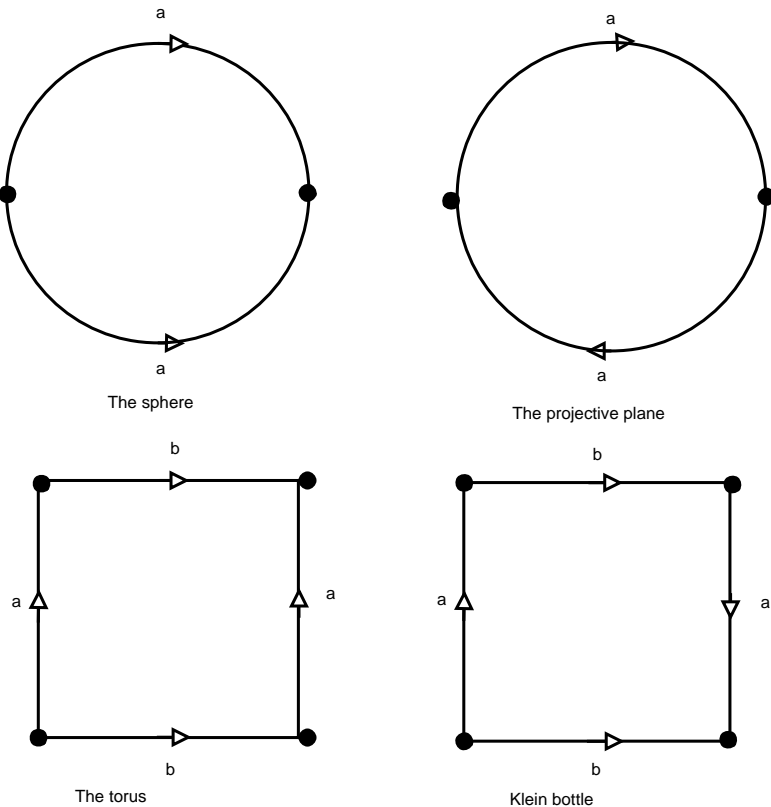


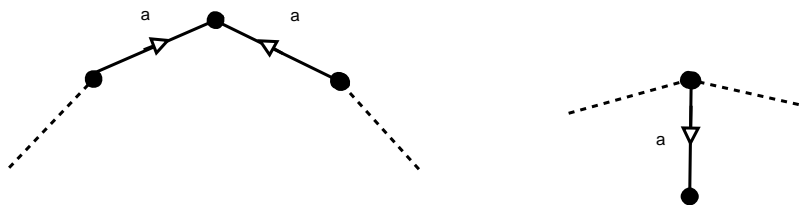
Figure 1: Examples of surfaces

We prove Theorem 1.2 by showing that every polygonal representation of  $S$  can be transformed to a *canonical representation* either bounded a closed walk of the form  $W = \prod_{i=1}^r a_i b_i a_i^{-1} b_i^{-1}$ , or of the form  $W = \prod_{i=1}^s a_i a_i$ . Note that given polygonal representation the surface is completely determined by the cyclic permutation of darts on the boundary walk. We shall use cut-and-glue surgery to transform a given polygonal representation into another one. This is done by introducing a new edge in the interior of the disk, cutting the surface along it, so that the surface is represented as a factor space defined on a union of two disks, and gluing the two disks along another boundary edge thus getting another polygonal representation. In each stage the underlying surface is completely described by the respective cyclic permutations of darts forming the boundary. We use capital letters  $A, B, C \dots$  to denote subwalks, while small letters  $a, b, c, \dots$  will denote darts.

**Lemma 1.4** STEP 1: Elimination of adjacent edges of the first kind.

If  $W$  contains a sub-walk  $U$  of the form  $xx^{-1}$ , or of the form  $x^{-1}x$  and  $W \neq U$  then there is a polygonal representation of the same surface  $S$  bounded by  $W'$ , where  $W'$  arises from  $W$  by deleting  $U$ .

PROOF Identifying  $x$  and  $x^{-1}$  the surface does not change, see Figure 1. Note that the terminal (initial) vertex of  $x$  appears on  $\partial P$  just once and after performing these operation it vanishes.



STEP 1: Elimination of adjacent darts of the first kind

Figure 2: Elimination of adjacent darts of the first kind

**Lemma 1.5** STEP 2: Transformation to a one-vertex embedding

*Every surface has a polygonal representation whose vertices form just one equivalence class.*

PROOF Assume there are at least two classes of equivalence of vertices of  $\mathcal{P}$ . Then there is a side  $b$  on  $\partial \mathcal{P}$  such that the two incident vertices  $u, v$  are not equivalent and let  $\deg(u) \leq \deg(v)$  in the underlying embedded graph. Then the boundary walk can be written as  $W = (AaBba^{-1})$  such that  $u$  is the head of  $b$  and  $v$  is the tail of it. Indeed, in the other cases we may change the preferred orientation of the edges  $a, b$  incident to  $u$  such that the respective subsequence changes to the form  $\dots ba^{-1} \dots$  with one exception, when  $a = b$  and the edge  $a = b$  is of the second kind, so that the sequence is  $\dots aa \dots$ , or  $\dots a^{-1}a^{-1} \dots$ . However, in this case  $u \equiv v$  contradicting the assumption. If  $a = b$  we have a consecutive pair of darts of the first kind and we apply STEP 1 to get rid of vertex  $u$ . In the other case we perform a move from  $\mathcal{P}$  to  $\mathcal{P}'$  first by adding a new edge  $c$  joining the tail of  $a$  to  $v$ , cutting the triangle  $T$  bounded by  $c, a$  and  $b^{-1}$ . This way we get a representation of  $S$  as quotient space of a union of two disks bounded by  $(cba^{-1})$  and  $(AaBc^{-1})$ . Gluing  $T$  to the remaining part of  $\mathcal{P}$  along  $a$  such that the orientation is preserved we get a polygon  $\mathcal{P}'$  bounded by  $(AcbBc^{-1})$ . Now  $a$  is placed in the interior of the polygon  $\mathcal{P}'$  and we delete it (see Fig. 3).

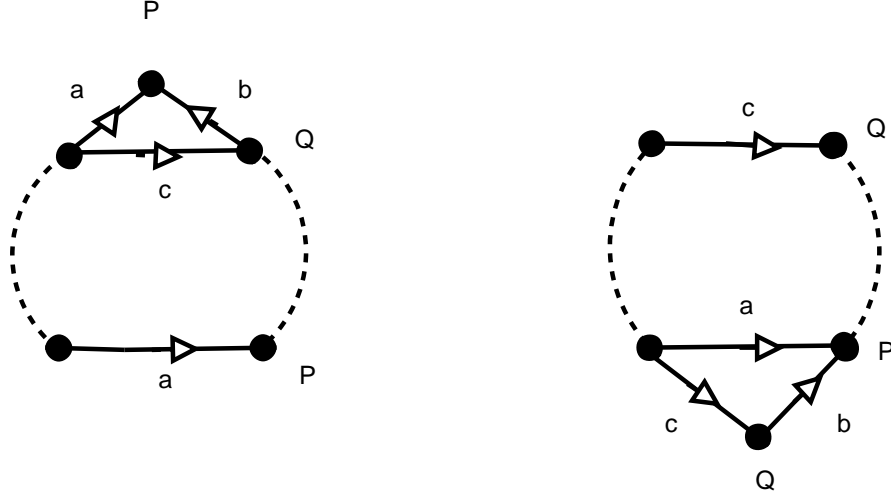


Figure 3: Step 2 Elimination of vertices

After performing this operation  $\deg_{\mathcal{P}'}(u) < \deg_{\mathcal{P}}(u)$ . One of the following cases happens for  $u$ .

- (a)  $\deg_{\mathcal{P}'}(u) = 1$  and there is a pair of adjacent edges of the first kind incident  $u$ ,
- (b) there is an edge  $b$  joining  $u$  to another equivalence class  $v$ .

In Case (a) we use STEP 1 to get rid of  $u$ , in Case 2 we repeat the above transformation on  $\mathcal{P}'$  getting  $\mathcal{P}''$  with  $\deg_{\mathcal{P}''}(u) < \deg_{\mathcal{P}'}(u) < \deg_{\mathcal{P}}(u)$ . It follows that repeating STEP 2 we finally end in Case (a), then using STEP 1, we get a polygonal representation with one less equivalence classes of vertices. It follows that repeating this process of elimination of equivalence classes of vertices we end with a polygonal representation possessing just one class.

**Lemma 1.6** STEP 3: Making pairs of darts of the second kind adjacent.

*A polygonal representation of  $S$  can be transformed to a polygonal representation  $\mathcal{P}$  of  $S$  such that all the pairs of darts of the second kind are consecutive on  $\partial\mathcal{P}$ .*

PROOF If  $W = (AbBb)$  is a closed boundary walk of a polygonal representation then we perform a move first introducing a new edge  $a$  joining heads of the two appearances of  $b$  in  $W$ . Cutting  $W$  along  $a$  we get a representation of  $S$  as a factor space of two disks  $D_1, D_2$  bounded by  $(Aab)$  and  $(ab^{-1}B^{-1})$ . Gluing  $D_1, D_2$  along  $b$  and deleting the edge  $b$  we get a polygonal representation  $aaAB^{-1}$  of  $S$  containing one less pair of non-adjacent darts of the second kind (see Fig. 4).

If there are no darts of the first kind we have a polygonal representation  $\mathcal{P}$  of  $S$  bounded by a closed walk of the form  $\prod_{i=1}^r a_i a_i$ .

**Lemma 1.7** STEP 4: Coupling pairs of darts of the first kind

*Let  $\mathcal{P}$  be a polygonal representation of  $S$  reduced with respect to Lemmas 1.4, 1.5, 1.6. If there are darts of the first kind in a polygonal representation of  $S$  then it can be transformed to a polygonal representation of  $S$  such that every pair of the first kind appears in a subsequence of the form  $\dots aba^{-1}b^{-1} \dots$*

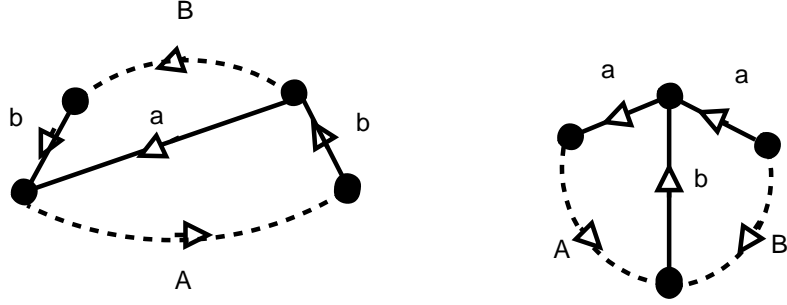


Figure 4: Making pairs of darts of the second kind adjacent

PROOF Let  $a, a^{-1}$  be a pair of darts of the first kind, and let  $\partial\mathcal{P} = (aAa^{-1}B)$ , where  $A$  does not consist of a single dart of first kind. Between all such darts  $a$  assume that  $a$  was chosen such that the length of  $A$  is minimal. By Lemma 1.4  $A$  is non-trivial. We claim that  $A$  contains a dart  $b$  of the first kind such that  $b^{-1} \in B$ . Assume this is not the case. By minimality, each dart  $c$  in  $A$  is of the second kind, or it is of the first kind forming (up to reversing orientation and changing  $c$  and  $d$ ) a sequence  $cdc^{-1}d^{-1} \subseteq A$ . By Lemma 1.6  $A$  is formed by adjacent pairs of darts of the second kind and by coupled pairs of darts of the first kind. Then the initial and terminal vertices of  $a$  are not equivalent contradicting Lemma 1.5. It follows that  $\partial\mathcal{P}$  is of the form  $(aAbBa^{-1}Cb^{-1}D)$ . We shall proceed by the cut-and-glue surgery depicted on Figure 5. The corresponding transformations of the boundary walks follow:

$$(aAbBa^{-1}Cb^{-1}D) \rightarrow (AbBc) \cup (a^{-1}Cb^{-1}Dac^{-1}) \rightarrow (ac^{-1}a^{-1}CBcAD),$$

$$(ac^{-1}a^{-1}CBcAD) \rightarrow (ad^{-1}AD) \cup (cdc^{-1}a^{-1}CB) \rightarrow (cdc^{-1}d^{-1}ADCB).$$

As a result the two (crossing) pairs  $a, a^{-1}, b, b^{-1}$  of darts of the first kind were replaced by a sequence of new darts of the first kind of the form  $cdc^{-1}d^{-1}$ . Repeating the argument we complete the proof. ■

If all the darts are of the first kind we have obtained the required form of polygonal representation of  $S$  with boundary of the form  $\partial\mathcal{P} = (\prod_{i=1}^r a_i b_i a_i^{-1} b_i^{-1})$ .

The proof of Theorem 1.2 will be completed by using the following lemma.

**Lemma 1.8** STEP 5: Connected sum of a torus and a projective plane is homeomorphic to the connected sum of three projective planes.

*Surfaces determined by the polygonal representations  $\mathcal{P}$  and  $\mathcal{P}'$  with  $\partial\mathcal{P} = (Aefe^{-1}f^{-1}dd)$ ,  $\partial\mathcal{P}' = (A^{-1}aabbcc)$  are homeomorphic.*

PROOF First we observe that a connected sum of two projective planes is homeomorphic to the Klein bottle. Indeed, the respective sequence of moves reads as follows:

$$(aabb) \rightarrow (bac^{-1}) \cup (cab) \rightarrow (cbcb^{-1}).$$

It follows that  $(A^{-1}aabbcc)$  can be replaced by  $(A^{-1}ab^{-1}abcc)$ . Next we perform the following sequence of moves (see Figure 6):

$$(A^{-1}ab^{-1}abcc) \rightarrow (d^{-1}b^{-1}ab^{-1}abc) \cup (dcA^{-1}a) \rightarrow (bcd^{-1}b^{-1}Ac^{-1}d^{-1})$$

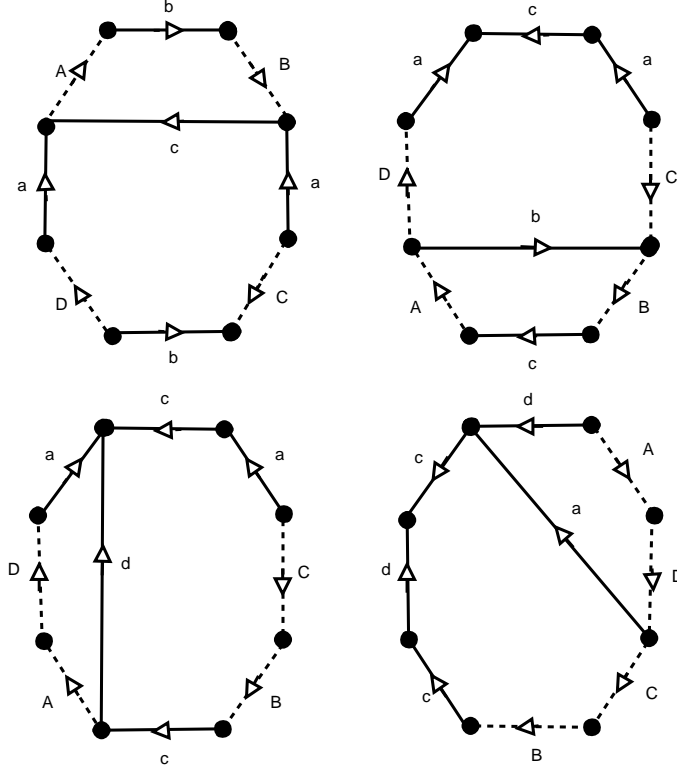


Figure 5: Coupling pairs of darts of the first kind

$$\begin{aligned}
&\rightarrow (e^{-1}c^{-1}d^{-1}bc) \cup (ed^{-1}b^{-1}A) \rightarrow (ce^{-1}c^{-1}d^{-1}Aed^{-1}) \rightarrow (fe^{-1}c^{-1}d^{-1}Ae) \cup (f^{-1}d^{-1}c) \\
&\rightarrow (Aefe^{-1}f^{-1}d^{-1}d^{-1}) = (Aefe^{-1}f^{-1}dd).
\end{aligned}$$

After making Steps 1-4 all the pairs of darts of the second kind are adjacent and all pairs of darts of the first kind coupled such that they form sequences  $\dots aba^{-1}b^{-1}\dots$ . If there are darts of both types on the boundary then a repetitive use of Lemma 1.8 allow us to transform the polygonal representation of  $S$  onto the canonical form  $(\prod_{i=1}^r a_i a_i)$ . ■

To complete the classification we need to distinguish the surfaces represented by the polygonal representations in canonical forms. We do it by introducing two integer invariants - the Euler characteristic and the orientability. An *Euler characteristic* of a topological map  $\mathcal{M}$  with  $e$  edges, no semiedges,  $v$  vertices and  $f$  faces is defined to be  $\chi(\mathcal{M}) = v - e + f$ . A (first) *barycentric subdivision* of  $\mathcal{M}$  is a triangulation  $\mathcal{T}(\mathcal{M})$  of the same surface arising by subdividing each edge of  $\mathcal{M}$  by a vertex, inserting a new vertex into the interior of each face, and triangulating each face  $F$  by joining the ‘facial vertex’ to each vertex on the boundary of  $F$ . If a vertex appear on the boundary  $\partial F$  more than once we join the respective facial vertex  $u$  to each copy of  $u$  on  $\partial F$ . Let us remark that there is an induced proper colouring  $\ell : V \rightarrow \{0, 1, 2\}$  of vertices of a barycentric subdivision  $\mathcal{B}(\mathcal{M})$  associating 0 to vertex-vertices, 1 to edge-vertices and 2 to face-vertices. We say that  $\mathcal{M}$  is *orientable* if the triangles of  $\mathcal{T}(\mathcal{M})$  can be coloured by two colours such that two triangles sharing an edge in common are coloured by different colours. Equivalently,  $\mathcal{M}$  is orientable if the dual of its barycentric subdivision is bipartite. If  $\mathcal{M}$  is not orientable, we say that it is *non-orientable*.

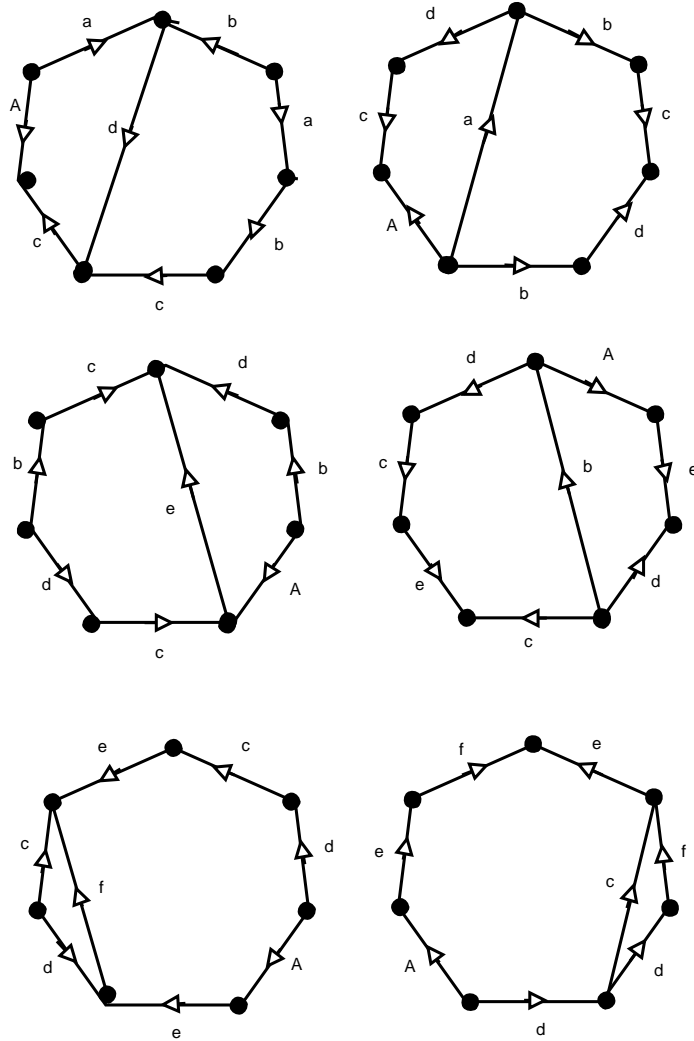


Figure 6: Connected sum of Klein bottle and a projective plane is homeomorphic to the connected sum of a torus and a projective plane

**Proposition 1.9** *Let  $S$  be a compact connected surface. Then every map on  $S$  has the same Euler characteristic and orientability.*

PROOF. Let  $\mathcal{M}$  be a map on  $S$ . Observe that the Euler characteristic and the orientability are invariant under the following elementary transformations of  $\mathcal{M}$ .

- (1) deleting or introducing a pendant edge in the interior of a face,
- (2) subdividing an edge by a vertex, or inversely, eliminating a vertex of degree two,
- (3) subdividing a face into two faces by introducing a new edge joining two vertices on its boundary, inversely, amalgamating two faces by removing an edge which appears at the boundaries of the two faces.

Employing Operation (3) repeatedly,  $\mathcal{M}$  transforms to a triangulation  $\mathcal{T}$  of  $S$ . If there are loops or multiple edges we can make the triangulation  $\mathcal{T}$  simplicial combining Operations (2) and (3).

By Lemma 1.3  $\mathcal{T}$  can be transformed to a one-face map  $\mathcal{P}$  using Operation (3). Inspecting the process of reduction done in the proof of Theorem 1.2 we see that the canonical representation is obtained by applying the elementary Operations (1) and (3). Since the Euler characteristic and the orientability is preserved, the original map and the resulting canonical map has the same Euler characteristic and the orientability. Counting the Euler characteristic and orientability for the canonical polygonal representations  $\chi(S) = 2$ ,  $\chi(S) = 2 - 2g$  and  $\chi(S) = 2 - \tilde{g}$  if  $S$  is the canonical representation for the sphere, of a connected sum of  $g$  tori and of a connected sum of  $\tilde{g}$  projective planes. Direct observation shows that the representatives are orientable in the first two cases and they are nonorientable in the last case. Assume now that there is a homeomorphism  $\Phi : S_g \rightarrow S$ , where or  $S = S_h$  or  $N_h$ . Then we have two one-vertex, one-face canonical maps  $\mathcal{M}_1$  and  $\mathcal{M}_2$  on  $S$ . If any two edges of the two maps intersects just finitely many times then the point-set union of the two embedded graphs determines a map  $\mathcal{M}$ , which can be reduced using the operations (1), (2) and (3) to both  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . Since the operations preserve the Euler characteristic and the orientability, the maps  $\mathcal{M}_1, \mathcal{M}_2$  coincide. The proof is finished by a slightly delicate argument stating that we can always move edges of  $\mathcal{M}_1$  and  $\mathcal{M}_2$  to avoid infinitely many intersections.

■

Given surface  $S$  we set its *Euler characteristic*  $\chi(S)$  and *orientability* to be the Euler characteristic and orientability of any map on  $S$ .

**Theorem 1.10** *A compact connected surface  $S$  is determined by its Euler characteristic and orientability. In particular, either  $S$  is orientable and  $\chi(S) = 2 - 2g$ , for some  $g \geq 0$ , or it is non-orientable and  $\chi(S) = 2 - \tilde{g}$ , for some  $\tilde{g} \geq 1$ .*

PROOF By Theorem 1.2 each surface is homeomorphic either to a sphere, or to a connected sum of  $g \geq 1$  tori with a canonical polygonal representation bounded by  $(\prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1})$ , or to a connected sum of  $\tilde{g}$  projective planes with a canonical polygonal representation bounded by  $(\prod_{i=1}^{\tilde{g}} a_i a_i)$ . By Proposition 1.9 the above surfaces are pairwise distinguished by the Euler characteristic and orientability. ■

The invariants  $g \geq 0, \tilde{g} \geq 1$  are called the (orientable) genus of  $S$ , non-orientable genus of  $S$ , respectively.

**Corollary 1.11** (Euler-Poincare formula) *Let  $\mathcal{M}$  be a map on a surface  $S$  with  $v$  vertices,  $e$  edges and  $f$  faces. Then*

$$v - e + f = \chi(S) = \begin{cases} 2 - 2g & \text{if } S \text{ has genus } g, \\ 2 - \tilde{g} & \text{if } S \text{ has non-orientable genus } \tilde{g}. \end{cases}$$

### Problems.

1\*. Describe the topological spaces (even spaces that are not surfaces) arising from a  $2n$ -gonal representations for  $n \leq 5$  such that each edge appear on the boundary exactly twice and the identification of vertices is prescribed.

2. Draw a figure proving that the connected sum of two projective planes is homeomorphic to Klein bottle.

## 2 Category of Maps and Ormaps

Recall that a topological map is a (finite) 2-cell decomposition of a surface. Maps are usually described by 2-cell embeddings of finite connected topological graphs into surfaces. An embedding  $i : X \rightarrow S$  of a graph  $X$  into a surface  $S$  is called a *2-cell embedding* if the connectivity components  $S - i(X)$  are homeomorphic to open discs (or equivalently, to Euclidean planes). The connectivity components will be called *faces*. Each face is bounded by a closed walk in  $X$ . Note that a boundary walk may not be a simple cycle. For instance, if  $T \rightarrow S$  is an embedding of a tree into the sphere then there is a unique boundary walk traversing each edge of  $T$  twice. The embedding is determined by the graph  $X$  and a finite collection  $\mathcal{F}$  of boundary walks. We say that an edge  $\{x, \lambda x\}$  is double-covered by  $\mathcal{F}$  if either  $x$ , or  $\lambda x$  appear twice in walks of  $\mathcal{F}$ , or both  $x$  and  $\lambda x$  appear in walks of  $\mathcal{F}$  exactly ones; and it is one-covered if exactly one of  $x$ ,  $\lambda x$  appears in a walk of  $\mathcal{F}$ . Given set of closed walks in a graph  $X$  we say that two darts  $x, y$  form an angle  $\alpha = (x, y) = \overrightarrow{xy}$ , or an angle  $\alpha = (y, x) = \overrightarrow{yx}$ , if  $I(x) = I(y)$  and there exists  $W$  in  $\mathcal{F}$  such that either  $\lambda x, y$ , or  $x, \lambda y$  are consecutive in  $W$ . Two darts  $x, y$  forming an angle  $\overrightarrow{yx}$  or  $\overrightarrow{xy}$  will be called elementary  $\mathcal{F}$ -equivalent. Generally, two darts  $x$  and  $y$  will be called  $\mathcal{F}$ -equivalent if there is a sequence of darts  $x_0, x_1, \dots, x_k$  such that  $x = x_0, y = x_k$  and  $x_{i-1}, x_i$  are elementary  $\mathcal{F}$ -equivalent for  $i = 1, 2, \dots, k$ .

The following statement gives a simple characterization of sets of walks giving rise to a map.

**Lemma 2.1** *Let  $i : X \rightarrow S$  be a 2-cell embedding of a connected finite graph into a surface  $S$ . Let  $\mathcal{F}$  be a collection of boundary walks and let  $X = (D, V; I, \lambda)$  be the associated combinatorial graph. Then*

- (1) *the edges of size two are double-covered by  $\mathcal{F}$ ,*
- (2) *semiedges are one-covered by  $\mathcal{F}$ ,*
- (3)  *$x$  and  $y$  are  $\mathcal{F}$ -equivalent if and only if  $I(x) = I(y)$ .*

*Conversely, for a collection  $\mathcal{F}$  satisfying (1), (2) and (3) there exists a 2-cell embedding  $i : X \rightarrow S$  whose faces are bounded by walks of  $\mathcal{F}$ .*

**Corollary 2.2** *A closed walk of a graph  $X$  determines a one face 2-cell embedding if and only if each edge is traversed twice and  $X$  is a one-vertex graph.*

As far as maps on surfaces are concerned, there are two essentially different approaches to their combinatorial description. The first approach, based on a rotation-involution pair acting on darts, involves the orientation of the supporting surface and so is suitable only for maps on orientable surfaces [?, ?]. The corresponding combinatorial structure is called a combinatorial (or, sometimes, algebraic) oriented map. The other approach, using three involutions acting on mutually incident (vertex, edge, face)-triples called flags, is orientation insensitive and thus allows us to represent maps on non-orientable surfaces as well [?]. The resulting combinatorial structure will be called a combinatorial nonoriented map. Accordingly, we shall usually employ the same notation for a topological map and for the corresponding combinatorial structure on it.

We start with necessary definitions concerning oriented maps. By a (combinatorial) *oriented map* we henceforth mean a triple  $(D; R, L)$  where  $D = D(\mathcal{M})$  is a non-empty finite set of *darts*, and  $R$  and  $L$  are two permutations of  $D$  such that  $L$  is an involution and the group

$\text{Mon}(\mathcal{M}) = \langle R, L \rangle$  acts transitively on  $D$ . The group  $\text{Mon}(\mathcal{M})$  is called the oriented *monodromy group* of  $\mathcal{M}$ . The permutation  $R$  is called the *rotation* of  $\mathcal{M}$ . The orbits of the group  $\langle R \rangle$  are the *vertices* of  $\mathcal{M}$ , and elements of an orbit  $v$  of  $\langle R \rangle$  are the darts *radiating* (or *emanating*) from  $v$ , that is,  $v$  is their initial vertex. The cycle of  $R$  permuting the darts emanating from  $v$  is the *local rotation*  $R_v$  at  $v$ . The permutation  $L$  is the *dart-reversing involution* of  $\mathcal{M}$ , and the orbits of  $\langle L \rangle$  are the *edges* of  $\mathcal{M}$ . The orbits of  $\langle RL \rangle$  define the face-boundaries of  $\mathcal{M}$ . The incidence between vertices, edges and faces is given by nontrivial set intersection. The vertices, darts and the incidence function define the *underlying graph*  $\mathcal{M}$ , which is always connected due to the transitive action of the monodromy group. It is easily seen that the cycles of  $RL$  satisfy the conditions (1),(2) and (3) in Lemma 2.1, hence with each combinatorial map there is an associated topological map.

An oriented map can be equivalently described as a pair  $(\mathcal{G}; R)$  where  $\mathcal{G} = (D, V; I, L)$  is a connected graph and  $R$  is a permutation of the dart-set of  $\mathcal{G}$  cyclically permuting darts with the same initial vertex, that is,  $IR(x) = I(x)$  for every dart  $x$  of  $\mathcal{G}$ .

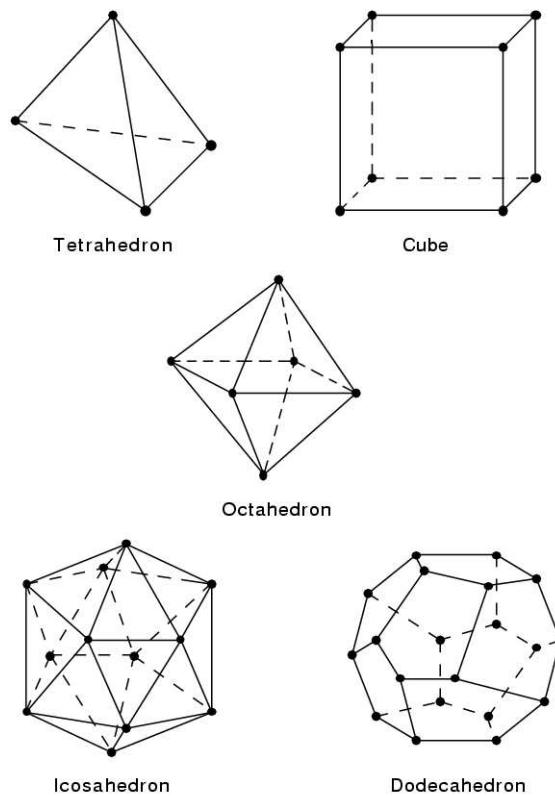


Figure 7: The five Platonic solids

Combinatorial nonoriented maps are built from three involutions acting on a non-empty finite set  $F$  of *flags* [?]. Flags can be viewed as the angles of the associated topological map. Given map  $\mathcal{M}$  there are three involutions acting as follows:  $\rho(\overrightarrow{xy}) = \overrightarrow{yx}$ ,  $\tau(\overrightarrow{xy}) = \overrightarrow{xz}$ ,  $\lambda(\overrightarrow{xy}) = \overrightarrow{x^{-1}z}$ . Given dart  $x$ ,  $z$  is the unique dart  $z \neq y$  forming an angle with  $x$  ( $x^{-1}$ ). A (combinatorial) *nonoriented map* is a quadruple  $(F; \lambda, \rho, \tau)$  where  $\lambda$ ,  $\rho$  and  $\tau$  are fixed-point free involutory permutations of  $F = F(\mathcal{M})$  called the *longitudinal*, the *rotary* and the *transversal involution*, respectively, which satisfy the following conditions:

- (i)  $\lambda\tau = \tau\lambda$ ; and
- (ii) the group  $\langle \lambda, \rho, \tau \rangle$  acts transitively on  $F$ .

This group is the nonoriented *monodromy group*  $\text{Mon}(\mathcal{M})$  of  $\mathcal{M}$ .

We define the *vertices* of  $\mathcal{M}$  to be the orbits of the subgroup  $\langle \rho, \tau \rangle$ , the *edges* of  $\mathcal{M}$  to be the orbits of  $\langle \lambda, \tau \rangle$ , and the *face-boundaries* to be the orbits of  $\langle \rho, \lambda \rangle$  under the action on  $F$ , the incidence being given by nontrivial set intersection. Note that each orbit  $z$  of  $\langle \lambda, \tau \rangle$  has cardinality 2 or 4 according to whether  $z$  is a semiedge or not.

**Example 2.3** *The underlying graph of the tetrahedron is the complete graph  $K_4 = (D, V; I, L)$  on 4 vertices. We may set  $V = \{1, 2, 3, 4\}$ ,  $D = \{12, 21, 13, 31, 14, 41, 23, 32, 24, 42, 34, 43\}$ ,  $L(ij) = ji$  and  $I(ij) = i$ , for any  $ij \in D$ . Then the rotation at vertices compatible with the counterclockwise global orientation in Figure 8 is  $R = (12, 13, 14)(23, 21, 24)(31, 32, 34)(41, 43, 42)$ . Vice-versa, having  $R$  we can identify the triangular faces of the map  $(D; R, L)$  via cycles of the permutation  $RL = (12, 24, 41)(21, 13, 32)(31, 14, 43)(23, 34, 42)$ . Figure 9 shows the same topological map (the tetrahedron) described by means of 3 involutions acting on 24 flags as a map  $(F; \lambda, \rho, \tau)$  in the category of Maps.*

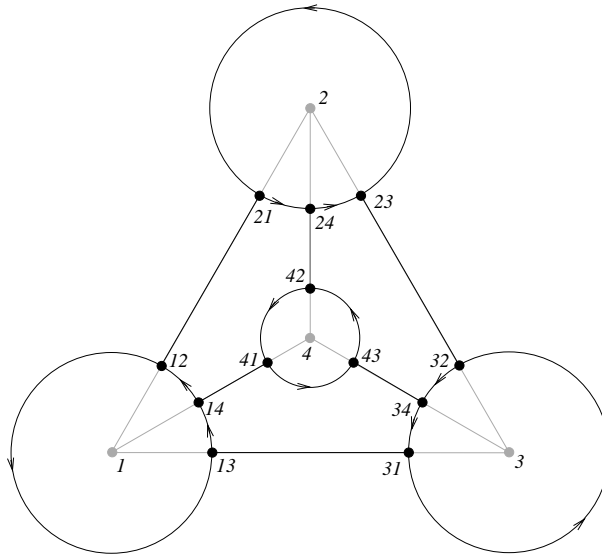


Figure 8: The tetrahedron described as an oriented map  $(D; R, L)$

The meaning of the condition on  $\lambda, \rho, \tau$  requiring these involutions to be fixed-points free becomes clear, if one decides to extend our theory onto *maps on surfaces with boundary*. This was done by Bryant and Singerman in [?]. To do the generalisation we have to allow fixed points of  $\lambda, \rho, \tau$ . The underlying surface of a map has a non-empty boundary if and only if at least one of  $\lambda, \rho, \tau$  fixes a flag. In fact, the category of nonoriented maps is not complete if we do not consider maps on surfaces with boundary, since a homomorphic image of a map on a closed surface can be a map on a surface with a non-empty boundary. As an example, consider an embedding of a cycle in the sphere and its quotient on the disk given by the reflection interchanging the two faces of the map and fixing the embedded graph point-wise.

Clearly, the even-word subgroup  $\langle \rho\tau, \tau\lambda \rangle$  of  $\text{Mon}(\mathcal{M})$  has always index at most two. If the index is two, then  $\mathcal{M}$  is said to be *orientable*.

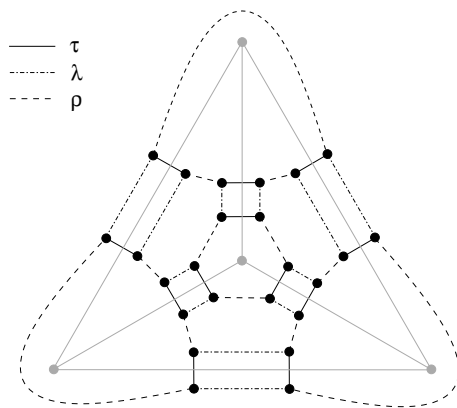


Figure 9: The tetrahedron described as a map  $(F; \lambda, \rho, \tau)$

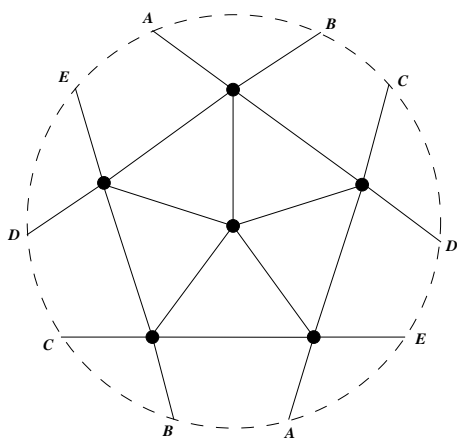


Figure 10:  $K_6$  in the Projective plane

With every oriented map  $(D; R, L)$  we associate the *corresponding nonoriented map*  $\mathcal{M}^\natural = (F^\natural; \lambda^\natural, \rho^\natural, \tau^\natural)$  by setting  $F^\natural = D \times \{1, -1\}$  and defining for a flag  $(x, j) \in D \times \{1, -1\}$ :

$$\lambda^\natural(x, j) = (L(x), -j), \quad \rho^\natural(x, j) = (R^j(x), -j), \quad \text{and} \quad \tau^\natural(x, j) = (x, -j).$$

Conversely, from an orientable nonoriented map  $\mathcal{M} = (F; \lambda, \rho, \tau)$  we can construct a pair of oriented maps  $\mathcal{M}' = (D; R, L)$  and  $\mathcal{M}'' = (D; R^{-1}, L)$  that are the *mirror image* of each other. We take  $D$  to be the set  $F/\tau$  of orbits of  $\tau$  on  $F$ . Let us denote by  $F^+ \subset F$  one of the two orbits induced by the action of the even-word subgroup of  $\text{Mon}(\mathcal{M})$ . For a dart  $\{z, \tau(z)\} = [z]$ , where  $z \in F^+$ , we set  $R([z]) = [\rho\tau(z)]$  and  $L([z]) = [\lambda\tau(z)]$ . Instead of  $R$  we could have taken the rotation  $R'([z]) = [\tau\rho(z)]$ , but since  $R' = R^{-1}$  we get nothing but the mirror image – as expected.

### Test of orientability

Let  $\mathcal{M} = (F; \lambda, \rho, \tau)$  be a nonoriented map. We want to determine whether the respective supporting surface  $S$  is orientable, or not. The following simple algorithm is well-known, see [?, ?]. Consider the associated 3-valent graph  $\mathcal{G} = \mathcal{G}(\lambda, \rho, \tau)$ , whose set of vertices is  $F$ , each vertex  $f \in F$  is incident with darts  $(f, \lambda)$ ,  $(f, \rho)$  and  $(f, \tau)$  and the dart reversing involution takes

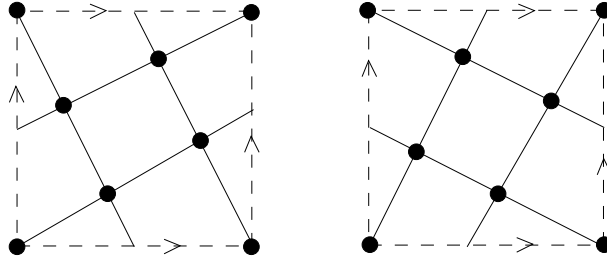


Figure 11: Regular embedding of  $K_5$  in the torus appears in two ‘enantiomers’, in the category of ORMAPS they are not isomorphic.

$(f, \lambda) \mapsto (\lambda f, \lambda)$ ,  $(f, \rho) \mapsto (\rho f, \rho)$ ,  $(f, \tau) \mapsto (\tau f, \tau)$ . Note that  $\mathcal{G}$  is nothing but the dual of the barycentric subdivision of  $\mathcal{M}$ . The result follows.

**Proposition 2.4** *A map  $\mathcal{M} = (F; \lambda, \rho, \tau)$  is orientable if and only if the associated 3-valent graph  $\mathcal{G}$  is bipartite.*

### Problems

1. Prove that two 4-gonal embeddings of  $K_5$  depicted on Figure 11 are not isomorphic in the category of Ormaps, but they are isomorphic in the category of Maps.
2. How many isomorphism classes of oriented embeddings has  $K_4$ ? For each class determine a rotation and genus.
3. There is a pentagonal embedding of the Petersen graph  $P$  in the projective plane such that faces are bounded by simple cycles. Is there an embedding of  $P$  in torus whose faces are bounded by simple cycles?

## 3 Graph Embeddings, Embedding schemes

Every topological map determines a 2-cell embedding of a graph  $X$  into a surface. Generally, an embedding  $i : X \hookrightarrow S$  of a graph into a surface  $S$  is called a 2-cell embedding if each connectivity component of  $S - i(X)$  is homeomorphic to an open disc. As mentioned above embeddings of a given graph  $X$  into orientable surfaces can be described by means of rotations permuting the set of darts of  $X$ . Our aim is to describe embeddings of graphs into both orientable and non-orientable surfaces in such a way that flags will not be referred. Recall that each vertex in an embedded graph corresponds to an orbit of  $\langle \rho, \tau \rangle$ , which can be viewed as an alternating cycle coloured by  $\rho$  and  $\tau$  in  $\mathcal{G}(\lambda, \rho, \tau)$ . Similarly, each edge is in a correspondence with an orbit of  $\langle \lambda, \tau \rangle$ , which can be viewed as an alternating cycle coloured by  $\lambda$  and  $\tau$  in  $\mathcal{G}(\lambda, \rho, \tau)$ . A (simple) cycle  $C$  in an embedded graph  $X \hookrightarrow S$  will be called *orientable* if the subgraph of  $\mathcal{G}$  induced by the edges of  $\mathcal{G}$  of alternating cycles corresponding to vertices and edges of  $C$  is bipartite. Otherwise  $C$  is called *non-orientable*. Choose a spanning tree  $T$  and set  $\xi(x) = 1 \in \{\pm 1, \cdot\}$  if  $x$  belongs to  $T$ . If  $x$  is a cotree dart we set  $\xi(x) = 1 \in \{\pm 1, \cdot\}$  if the corresponding fundamental cycle in  $T \cup \{x, x^{-1}\}$  is orientable, and  $\xi(x) = -1 \in \{\pm 1, \cdot\}$  otherwise. This way a Cayley voltage space  $(\{\pm 1, \cdot\}, \{\pm 1, \cdot\}, \xi)$  on  $X$  is determined. Clearly,  $\xi(x) = 1$  for all  $x \in D$  if and only if  $S$  is orientable. Given embedding of a graph into a closed surface  $S$  there is an associated set of closed walks satisfying Lemma 2.1. Take  $R$  to be any permutation of darts such that  $(x_0, x_1, \dots, x_k)$  is a cycle in  $R$  if and only if  $I(x_i) = I(x_j) = v$ ,  $0 \leq i < j \leq k$ ,  $k$  is the valency of  $v$  and  $\overrightarrow{x_i x_{i+1}}$  is an

angle, for every  $i = 0, 1, \dots, k$ . Note that in general,  $R$  is not uniquely determined, since replacing any cycle of  $R$  by its inverse we get another permutation satisfying the condition. The above permutation  $R$  permuting at each vertex  $v$  cyclically darts incident  $v$  will be called a *rotation*.

Given connected graph  $X = (D, V; I, L)$ , an *embedding scheme* is a triple  $(X; R, \xi)$ , where  $R$  is a rotation and  $\xi : D \rightarrow \{\pm 1, \cdot\}$  is a voltage assignment. We define the *associated set*  $\mathcal{F}$  of closed walks given by the following rules:

- (1)  $x$  and  $y$  are consecutive in  $W \in \mathcal{F}$  if  $y = R^{\xi(x)}(x^{-1})$ ,
- (2) for each  $x \in D$  either  $\xi(x) = 1$  and both  $x, x^{-1}$  appear in the walks of  $\mathcal{F}$  exactly once, or  $\xi(x) = -1$  and  $x$  or  $x^{-1}$  (but not both) appear in the walks of  $\mathcal{F}$  exactly twice,

We summarise the above discussion in the following proposition.

**Proposition 3.1** *Let  $(X; R, \xi)$  be an embedding scheme. Then the set  $\mathcal{F}$  of closed walks given by the rules (1) and (2) satisfies the conditions in Lemma 2.1, in particular it defines faces of a topological map. Conversely, for every 2-cell embedding of  $X$  there is an embedding scheme  $(X; R, \xi)$  such that the boundary walks of faces are associated with the scheme.*

A disadvantage of the description of graph embeddings by embedding schemes consists in fact that different schemes code the same topological map. A partial solution of this problem follows. Let  $(X; R, \xi)$  be an embedding scheme. We say that  $(X; R, \eta)$  arises by an elementary vertex-switch  $(X; R, \xi)$  if there exists a vertex  $v$  in  $V$  such  $v = I(x)$  implies  $I(x) \neq I(x^{-1})$ ,  $\eta(x) = -\xi(x)$  if  $I(x) = v$  or  $I(x^{-1}) = v$  and  $\eta(x) = \xi(x)$  otherwise. Two schemes are equivalent if one can be obtained from the other by a finite sequence of elementary vertex-switches.

**Proposition 3.2** *Equivalent schemes define the same topological map. In particular, a map is orientable if and only if it can be described by an embedding scheme with the voltage assignment constantly equal  $\xi(x) = 1$ .*

**Corollary 3.3** *Topological maps on orientable surfaces can be described by means of combinatorial maps. There is one-to-two correspondence, where one topological map corresponds to two combinatorial maps  $(D; R, L)$ ,  $(D; R^{-1}, L)$ .*

## Interpolation Theorems

**Theorem 3.4** *Let  $X$  be a graph. Let  $i : X \hookrightarrow S_g$ ,  $j : X \hookrightarrow S_h$  be 2-cell embeddings of  $X$  into orientable surfaces of genera  $0 \leq g < h$ . Then for every integer  $\gamma$ ,  $g \leq \gamma \leq h$  there exists a 2-cell embedding of  $X$  into an orientable surface of genus  $\gamma$ .*

PROOF Given rotation  $R$  of darts of  $X$  we may form a new rotation  $R'$  by switching the order of two consecutive darts in a cycle of  $R$ , so that a cycle  $(x_1, x_2, \dots, x_k)$  is replaced by  $(x_2, x_1, x_3, \dots, x_k)$ . We want to show that the number of faces of the new embedding of  $X$  described by  $R'$  either remained the same, or it has changed by  $\pm 2$ . It is enough to see how the faces containing at the boundary the three consecutive pairs of darts  $(x_k^{-1}, x_1)$ ,  $(x_1^{-1}, x_2)$  and  $(x_2^{-1}, x_3)$  have been changed. According to the distribution of the above pairs in faces of  $(X; R)$ , there are six cases to discuss. The respective transformations of faces are indicated in the following table.

original faces	faces after transformation	No. of faces:
$(x_k^{-1}, x_1, A), (x_1^{-1}, x_2, B), (x_2^{-1}, x_3, C)$	$(x_k^{-1}x_2, B, x_1^{-1}, x_3, C, x_2^{-1}x_1A)$	$f' = f - 2$
$(x_k^{-1}x_1, A, x_1^{-1}, x_2, B, x_2^{-1}x_3C)$	$(x_k^{-1}x_2, B, x_2^{-1}, x_1, A, x_1^{-1}x_3C)$	$f' = f$
$(x_k^{-1}x_1, A, x_2^{-1}x_3, B, x_1^{-1}, x_2, C)$	$(x_k^{-1}, x_2, C), (x_2^{-1}, x_1, A), (x_1^{-1}, x_3, B)$	$f' = f + 2$
$(x_k^{-1}x_1, A, x_1^{-1}, x_2, B)(x_2^{-1}, x_3, C)$	$(x_k^{-1}, x_2, B)(x_1^{-1}, x_3, C, x_2^{-1}, x_1, A)$	$f' = f$
$(x_k^{-1}x_1, A, x_2^{-1}, x_3, B)(x_1^{-1}, x_2, C)$	$(x_k^{-1}, x_2, C, x_1^{-1}, x_3, B)(x_2^{-1}, x_1, A)$	$f' = f$
$(x_k^{-1}x_1, A)(x_1^{-1}, x_2, B, x_2^{-1}, x_3, C)$	$(x_k^{-1}, x_2, B, x_2^{-1}, x_1, A)(x_1^{-1}, x_3, C)$	$f' = f$

The embeddings  $i$  and  $j$  can be described by rotations  $P$  and  $Q$ , respectively. Clearly, the rotation  $Q$  can be obtained from  $P$  applying a finite number of elementary transpositions of consecutive elements. By the Euler-Poincare theorem in each step the genus of the underlying surface either remains the same, or it changes by  $\pm$ . It follows that the integer interval  $[g, \dots, h]$  is covered by the genera of embeddings of  $X$ . ■

**Theorem 3.5** *Let  $X$  be a graph. Let  $i : X \hookrightarrow N_g, j : X \hookrightarrow N_h$  be 2-cell embeddings of  $X$  into non-orientable surfaces of genera  $0 \leq g < h$ . Then for every integer  $\gamma, g \leq \gamma \leq h$  there exists a 2-cell embedding of  $X$  into an non-orientable surface of genus  $\gamma$ .*

PROOF We may assume that  $i$  be an embedding of  $X$  into a non-orientable surface with maximal number of faces. Then  $i$  can be described by an embedding scheme  $(X; R, \xi)$ . Assume there are at least two faces in the embedding. Let  $e = \{x, x^{-1}\}$  be an edge separating the two faces. Set  $\xi'(x) = -\xi(x)$  and  $\xi'(x) = \xi(x)$ , otherwise. Either the boundary walks of the two faces are of the form  $(Axy), (Bx^{-1}z)$ , or of the form  $(Axy), (Bxz)$ . In the new embedding the two faces transform to one faces bounded by  $(Axz^{-1}B^{-1}xy)$  in the first case, and by  $(AxzBxy)$  in the second case. Repeating the above process we end with an one-face embedding of  $X$ . Since the closed walk  $W = (Axy)$  bounds a face,  $\xi(W) = 1$ , thus  $\xi'(W) = -1$ . Hence the non-orientability of the surface is preserved. The proof is complete. ■

By an orientable *minimum and maximum genus* of a given graph  $X$  we mean the integers  $g \leq h$  such that if  $X$  has a 2-cell embedding into  $S_g$  and  $S_h$  and orientable genus  $\gamma$  of every 2-cell embedding of  $X$  satisfies  $g \leq \gamma \leq h$ . The minimum and maximum non-orientable genera of  $X$  are defined similarly. Given graph  $X$  with  $e$  edges,  $v$  vertices and no semiedges denote by  $\beta(X) = e - v + 1$  the Betti number of  $X$ . We have the following proposition.

**Proposition 3.6** *The nonorientable maximum genus of  $X$  is equal  $\beta(X)$ . The orientable maximum genus  $g_{MAX}$  of  $X$  is bounded by  $g_{MAX} \leq \left\lceil \frac{\beta(X)}{2} \right\rceil$ .*

PROOF In the proof of Theorem 3.5 it is proved that every graph has a one-face 2-cell embedding into a nonorientable surface. By Euler-Poincare theorem  $v - e + 1 = 2 - g$ , where  $g$  is the non-orientable genus of the embedding. It follows  $g = \beta(X)$ . If  $i$  is any embedding of  $X$  into a non-orientable surface  $N_h$  then it has  $f \geq 1$  faces. Inserting  $f \geq 1$  in the Euler-Poincare equation we derive  $h \leq g$ . Similarly, inserting  $f \geq 1$  into the Euler-Poincare equation in the orientable case we get  $g \leq \beta(X)/2$ . ■

In case of embeddings of graphs into orientable surfaces the natural upper bound  $[\beta(X)/2]$  may not be reached. For instance, a 2-vertex graph  $X$  cubic graph consisting from two loops joined by an edge has  $\beta(X)/2 = 1$  but it does not admit a 2-cell embedding into the torus (as can be easily checked by inspecting all the four possible rotations). The problem of determining the (orientable) maximum genus of a graph arises.

## Maximum Genus of a Graph

The aim of this subsection is to prove Xuong's characterisation theorem. First we introduce a graph invariant. Given spanning tree  $T$  of a graph  $X$  denote by  $\zeta(X, T)$  the number of connectivity components of  $X - T$  with an odd number of edges. Set  $\zeta(X) = \min \zeta(X, T)$ , where the minimum is taken through all spanning trees of  $X$ . The following theorem by Xuong gives a characterisation of maximum genus. Observe that both the minimum number of faces in an embedding of  $X$  and  $\zeta(X)$  are preserved when subdividing edges of  $X$  by inserting vertices of degree 2.

**Theorem 3.7** *Let  $X$  be a connected graph without semiedges. Then minimum number of faces of a 2-cell embedding of  $X$  into an orientable surface is  $\zeta(X) + 1$ . In particular, the maximum genus of  $X$  is equal  $(\beta(X) - \zeta(X))/2$ .*

PROOF First we show that there exists an embedding of  $X$  with at most  $\zeta(X) + 1$  faces. We may assume that  $X$  is a simple graph, otherwise we can subdivide each edge by at most two vertices to make the graph simple. Let  $T$  be a spanning tree such that  $\zeta(X, T) = \zeta(X)$ . First we construct a spherical one-face embedding of  $T$ . The co-tree edges in even connectivity components can be arranged into couples sharing a vertex vertex in common, and the co-tree edges in each odd connectivity component can be, except one, arranged into couples sharing at least one vertex. Denote by  $\mathcal{R} = \{e_1, e_2, \dots, e_{\zeta(X)}\}$  the set of residual edges in the odd components. Let  $\{x, x^{-1}\}, \{y, y^{-1}\}$  be a couple of incident co-tree with  $v = I(x^{-1}) = I(y^{-1})$ ,  $u = I(x)$  and  $w = I(y)$ . Since the embedding of  $T$  is one-face, vertices  $v$ ,  $u$  and  $w$  appear at the boundary walk  $W$  of the face at least once. It follows that  $W$  can be written in the form  $W = (ABC) = (vAuBwC)$ . We form a one-face embedding of  $T \cup \{x, y\}$  transforming  $W$  onto  $W' = (Axy^{-1}Cx^{-1}By)$ . Since each dart is of the first type the embedding is orientable. Repeating the above procedure, we get a one-face embedding of  $Y = X - \mathcal{R}$ . Let  $e_i, e_j$  be two residual edges incident with vertices  $u, v$  and  $z, w$ , respectively. We say that they are in a crossing position with respect to a one face embedding of  $Y$  if the boundary walk of the face can be expressed as  $W = (ABCD)$  where  $u = I(A), z = I(B), v = I(C)$  and  $w = I(D)$ . In such case we may form a one-face embedding of  $Y \cup \{e_i, e_j\}$  by setting  $W = (Ae_jDe_iCe_j^{-1}Be_i^{-1})$ , here we have identified  $e_i = \{e_i, e_i^{-1}\}$ ,  $e_j = \{e_j, e_j^{-1}\}$  and we assume that  $I(e_i) = z$  and  $I(e_j) = u$ . It follows that whenever we find a crossing pair of residual edges we may add the two edges to a one-face embedded graph and form a one-face embedding of the new graph. The procedure stops when each pair of the remaining residual edges is non-crossing. In such case we may draw them in the interior of the face forming a 2-cell embedding with at most  $\zeta(X) + 1$  faces.

Let  $i$  be an embedding with  $f$  faces. We prove the inequality  $f \geq \zeta(X) + 1$ . There are edges  $e_1, e_2, \dots, e_{f-1}$  such that  $Y = X - \{e_1, \dots, e_{f-1}\}$  is a one-face embedded. Since  $\zeta(X) \leq \zeta(Y) + f - 1$ , it is sufficient to prove  $\zeta(Y) = 0$  for any one-face embeddable graph  $Y$ . To see it we use induction by the number of edges of  $Y$ . The statement clearly holds for a trivial graph without edges. Since all the edges of  $Y$  are of the first kind, there is a dart  $x$  at the boundary walk  $W$  such that the distance of  $x$  and  $x^{-1}$  is minimal. It follows that  $W$  can be written in the form  $W = (xAx^{-1}B)$ . If  $A$  is an empty word then  $e = \{x, x^{-1}\}$  is a pendant edge, removal of such an edge does not change  $\zeta(Y)$  and so by induction  $\zeta(Y) = \zeta(Y - e) = 0$ . If  $y \notin \{x, x^{-1}\}$  is the first dart  $y \in A$  then  $y^{-1} \in B$ . It follows that  $W$  can be written as  $W = (xyA_1x^{-1}B_1y^{-1}B_2)$ , where  $A = xA_1$  and  $B = B_1y^{-1}B_2$ . Now  $W' = (A_1B_2B_1)$  gives a one-face embedding of  $Z = Y - \{x, x^{-1}, y, y^{-1}\}$ . By induction hypothesis  $\zeta(Y) = \zeta(Z) = 0$ .

The expression  $g_{MAX} = (\beta(X) - \zeta(X))/2$  can be obtained from the Euler-Poincare formula inserting  $f = \zeta(X) + 1$  for the number of faces. ■

Graph  $X$  is called upper-embeddable if  $g_{MAX} = \lceil \beta(X)/2 \rceil$ , equivalently,  $X$  is upper-embeddable if it admits a 2-cell embedding with at most two faces. The following statement shows that most

of simple graphs are upper embeddable.

**Theorem 3.8** *A 4-edge-connected simple graph is upper-embeddable.*

PROOF By a theorem of Kundu a 4-edge-connected graph  $X$  has two edge-disjoint spanning trees. It follows that  $\zeta(X) \leq 1$  and by Theorem 3.7  $X$  is upper-embeddable.

It is easy to construct graphs with bridges having arbitrarily large  $\zeta(X)$ . We just take an arbitrary tree and attach loops to each pendant vertex. To calculate maximum genus (or  $\zeta(X)$ ) for a 3-edge-connected simple graph is less trivial, since each spanning tree gives us just an upper bound for  $\zeta(X)$ . Indeed, we need a tool to verify that a given tree is optimal. Such a tool gives a max-like characterisation proved by Nebesky.  $(\beta(X) - \zeta(X))/2$ . Let  $X$  be a connected graph and  $A \subseteq E(X)$  be a set of edges. Denote by  $c(A)$  the number of connectivity components in  $X - A$  and by  $b(A)$  the number of components in  $X - A$  with odd Betti number.

**Theorem 3.9** *Let  $X$  be a connected graph. Then  $\zeta(X) = \max(c(A) + b(A) - |A| - 1)$ , where the maximum is taken through all subsets  $A \subseteq E(X)$ .*

**Example.** Let  $X$  be the cubic graph depicted on Figure 12. Figure 12 shows a spanning tree  $T$  (the edges of  $T$  are drawn with bolded lines), with  $\zeta(X, T) = 2$ . On the other hand the set  $A$  of nine edges of  $X$  which does not lie in a triangle, separate 6 triangles. Hence  $c(A) = b(A) = 6$  and  $c(A) + b(A) - |A| - 1 = 2$ . It follows that  $\zeta(X) = 2$  and the maximum genus is  $(\beta(X) - 2)/2 = 4$ .

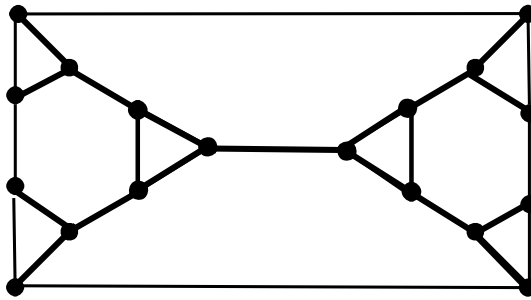


Figure 12: 3-edge-connected non-upper embeddable graph

An existence of characterisation of max-min type indicates that there could be an algorithm computing the maximum genus of a graph in a polynomial time. Indeed such an algorithm does exist. In contrast Thomassen proved that the problem of determining minimum genus of a graph is NP-complete. If a simple graph  $X$  has a triangular embedding into an orientable (non-orientable) surface  $S$  then the orientable (non-orientable) minimum genus of  $X$  is equal to the genus of  $S$ , which can be computed by the E-P formula by setting  $f = 2e/3$ . Similarly, quadrangular embeddings of bipartite graphs are necessarily minimal genus embeddings.

### Genus of $K_n$

The following definition gives a combinatorial counterpart of smooth coverings between surfaces considered in topology and complex analysis. Let  $\tilde{\mathcal{M}}$  and  $\mathcal{M}$  be two maps with the respective underlying graphs  $\tilde{X}$ ,  $X$ . We say that  $\tilde{\mathcal{M}}$  *smoothly covers*  $\mathcal{M}$  if there is a covering projection  $p : \tilde{X} \rightarrow X$  such that every boundary walk  $\tilde{W}$  of  $\tilde{\mathcal{M}}$  is a lift of a boundary walk  $W$  in  $\mathcal{M}$  and  $|\tilde{W}| = |W|$ . Any  $p : \tilde{X} \rightarrow X$  defined over an embedded graph  $X \rightarrow S$  with a set of boundary

walks  $\mathcal{F}$  determines an lifted embedding of  $\tilde{X}$  into a surface  $\tilde{S}$  with the set of boundary walks made from the all lifts of boundary walks of the base map,  $\tilde{\mathcal{F}} = \{\tilde{W} | p(\tilde{W}) \in \mathcal{F}\}$ . If  $p$  is given by an associated voltage space on  $X$  then the lifted map  $\tilde{\mathcal{M}}$  is called *the derived map*. The following statement is known as Kirchoff voltage law (KVL).

**Proposition 3.10** *Let  $\mathcal{M}$  be a map defined by a 2-cell embedding of connected graph  $X$ . Let  $(F, G, \xi)$  be a voltage space on  $X$ . Then the derived map smoothly covers  $M$  if and only if  $\xi(W) = 1$  for every boundary walk  $W$  of  $\mathcal{M}$ .*

**Example** Figure 13 defines a Cayley voltage space  $(\mathbf{Z}_{12s+7}, \mathbf{Z}_{12s+7}, \xi)$  on an triangular embedding of bouquet  $B_{6s+3}$ . To see it, it is sufficient to verify that the induced one-face embedding of a graph, whose edges form the  $4s+4$ -sided polygon (with opposite edges identified), gives rise to a surface. The latest statement is equivalent with the fact that the embedded graph is one-vertex. Since  $\xi : D \rightarrow \mathbf{Z}_{12s+7}^*$  is a bijection, the derived graph is  $K_{12s+7}$ . Moreover, the sum of voltages at each triangle is 0. Thus the voltage space satisfies the Kirchoff voltage law, and consequently, the base embedding lifts to a triangular embedding of  $K_{12s+7}$ . The above construction gives a solution of the particular instance ( $n \equiv 7 \pmod{12}$ ) of the Heawood map coloring problem.

Using a theory of current graphs which is in a sense dual theory to the theory of voltage spaces Ringel and Youngs proved the following theorem.

**Theorem 3.11** *The genus of  $K_n$ ,  $n \geq 3$ , is equal*

$$g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil,$$

*the non-orientable genus of  $K_n$ ,  $n \geq 3$ ,  $n \neq 7$ , is equal*

$$\tilde{g}(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil.$$

### Problems:

1. Determine the orientable and non-orientable surfaces admitting a 2-cell embedding of the Petersen graph. For each admissible surface determine a rotation or an embedding scheme describing it.
2. Prove that the difference  $\beta(X) - \zeta(X)$ , called sometimes the Betti deficiency, can be arbitrarily large even for planar 3-connected simple graphs.
3. Determine by means of rotations a minimum genus embedding of  $K_5$ ,  $K_6$ ,  $K_7$  and  $K_8$ .
- 4\*. Determine the minimum and maximum genus of the  $n$ -dimensional cube.
- 5\*. Determine the minimum and maximum genus of the complete 3-partite graphs  $K_{n,n,n}$ .

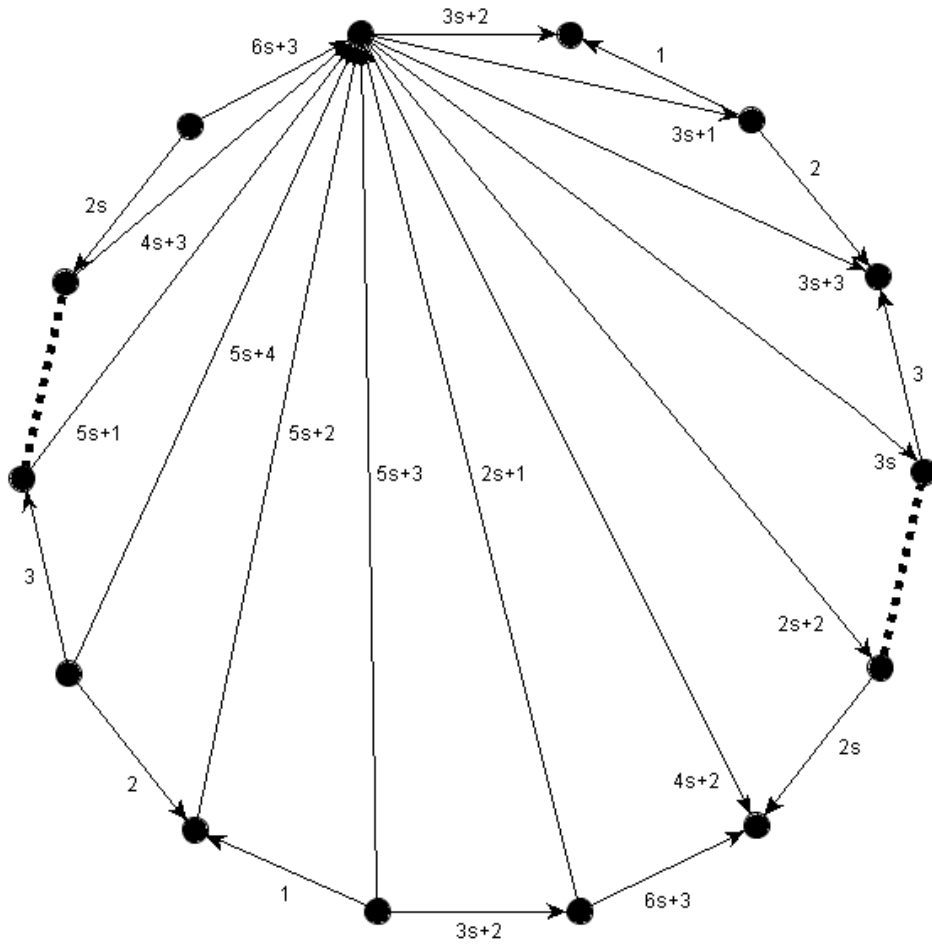


Figure 13: Triangular embedding of  $K_{12s+7}$  defined by a Cayley voltage space  $(\mathbb{Z}_{12s+7}, \mathbb{Z}_{12s+7}, \xi)$  on an triangular embedding of bouquet  $B_{6s+3}$ . (Figure comes from Gross and Tucker, page 227)