On regular map homomorphisms

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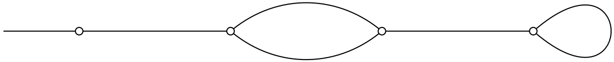
Graphs

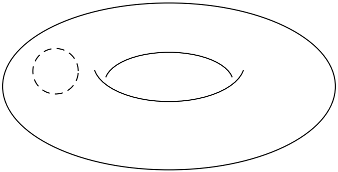
Graph - connected, undirected graph (loops, semi-edges and parallel edges are allowed)



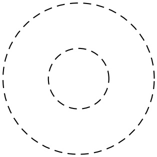
a semi-edge parallel edges a loop

Surfaces

Surface - 2-dimensional manifold - orientable or not, with or without boundary



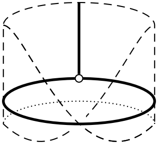
Torus with one hole



2-sphere with two holes

Maps

Map M - a graph G embedded in a surface $\mathcal{S}$ such that every face of the embedding is homeomorphic to an open disc or to a half-disc $\{[x, y] \in E_2 | x^2 + y^2 < 1, y \geq 0\}$



A single vertex graph with one loop and one semi-edge cellularly embedded into the Möbius band

Edges, darts, flags

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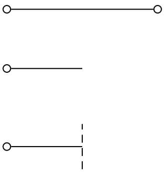
Edges



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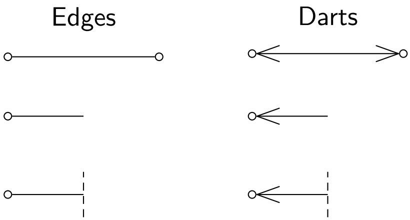

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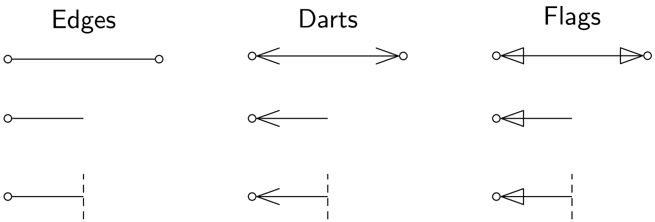
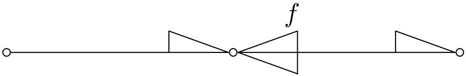
- Flags - a dart gives rise to two flags corresponding to the two "sides" of the dart



Description of maps

- Involutions τ, λ and ρ acting on the flag set of the map



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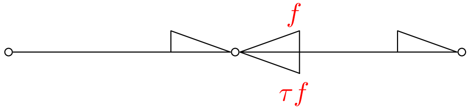

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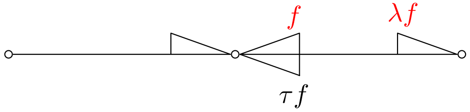

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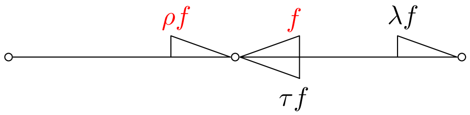

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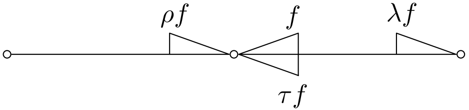

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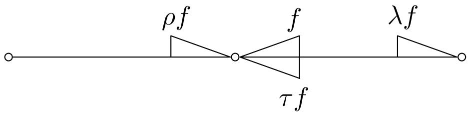

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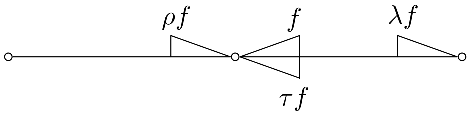

- Involutions τ and ρ are fixed free
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- We simply write $M = M(F; \tau, \lambda, \rho)$

Map homomorphisms

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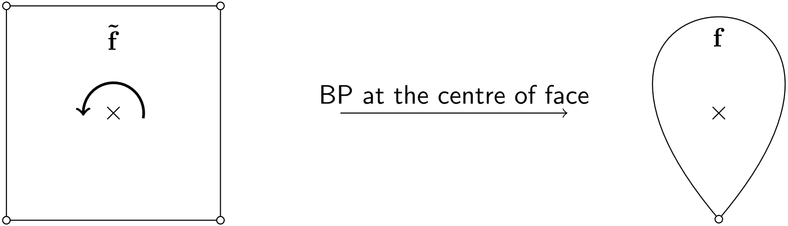
- A map homomorphism - a branched and folded covering from the supporting surface of $\tilde{M}$ to the supporting surface of M with branch points at the
  - centers of faces
  - dangling ends of semi-edges
  - vertices

  of the map $\tilde{M}$.

Branch points



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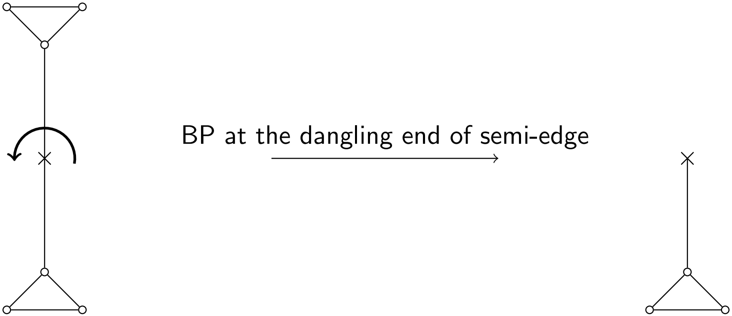
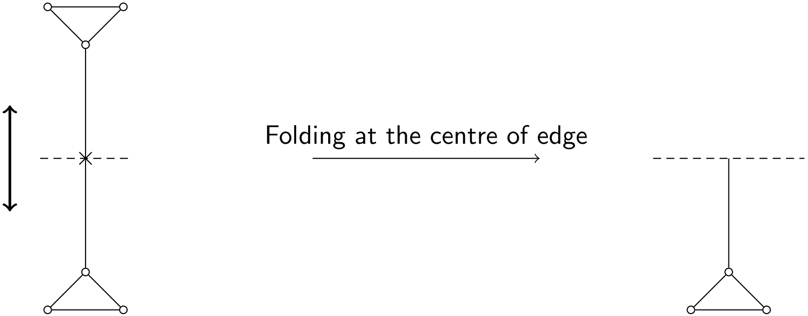


Longitudinal fold



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  that is, there are no folds at the vertices of $\tilde{M}$.

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- A map isomorphism $\varphi : \tilde{M} \rightarrow M$ is a map **automorphism** if $M = \tilde{M}$
- $Aut(\tilde{M})$ is the automorphism group of $\tilde{M}$ - the set of all automorphisms of $\tilde{M}$ under composition of mappings

Regular map homomorphisms

- A homomorphism $\varphi : \tilde{M} \to M$ without transversal and corner folds is **regular** if there exists a subgroup Γ of $Aut(\tilde{M})$ such that for any two flags $\tilde{f}_1, \tilde{f}_2$ of $\tilde{M}$ we have $\varphi(\tilde{f}_1) = \varphi(\tilde{f}_2)$ if and only if $\tilde{f}_2 = \gamma(\tilde{f}_1)$ for some map automorphism $\gamma \in \Gamma$.

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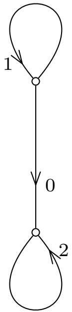
- If $\varphi : \tilde{M} \to M$ is a regular map homomorphism then M is isomorphic to the quotient map $\tilde{M}/\Gamma$.

Lifts of graphs



Dumbbell graph

Voltage group is $\mathcal{Z}_5$

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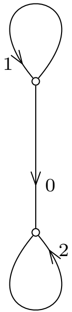
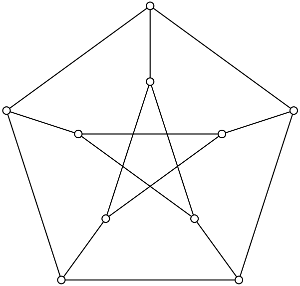


$$\underset{\text{Petersen}/\mathcal{Z}_5}{\overset{\text{Lift by } \mathcal{Z}_5}{\rightleftarrows}}$$



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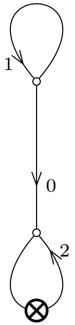
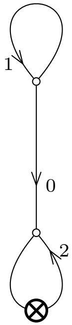
Petersen graph
The lift of dumbbell graph

Lifts of maps



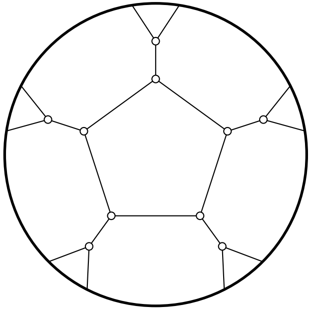
An embedding of DG
in projective plane

Lifts of maps



Lift by $\mathcal{Z}_5$

P in PP/$\mathcal{Z}_5$



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- $\alpha : F(M) \to \Gamma$ - a voltage assignment on M which realises the map $\tilde{M}$

Composition of regular homomorphisms - part 1

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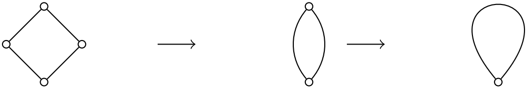

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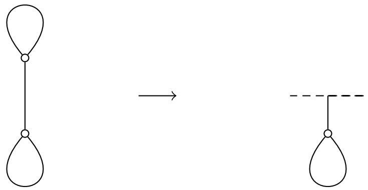
  - B: non-regular

An embedding of P in double torus $\longrightarrow$



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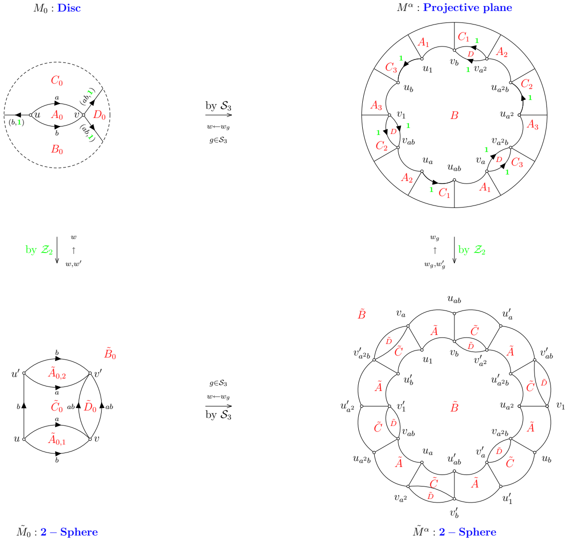
  Then $(M^{\alpha_1})^{\alpha_2} \cong (M^{\alpha_2})^{\alpha_1} \cong M^{\alpha}$

An application




. . . minutes to lunch . . .