Primer hypermaps and their role in the classification of regular hypermaps with p (prime) hyperfaces

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Abstract

A (face-)primer hypermap is a regular hypermap with no regular proper quotients with the same number of hyperfaces. Primer hypermaps are then regular hypermaps whose automorphism group induce a faithful action on its hyperfaces. This has been important in the classification of the orientably-regular hypermaps with p (prime) hyperfaces. In this talk we speak about the classification of primer hypermaps with p hyperfaces and the role that primer hypermaps have in the previously mentioned classification.

Connections

Wilson and Breda - Surfaces with no regular hypermaps (Discrete Math. 277 (2004) 241-274)
(Actions on faces)

Nedela and Breda -Chiral hypermaps with few hyperfaces (Math. Slovaca, (53) 2003 (2) 107-128)
(Action of the automorphism group on the hyperfaces)

Skoviera - Regular Maps with Multiple Edges (SIGMAC'02)
(Shadow of a map)

S.F. Du, J.H. Kwak and R. Nedela - Regular embeddings of complete multipartite graphs (European J. Combin., 26 (2005) 505-519)
(Extension)

Notations

Regular oriented hypermaps

$\mathcal{H} = (G; \mathrm{a}, \mathrm{b})$

$\mathrm{a} \rightsquigarrow$ permutation cycling darts 1 step CCW about hyperfaces.

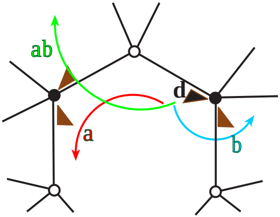
$$\mathrm{a} = r_0 r_1 = (RL)^{-1}$$

$\mathrm{b} \rightsquigarrow$ permutation cycling darts 1 step CCW about hypervertices.

$$\mathrm{a} = r_1 r_2 = R$$

$\mathrm{ab} \rightsquigarrow$ permutation cycling darts 1 step CW about hyperedges.

$$\mathrm{ab} = r_0 r_2 = L^{-1}$$



Primer hypermaps

(face -) Primer hypermap

= regular oriented hypermap with no proper quotients with the same number of hyperfaces

$$\begin{array}{rccl} \rho : \Delta^+ = \langle a,b \rangle = C_\infty * C_\infty & \longrightarrow & G = Mon(\mathcal{H}) = \langle \mathrm{a},\mathrm{b} \rangle \\ a & \longmapsto & \mathrm{a} \\ b & \longmapsto & \mathrm{b} \end{array}$$

$H = Ker(\rho)$ = hypermap-subgroup of $\mathcal{H}$.

????????????????????${}_?$?????????.

Chirality group: $X(\mathcal{H}) = H\overline{H}/H\,,\ \ (\overline{H} = H^{r_0})$

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$\mathcal{H}$ is primer $\quad \Leftrightarrow \quad Core_{\Delta^{+}}(\langle a \rangle) \subset H$.

Chirality group: $X(\mathcal{H}) = H\overline{H}/H$, $\quad (\overline{H} = H^{r_0})$

The Primer hypermap of $\mathcal{H}$

$$\mathcal{H} \text{ not primer } \Rightarrow \; Core_{\Delta^+}(\langle a \rangle) \not\subset H$$

$$\Downarrow$$

$$Core_{\Delta^+}(\langle a \rangle)H \triangleleft \Delta^+ \rightsquigarrow \text{ reg. orient. hypermap } \mathcal{P}(\mathcal{H})$$

$\mathcal{P}(\mathcal{H})$ is primer = Primer hypermap of $\mathcal{H}$.

If $A =$ permutation of the hyperfaces induced by the automorphism one step rotation about a hyperface then

$$Core_{\Delta^+}(\langle a \rangle)H = \langle a^{|A|} \rangle H$$

How to construct $P(\mathcal{H})$

$\mathcal{H} = (G; \mathrm{a}, \mathrm{b})$.

$Aut(\mathcal{H}) = \langle \phi_{\mathrm{a}}, \phi_{\mathrm{b}} \rangle$ acts transitively on $\mathcal{F}$ (hyperfaces)

ϕ_{a} and ϕ_{b} induces permutations A and B of $\mathcal{F}$.

Then

$P(\mathcal{H}) = (P; A^{-1}, B^{-1})$, where $P = \langle A, B \rangle$.

Proprieties of the primer hypermaps

- $\#$ hyperfaces of $\mathcal{H} \;=\; \#$ hyperfaces of $\mathcal{P}(\mathcal{H})$
- $\mathcal{P}(\mathcal{P}(\mathcal{H})) = \mathcal{P}(\mathcal{H})$
- If $Mon(\mathcal{H}) = \langle a, b \mid R \rangle$ then $Mon(\mathcal{P}(\mathcal{H})) = \langle a, b \mid R, a^{|A|} \rangle$
- The chirality group $X(\mathcal{P}(\mathcal{H})) = X(\mathcal{H})/L$
- $\mathcal{P}(\mathcal{H})$ chiral $\Rightarrow$ $\mathcal{H}$ chiral.
- The converse is not true.

If $\mathcal{H}$ = metacyclic hypermap $M(n, m, r, t); x, y$) $\rightsquigarrow$ chiral and reflexible
($M(n, m, r, t) = \langle x, y \mid x^n = 1, y^m = x^r, x^y = x^t \rangle$)
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p-primer groups (*p* prime)

p-primer hypermap = primer hypermap with *p* hyperfaces

p-primer group = monodromy group of a *p*-primer hypermap

Theorem

If $\mathcal{P} = (P; A, B)$ is a primer hypermap with p (prime) hyperfaces then only one of the situations can occur: $A = 1$ or the support of A is $\{2, \dots, p\}$. Moreover, the permutation A is either 1, a cycle of length $p - 1$, or a product of cycles each of length $|A|$.

Theorem

If $P = \langle A, B \rangle$ is a p-primer group then

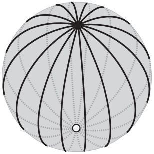
(1) *$|P| = |A|p$ and $|A|$ is a divisor of $p - 1$;*

(2) *P is a semidirect product $\langle A \rangle \ltimes \langle \sigma \rangle$, where σ is a permutation of order p, and hence P is a metacyclic group;*

(3) *P is primitive.*

Before the classification I

- $M(p,\ell,0,t) = \langle x,y \mid x^p = y^\ell = 1, x^y = x^t \rangle = \langle x \rangle \rtimes \langle y \rangle$,
- $t^\ell = 1 \mod p$.
- $|y| = \ell$ so $M(p,\ell,0,t)$ = monodromy group of a primer hypermap $\Rightarrow \ell$ is a divisor of $p-1$.
- Denote $\mathcal{M}_k^{p,\ell,t} := (M(p,\ell,0,t); y, xy^k)$.
- $\ell = 1 \Rightarrow (t=0)$ $\mathcal{M}_k^{p,1,0} = (M(p,1,0,0); 1, x) =$ spherical dihedral hypermap δ_p with p hyperfaces. This is clearly primer.



Before the classification II

Theorem

For each $k \in \{0, ..., \ell - 1\}$,

$$\mathcal{M}_k^{p,\ell,t} = (M(p,\ell,0,t); y, xy^k)$$

has p hyperfaces, each of valency ℓ. *Moreover,*

$\mathcal{M}_k^{p,\ell,t}$ *is primer*

$$\Updownarrow$$

(1) ℓ *is a divisor of* $p - 1$

(2) *If* $\ell > 1$, $t^i \neq 1 \pmod{p}$ *for each* $i \in \{1, 2, ..., \ell - 1\}$

Classification of the p-primer hypermaps

Denote by $\mathcal{P}_k^{p,\ell,t}$ the primer hypermap $\mathcal{M}_k^{p,\ell,t}$ before,

where $t \in \{0, ..., p-1\}$ and ℓ satisfy the two conditions given before.

Theorem

$\mathcal{P}$ = primer hypermap with p hyperfaces $\Leftrightarrow$ $\mathcal{P} \cong \mathcal{P}_k^{p,\ell,t}$ for some $k \in \{0, ..., \ell - 1\}$.

Different parameters ℓ, t and k correspond to non-isomorphic hypermaps with p hyperfaces of valency ℓ.

Shadow maps versus primer maps

Multiple edges reg. maps $\underset{\text{dual}}{\rightsquigarrow}$ face multiple edges maps

(incident faces sharing more than 1 edge)

$\downarrow$ $\downarrow$

Shadow map Primer map

$\downarrow$ (generalises)

primer hypermaps

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