On the classification of regular embeddings of hypercubes

joint work with Marston D.E. Conder, Shao Fei Du, Young Soo Kwon, Roman Nedela and Stephen E. Wilson

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1. Nedela and Škoviera:
   For every solution of the congruence $e^2 \equiv 1 \pmod{n}$ *there is a regular embedding of* Q_n*, with different solutions giving rise to non-isomorphic embeddings.*
2. Du, Kwak and Nedela:
   There are no other regular embeddings of Q_n *into orientable surfaces when n is odd.*
3. Kwon and Nedela:
   There are no regular embeddings of Q_n *into non-orientable surfaces, for all* $n > 2$*.*

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- $\mathcal{G}$ connected simple graph
- $\mathcal{S}$ closed (compact and connected) orientable surface.

Definition

An embedding of $\mathcal{G}$ in $\mathcal{S}$ is a cellular subdivision $\mathcal{M} = (V, E, F)$ of $\mathcal{S}$ such that $(V, E) \cong \mathcal{G}$. An automorphism of $\mathcal{M}$ is a self homeomorphisms of $\mathcal{S}$ preserving $\mathcal{G}$ and orientation.

$$\boxed{Aut(\mathcal{M}) \leqslant Aut(\mathcal{G})} \quad \text{and} \quad \boxed{|Aut(\mathcal{M})| \leqslant 2|E|}$$

Definition

If $|Aut(\mathcal{M})| = 2|E|$ then $\mathcal{M}$ is called (orientably) regular.

- Vertex set $V = \mathbb{Z}_2{}^n$
- Edge set E: $\{u, v\} \in E \Leftrightarrow u + v \in S$

  ($S = \{e_0, ..., e_{n-1}\}$ standard basis of $\mathbb{Z}_2{}^n$)
- Dart set $D = V \times \mathbb{Z}_n$ $(D = V \times S)$
- Dart reversing involution
  $L : D \to D,\ (v, i) \mapsto (v + e_i, i)$

Definition

$Q_n = (V, E)$ or $Q_n = (D, V, L)$ or $Q_n = \mathcal{C}(V, S)$ or $Q_n = H(n, 2)$ or ...

- for $a \in V$: $(v,i)^a = (v+a,i)$ **Translations**
- for $\alpha \in S_n = Sym(\mathbb{Z}_n)$: extend $e_i \mapsto e_{i^\alpha}$ linearly from S to V and set $(v,i)^\alpha = (v^\alpha, i^\alpha)$ **Permutations**

Then $Aut(Q_n) = V \rtimes S_n$ with $a\alpha = \alpha a^\alpha$

Remark

$Aut(Q_n) = \mathbb{Z}_2 \wr S_n$

$\rho : \mathbb{Z}_n \to \mathbb{Z}_n, \ i \mapsto i + 1.$

Lemma (Kwon)

For any involution $\sigma \in S_n$ fixing 0 the group $G(\sigma) = \langle e_0\sigma, \rho\rangle \leqslant Aut(Q_n)$ acts transitively on darts.

Call σ admissible if $G(\sigma)$ acts **regularly** on darts. Then the stabilizer of vertex $v = 0$ is $\langle\rho\rangle$ (cyclic) $\overset{\text{GNSS}}{\Rightarrow}$ $G(\sigma) = Aut(\mathcal{M}(\sigma))$ for some regular embedding $\mathcal{M}(\sigma)$ of Q_n. Reciprocally,

Theorem (Kwon)

For any regular embedding $\mathcal{M}$ of Q_n there is an admissible involution $\sigma \in S_n$ such that $Aut(\mathcal{M}) = G(\sigma)$. Moreover, $\mathcal{M}(\sigma) \ncong \mathcal{M}(\omega)$ for admissible involutions $\sigma \neq \omega$.

Conclusion

Classification of regular embedding of Q_n
$=$ Classification of admissible involutions $\sigma \in S_n$.

$\tau : \mathbb{Z}_n \to \mathbb{Z}_n,\ i \mapsto -i$

Theorem

Let $\sigma \in S_n$ be an involution fixing 0. Then σ is admissible iff

$$\rho^{i}(e_0\sigma)\rho^{i^{\sigma\tau}}(e_0\sigma)\rho^{i^{(\sigma\tau)^2}}(e_0\sigma)\rho^{i^{(\sigma\tau)^3}}(e_0\sigma) = 1\,, \quad i \in \mathbb{Z}_n \setminus \{0\}$$
$$\Leftrightarrow$$
$$\rho^{i}\sigma\rho^{i^{\sigma\tau}}\sigma\rho^{i^{(\sigma\tau)^2}}\sigma\rho^{i^{(\sigma\tau)^3}}\sigma = 1\,, \qquad i \in \mathbb{Z}_n \setminus \{0\}\,. \qquad (*)$$

We call $(*)$ quadrilateral identities.

Let e be a square root of 1 in $\mathbb{Z}_n$ ($e^2 \equiv 1 \bmod n$).

Example (Nedela, Škoviera)

$\sigma : \mathbb{Z}_n \to \mathbb{Z}_n,\ i \mapsto ei$ is an admissible involution.

Example (Kwon)

$n = 2m$, $A \subseteq \mathbb{Z}_n \setminus \{0\}$ with characteristic function χ_A satisfying

$$\chi_A(ei) = \chi_A(i) \quad \text{and} \quad \chi_A(i+m) = \chi_A(i)\,.$$

Then $\sigma : \mathbb{Z}_n \to \mathbb{Z}_n,\ i \mapsto ei + m\,\chi_A(i)$ is an admissible involution (called K-involution).

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Let now $n = 2m$ divisible by 16 and e a square root of $1+m$ in $\mathbb{Z}_n$ ($e^2 \equiv 1 + m \mod n$).

Example (– and Nedela)

$A \subseteq \mathbb{Z}_n \setminus \{0\}$ with characteristic function χ_A satisfying

$$\chi_A(ei) \equiv i + \chi_A(i) \pmod 2 \quad \text{and} \quad \chi_A(i+m) = \chi_A(i)\,.$$

Then $\boxed{\sigma : \mathbb{Z}_n \to \mathbb{Z}_n,\ i \mapsto ei + m\chi_A(i)}$ is an admissible involution (called CN-involution).

Remark

In case $n = 2m$ we have $\sigma\rho^m = \rho^m\sigma$ in both examples.

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Remark

In case $n = 2m$ we have $\sigma\rho^m = \rho^m\sigma$ in both examples.

Theorem

$\sigma\rho^m = \rho^m\sigma$ for any admissible involution $\sigma \in S_{2m}$.

$\langle\sigma, \rho\rangle$ is a permutation group of degree $2m$ containing a regular element ρ such that point-stabilizers are 2-groups.

$$\Rightarrow \boxed{\{i, i+m\}^\sigma = \{i^\sigma, i^\sigma + m\}}$$

Corollary (Projection)

For any admissible involution $\sigma \in S_{2m}$

$$\overline{\sigma} : \mathbb{Z}_m \to \mathbb{Z}_m,\ i \mapsto i^\sigma \ (mod\ m)$$

is an admissible involution in S_m.

Definition

An admissible involution $\sigma \in S_{2m}$ is called a lift of the admissible involution $\omega \in S_m$ if $\overline{\sigma} = \omega$.

We have seen: Every admissible involution $\sigma \in S_{2m}$ is a lift of some admissible involution $\omega \in S_m$ (namely $\omega = \overline{\sigma}$). If m is odd then σ is a K-involution (Jing Xu). Hence we may assume $m = 2k$. Let $\sigma \in S_{2m}$ be a lift of

$$\omega : \mathbb{Z}_m \to \mathbb{Z}_m, \; i \mapsto ei + k\chi_A(i) \quad \text{K- or CN-involution.}$$

Then $\boxed{\sigma : \mathbb{Z}_{2m} \to \mathbb{Z}_{2m}, \; i \mapsto ei + k\psi(i)}$ for some $\psi : \mathbb{Z}_{2m} \to \mathbb{Z}_4$ satisfying the lifting conditions:

1. $\overline{\sigma} = \omega$ ($\Leftrightarrow$ conditions on ψ)
2. σ is an involution fixing 0 ($\Leftrightarrow$ conditions on ψ).
3. σ satisfies the quadrilateral identities ($\Leftrightarrow$ conditions on ψ).
4. $\sigma\rho^m = \rho^m\sigma$ ($\Leftrightarrow$ conditions on ψ).

Theorem

$\psi(i+j) \equiv \psi(i) + \psi(j)$ mod 2.

$\Leftrightarrow (i+j)^\sigma \equiv i^\sigma + j^\sigma$ mod m
$\Leftrightarrow \sigma$ is a K- or CN-involution.

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