On Siamese Association Schemes

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Overview

- Introduction
- Siamese association schemes
- Constructions
- Results
- Open problems

Related objects

- Association schemes; strongly regular graph (SRG), distance regular graph (DRG).

- Incidence structures; generalized quadrangle (GQ), affine plane (AP), projective plane (PP), Steiner system, group divisible design (GDD).

- Matrices; adjacency matrix, incidence matrix, permutation matrix, lift of a matrix, balanced generalized weighing matrix (BGW)

Siamese association schemes

Siamese color graphs

Let $SRG(q) = SRG((q+1)(q^2+1), q(q+1), q-1, q+1)$.

A *spread* in a $SRG(q)$ is a system of q^2+1 pairwise disjoint cliques of size $q+1$.

Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_{q+1}$ be $SRG(q)$s with a common spread Σ such that each edge of the complete graph on $(q+1)(q^2+1)$ vertices not belonging to Σ belongs to exactly one Γ_i.

Let $\Delta_i = \Gamma_i - \Sigma$. Then, the system $\Sigma, \Delta_1, \ldots, \Delta_{q+1}$ is a *Siamese color Graph* on $(q+1)(q^2+1)$ vertices ($SCG(q)$).

We will usually work with adjacency matrices $S, R_1, \ldots, R_{q+1}$.

Why Siamese color graphs

- R.C. Bose (1963): Point graph of a $GQ(q)$ is a $SRG(q)$ (*geometric*).

- A. Brouwer (1984): $SRG(q) - \Sigma$ is a distance regular graph, antipodal with respect to Σ ($= DRG(q)$).

- *Geometric* $DRG(q)$, $SCG(q)$.

- S. Reichard (2003): Union of all blocks of all $GQ(q)$s in a geometric $SCG(q)$ is a Steiner system $S(2, q+1, (q+1)(q^2+1))$.

Siamese association schemes

Let $W = \{S_1, \ldots, S_n, R_1, \ldots, R_{q+1}\}$ be an association scheme on $(q+1)(q^2+1)$ vertices. We say that W is a *Siamese association scheme* of order q ($SAS(q)$) if $\{S, R_1, \ldots, R_{q+1}\}$ is a $SCG(q)$.

- $\sum S_i = S + I$.

- W may by non-commutative.

History

- 2003: H. Kharaghani and R. Torabi – an infinite family of Siamese color graphs.

- 2003: S. Reichard (Thesis) – an infinite family of Siamese association schemes (these two families may coincide).

- 2005: M. Klin, S. Reichard and A. Woldar – classification of Siamese color graphs for $q = 2$ (2 color graphs, 1 scheme). Hundreds of geometric Siamese color graphs for $q = 3$.

- 2015: M. Klin, M.M. – classification of Siamese color graphs for $q = 3$ (25245 color graphs, 2 schemes).

Constructions

Balanced generalized weighing matrices

Let $(G, \cdot)$ be a group not containing 0 and let $\overline{G} = G \cup \{0\}$. A *balanced generalized weighing matrix* with parameters (v, k, μ) over G, shortly $BGW(v, k, \mu, G)$ is a $v \times v$ matrix $M = [g_{ij}]$ over $\overline{G}$ such that each column contains exactly k non-zero elements and for any $a, b \in \{1, \dots, v\}$, $a \neq b$ the multiset

$$\{g_{ai} g_{bi}^{-1} : \; 1 \leq i \leq v, \; g_{ai} \neq 0, g_{bi} \neq 0\}$$

contains each elements of G exactly $\mu / |G|$ times.

BGW's have many applications in combinatorics. In particular, they represent GDDs on which G acts semi-regularly on points and lines (Jungnickel).

Lift of a BGW

Let G be a group with elements $\{g_q, \ldots, g_n\}$ and let M be a matrix with coefficients in $G \cup \{0\}$ (e.g. a $BGW(v, k, \mu, G)$). The *lift* of M the $vn \times vn$ matrix $L(M)$ obtained from M by replacing each 0 in M by the $n \times n$ zero matrix and each g_i by the permutation matrix P_i corresponding to $x \mapsto x * g_i$.

If M is a BGW then $L(M)$ is the incidence matrix of a GDD.

Siamese matrices

Let G be a group of order $q+1$. We say that a matrix $M = BGW(q^2+1, q^2, q^2-1, G)$ is a *Siamese matrix* of order q ($SM(q)$) over G if all the diagonal elements are equal to 0 and $m_{ij} = m_{ji}$ for any i, j (note that all the off-diagonal elements are non-zero).

Abelian, cyclic Siamese matrices.

Cyclic $SM(q)$ to $SCG(q)$

Theorem (H. Kharaghani and R. Torabi, 2003)**.** *Let* $q > 1$ *be a positive integer, let* $(G, .)$ *be a* **cyclic** *group of order* $q + 1$ *with elements* $g_1, g_2, \ldots, g_{q+1}$ *and let* M *be a Siamese matrix over* G*. Let* $\imath$ *be the involutory permutation of* $\{1, 2, \ldots, q + 1\}$ *given by* $i^\imath = j$ *iff* $g_i^{-1} = g_j$*. For any* $g_i \in G$ *let*

$$R_i = L(M) \cdot L(D_i^{q^2+1}) \cdot L(D_{g_i}^{q^2+1})$$

and let $S = J - I - \sum R_i$*. Then* $W = \{S, R_1, \ldots, R_{q+1}\}$ *is a* **Siamese color graph***.*

$GF(q^2)$ to cyclic $SM(q)$

H. Kharaghani and R. Torabi (2003)presented a construction of a cyclic $SM(q)$ from a finite field of order q^2. They presented it as sa special case of a construction of P. Gibbons and R. Mathon (1987) which will be introduced later.

Abelian $SM(q)$ to $SAS(q)$

Theorem. *Let $q > 1$ be a positive integer, let $(G, .)$ be an* **abelian** *group of order $q + 1$ with elements $g_1, g_2, \ldots, g_{q+1}$ and let M be a Siamese matrix over G. Let $\imath$ be the involutory permutation of $\{1, 2, \ldots, q + 1\}$ given by $i^\imath = j$ iff $g_i^{-1} = g_j$. For any $g_i \in G$ let*

$$S_i = L(D_{g_i}^{q^2+1}),$$
$$R_i = L(M) \cdot L(D_\imath^{(q^2+1)}) \cdot S_i.$$

Then $W = \{S_1, \ldots, S_{q+1}, R_1, \ldots, R_{q+1}\}$ is a **Siamese association scheme** *(if $q_i = e_G$ then $S_i = I$).*

The fact that $\imath$ is a group automorphism is crucial.

Affine plane to $SM(q)$ (P. Gibbons and R. Mathon (1987))

Let A be an affine plane of order q (we are not assuming that q is a prime power) with points $\{p_1, \ldots, p_{q^2}\}$ and parallel classes $c_1, c_2, \ldots, c_{q+1}$.

Let $N = (n_{ij})_{q^2 \times q^2}$ be the color graph of A, that is $n_{ii} = 0$ and for $i \neq j$ n_{ij} is the parallel class c_k which contains unique line in A through p_i and p_j.

Let $(G, .)$ be a group of order $q + 1$ and let φ be any bijection between $\{c_1, \ldots, c_{q+1}\}$ and G. Then the $q^2 + 1 \times q^2 + 1$ matrix $M = M(G, A, \varphi)$ defined by $m_{ii} = 0$ for any i; $m_{ij} = n_{ij}^{\varphi}$ for $1 \leq i, j \leq q^2$, $i \neq j$; and $m_{i(q^2+1)} = m_{(q^2+1)i} = e_G$ for $1 \leq i \leq q^2$, is a $SM(q)$ over G.

Some notation

- $M = BGW(A, G, \varphi)$, $W = SAS(M, G)$, $W = SAS(A, G, \varphi)$,

- $W = SAS(A, G, \varphi)$ is *affine*,

- $\Gamma = SRG(q)$ or $\Delta = SRG(q)$ is *affine* if it appears in an affine SAS,

- SAS $W = \{S_1, \ldots, S_{q+1}, R_1, \ldots, R_{q+1}\}$ is *thin*.

Results

Theoretical results

- Natural sufficient condition for $SAS(A, G, \varphi)$ and $SAS(A', G, \varphi')$ to be isomorphic.

- Sufficient and necessary condition for $SAS(M, G)$ and $SAS(M', G)$ to be isomorphic.

- Each thin SAS is a $SAS(M, G)$.

- Families of H. Kharaghani and R. Torabi and of S. Reichard are isomorphic.

Computational results

All the affine planes of order $q \leq 10$ are known. Here are the numbers of corresponding affine objects.

q	$planes$	$groups$	$schemes$	$DRGs$	$SRGs$	GQs
2	1	1	1	1	1	1
3	1	2	2	$2+1^*$	2	1
4	1	1	1	1	1	1
5	1	1	3	3	3	1
6	0	1	0	0	0	0
7	1	3	24	$29+3^*$	29	1
8	1	2	14	14	14	1
9	7	1	1517	2899	2899	1
10	0	1	0	0	0	0
11	1?	2	10955?	25753?	25753?	1?

Different groups

q	$Group$	$schemes$	$DRGs$	$SRGs$
3	$\mathbb{Z}_4$	1	1	1
3	$\mathbb{Z}_2^2$	1	$1+1^*$	1
7	$\mathbb{Z}_8$	11	14	14
7	$\mathbb{Z}_4 \times \mathbb{Z}_2$	10	12	12
7	$\mathbb{Z}_2^3$	3	$3+3^*$	3
8	$\mathbb{Z}_9$	11	11	11
8	$\mathbb{Z}_3^2$	3	3	3
11	$\mathbb{Z}_{12}$	8201?	15550?	15550?
11	$\mathbb{Z}_6 \times \mathbb{Z}_2$	2754?	10203?	10203?

* Some affine SRG's contain also a non-affine DRG's

? numbers only for the Desarguesian plane

Different planes

There exist 4 non-isomorphic projective and 7 non-isomorphic affine planes of order 9. It turns out that non-isomorphic affine planes of order 9 give rise to non-isomorphic affine Siamese objects. In the following table we give the numbers of Siamese objects for each plane.

nr	$proj.plane$	$schemes$	$DRGs$	$SRGs$
1	$Desargue$	85	139	139
2	$Hall$	60	104	104
3	$Hall$	214	428	428
4	$dualHall$	60	104	104
5	$dualHall$	214	428	428
6	$Hughes$	214	428	428
7	$Hughes$	670	1268	1268

Symmetries of affine $DRG(9)$s

$\|Aut(\Delta)\|$	$graphs$
2	416
4	92
8	19
16	418
20	2
32	4
36	1252
40	1
64	8
72	16
144	416

$\|Aut(\Delta)\|$	$graphs$
288	4
324	88
576	8
648	11
1296	122
1620	2
2592	6
3240	1
5184	11
6480	1
2125440	1

Comments on affine objects (for $q \leq 10$)

- There are only natural isomorphisms, different APs give different SASs.

- SRG appear in unique SAS, with unique DRG.

- SAS may contain different SRGs/DRGs.

- Elementary Abelian 2-groups force non-affine DRGs (Mersenne primes).

- Just the classical GQ.

Open problems

Understand the construction

- Explain the role of $AP, G, SM, GDD \ldots$
- Prove the computations (for all q's).
- Predict new results.
- Double covers of $DRG(q)$s?

Reverse implications

$$
\begin{array}{rcl}
 & \text{prime power} & \Rightarrow \\
\Rightarrow & \text{affine plane} & \Rightarrow \\
\Rightarrow & \text{(Abelian) Siamese matrix} & \Leftrightarrow \\
\Leftrightarrow & \text{thin SAS} & \Rightarrow \\
\Rightarrow & \text{Siamese association scheme} &
\end{array}
$$

Which of the implications can be reversed?

Covers and lifts for incidence structures?

The theory of covers and lifts is used to study semi-regular actions of groups on graphs.

Are there applications for incidence structures?

Thank You

Generalized Quadrangles

A *generalized quadrangle* (GQ) of order q is an incidence structure such that:

Each block (line) contains $q+1$ points;

Each point lies on $q+1$ lines;

For each line l and point $P \notin l$ $\exists!$ line through P intersecting l.

We write $GQ(q)$. Dual structure to $GQ(q)$ is also a $GQ(q)$.

Point graph of a GQ has vertices points of GQ, two vertices are adjacent iff they are collinear.

A *spread* S in a generalized quadrangle $GQ(q)$ is a partition of its vertex set into q^2+1 classes of size $q+1$, such that points in each class are pairwise collinear.

Affine planes

An *affine plane* is an incidence structure such that:

each pair of points lies on a unique line;

for any line l and any point P not incident with l, there is exactly one line incident with P that does not meet l;

there are four points such that no line is incident with more than two of them.

Affine plane is of order q if each line contains exactly q points.

If we remove points of a line from a projective plane we obtain an affine plane (a line at infinity). This construction can be reversed.