Recent progress in the study of regular maps

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— including joint work with Domenico Catalano, Shaofei Du, Young Soo Kwon, Roman Nedela, Jozef Širáň, Tom Tucker and Steve Wilson

Summary

- Introduction/background
- Extended census of regular maps of small characteristic
- Gaps in the genus spectrum of chiral maps
- Gaps in the genus spectrum of ‘simple’ regular maps
- Regular maps of characteristic $-3p$ for p prime
- Regular Cayley maps for cyclic groups
- Regular embeddings of n-dimensional cube graphs Q_n

Platonic solids: regular maps on the sphere

The Platonic solids. Scotland, ca. 2000 B.C. (Photo, courtesy Graham Challifour, from "Time Stands Still", K. Critchlow.)

... also called 'Neolithic Scots' (c. 2000BC?)

Regular maps

A map M is a 2-cell embedding of a connected graph or multigraph (graph with multiple edges) on a surface.

Every automorphism of a map M is uniquely determined by its effect on a given flag (incident vertex-edge-face triple), and it follows that $|\mathrm{Aut}\, M| \leq 4|E|$ where E is the edge set. When the upper bound is attained, the map M is regular.

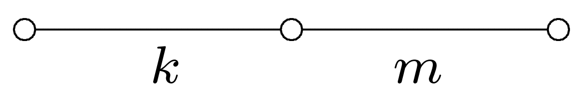
More generally, if M has two automorphisms that act like a single-step rotation about some face and some vertex, then M is called rotary. If also M has automorphisms that act (locally) like reflections, then M is reflexible and therefore regular. Otherwise, if M is rotary but not reflexible, then M must be orientable, with $|\mathrm{Aut}\, M| = 2|E|$, and M is chiral.

Transitivity, type and triangle groups

If M is a rotary map, then its underlying graph is vertex-transitive, edge-transitive and face-transitive.

In particular, every face of must have the same number of edges (say k) and every vertex must have the same valency (say m). In this case we say that M has type $\{k, m\}$.

Moreover, $\mathrm{Aut}\, M$ contains elements R and S that act as single-step rotations about a face and an incident vertex, and satisfy the relations $R^k = S^m = (RS)^2 = 1$, which define the $(2, k, m)$ triangle group — a subgroup of index 2 in the $[k, m]$ Coxeter group



Classification

Rotary/regular maps are usually viewed from one of three main perspectives:

- Classification by *surface*

— maps on an orientable surface of given genus
— maps on a non-orientable surface of given characteristic

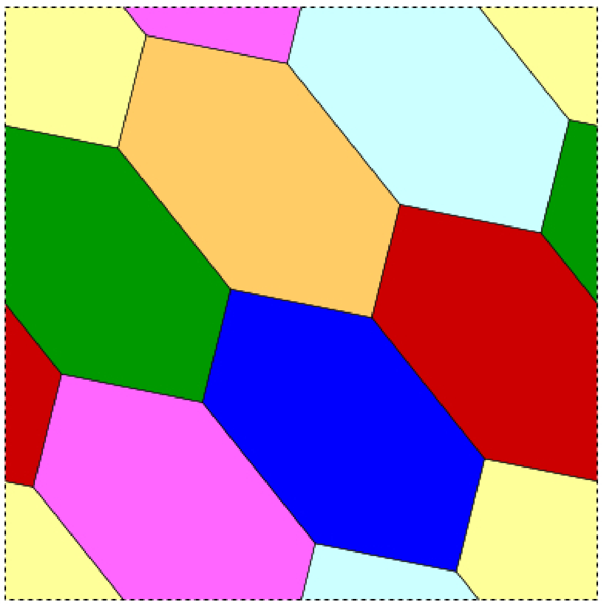
- Classification by *underlying graph*

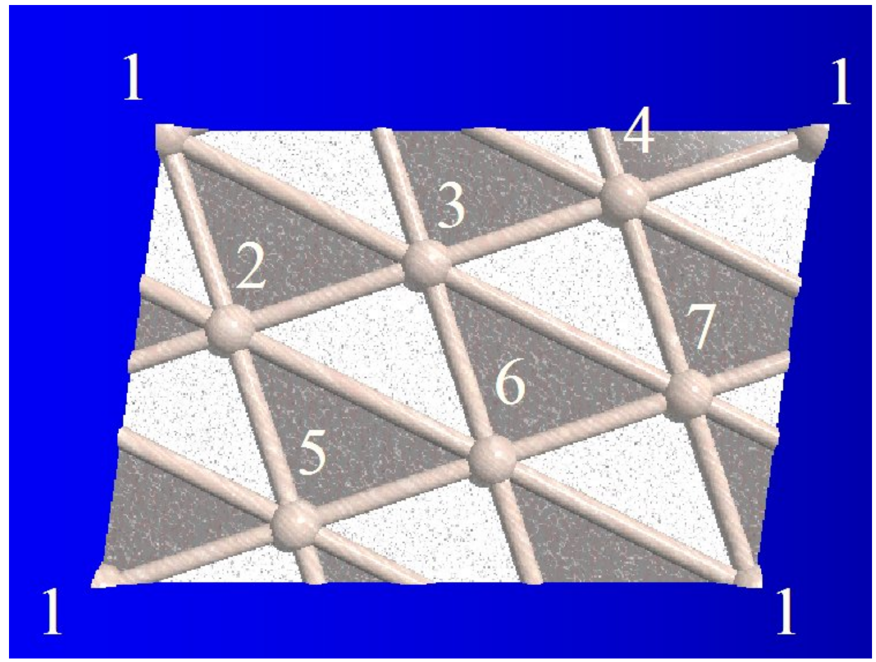
— e.g. rotary embeddings of K_n or $K_{n,n}$ or Q_n

- Classification by *group*

— rotation group or full automorphism group, e.g. $PGL(2, q)$.

Example: a map of type $\{6,3\}$ on the torus





This is chiral, with automorphism group $\mathbb{Z}_7 : \mathbb{Z}_6$, and is dual to the type $\{3,6\}$ triangulation of the torus by K_7 (RHS).

Natural questions about the genus spectrum

- Is there a rotary orientable map of every possible genus?

- Is there a non-orientable regular map of every genus?

- What are the genera of orientably-regular maps that have simple underlying graphs (with no multiple edges)?

- What are the genera of rotary but chiral maps?

Some known answers

• Is there a rotary orientable map of every possible genus?
Answer: Yes, for every $g > 1$ there exists a regular map of type $\{4g, 4g\}$ with dihedral automorphism group of order $8g$ — but with only one vertex and one face, and multiple edges

• Is there a non-orientable regular map of every genus?
Answer: No, e.g. there are none of genus 2 (Klein bottle) or genus 3. Moreover, Breda, Nedela and Siráň (2005) proved there is only one such map of genus $p+2$ when p is a prime congruent to 1 mod 12 (viz. one of genus 15)

• What are the genera of rotary but chiral maps?
Partial answer: There are no such maps on orientable surfaces of genus 2 to 6

Determination of regular maps of small genus

Genus 0: regular polyhedra (incl. "dihedra" and their duals)

Genus 1 and 2: Brahana [1927] and Coxeter [1957]

Genus 3: Sherk [1959]

Genus 4, 5 and 6: Garbe [1969]

MC & Peter Dobcsányi [2001]:
- All rotary maps of genus 2 to 15
- All non-orientable regular maps of genus 2 to 30.

MC [2006, from low index normal subgroups of Δ groups]:
- All rotary (hyper)maps of genus 2 to 101
- All non-orientable regular (hyper)maps of genus 2 to 202.

Method: Low index normal subgroups

Small homomorphic images of a finitely-presented group G can be found as the groups of permutations induced by G on cosets of subgroups of small index. This gives G/K where K is the core of H, but produces only images that have small degree faithful permutation representations.

Alternatively, the low index subgroups method can be adapted to produce only normal subgroups (of small index in G).

A new method has been developed recently by Derek Holt and his student David Firth, which systematically enumerates the possibilities for the composition series of the factor group G/K, for any normal subgroup K of small index in G.

Summary of results for small genus

Orientably-regular maps (up to isomorphism & duality)
Genus 2: 6 reflexible maps, 0 chiral
Genus 3: 12 reflexible maps, 0 chiral
Genus 4: 12 reflexible maps, 0 chiral
Genus 5: 16 reflexible maps, 0 chiral
Genus 6: 13 reflexible maps, 0 chiral
Genus 7: 12 reflexible maps, 2 chiral pairs
Genus 2 to 101: 3378 reflexible maps, 594 chiral pairs

Non-orientable regular maps (up to isomorphism & duality)
Genus 2 or 3: 0 maps
Genus 4: 2 maps
Genus 14: 3 maps
Genus 2 to 202: 862 maps

Observations

- There is no orientably-regular but chiral map of genus 2, 3, 4, 5, 6, 9, 13, 23, 24, 30, 36, 47, 48, 54, 60, 66, 84, 95, 108, 116, 120, 139, 150, 167, 168, 174, 180, 186 or 198

- There is no regular orientable map of genus 20, 32, 38, 44, 62, 68, 74, 80 or 98 with simple underlying graph

- A lot of these exceptional genera are of the form $p + 1$ where p is prime.

Theorems [joint work with Jozef Siráň & Tom Tucker]

- If M is an irreflexible (chiral) orientably-regular map of genus $p+1$ where p is prime, then
  either $p \equiv 1 \bmod 3$ and M has type $\{6,6\}$,
  or $p \equiv 1 \bmod 5$ and M has type $\{5,10\}$,
  or $p \equiv 1 \bmod 8$ and M has type $\{8,8\}$.

In particular, there are no such maps of genus $p+1$ whenever p is a prime such that $p-1$ is not divisible by 3, 5 or 8.

- There is no regular map M with simple underlying graph on an orientable surface of genus $p+1$ where p is a prime congruent to 1 mod 6, for $p > 13$.

In fact, even more ...

• A complete classification of all regular and orientably-regular maps M for which $|\mathrm{Aut}\, M|$ is coprime to the Euler characteristic χ (if χ is odd) or to $\chi/2$ (if χ is even)

This has all three main results to date as corollaries:

• No chiral orientably-regular maps of genus $p+1$ for primes p not congruent to 1 mod 3, 5 or 8

• No regular orientable maps with simple underlying graph and genus $p+1$ for primes $p > 13$ congruent to 1 mod 6,

• No non-orientable regular maps of genus $p+2$ for primes $p > 13$ congruent to 1 mod 12.

[This work to appear in J. Europ. Math. Soc. (2009)]

Coprime classification: $|G|$ coprime to χ or $\chi/2$

- Let X and Y generate stabilizers of a face and incident vertex, so that $\langle X, Y\rangle = G$ and $X^k = Y^m = (XY)^2 = 1$

- The coprime assumption gives us $|G| = 2\,\mathrm{lcm}(k,m)/t$ or $4\,\mathrm{lcm}(k,m)/t$ where $t = 1, 2$ or 4

- Every cyclic subgroup of G of odd order is conjugate to a subgroup of $\langle X\rangle$ or $\langle Y\rangle$, and so G is 'almost Sylow-cyclic'

- All 'almost Sylow-cyclic' groups are known [thanks to Zassenhaus (1936), Suzuki (1955) and Wong (1966)]

- When $\langle X\rangle \cap \langle Y\rangle$ is trivial, we have $|G| \geq |\langle X\rangle\langle Y\rangle| = km$, and this gives us only a small number of cases to consider, according to the values of $d = \gcd(k,m)$ and t ; these can be dealt with using fairly standard combinatorial group theory

- When $N = \langle X\rangle \cap \langle Y\rangle$ is non-trivial, it is central, so we can use the transfer homomorphism $h \mapsto h^{|G:N|}$ from G to N (and Schur's theorem, which says that the order of every element of the derived group G' divides the index $|G : Z(G)|$) to determine all possibilities in each of the five cases for G/N

- In all cases, the map M is reflexible

- The map M (or its topological dual) has simple underlying graph if and only if $\langle Y\rangle$ (resp. $\langle X\rangle$) is 'core-free' in G.

Type	Genus	$\lvert G\rvert$	Comments
$\{8n, 8n\}$	$2n$	$8n$	G cyclic
$\{4n+1, 8n+2\}$	$2n$	$8n+2$	G cyclic, $n \not\equiv 2 \bmod 3$
$\{2n, vn\}$	$v(n-1)/2$	$2vn$	$G \cong C_n \times D_v$, $n \equiv 1 \bmod 4$
$\{2rn, 2sn\}$	$rsn-r-s+1$	$4rsn$	G has quotient $C_2 \times C_2$
$\{4n, 3vn\}$	$6vn-3v-3$	$24vn$	G has quotient S_4
$\{8n, 3vn\}$	$12vn-3v-7$	$48vn$	G has genus 2 quotient
$\{3n, 3n\}$	$3n-3$	$12n$	$G \cong C_n \rtimes A_4$, n odd
$\{3n, 5n\}$	$15n-15$	$60n$	$G \cong C_n \times A_5$, $\gcd(n, 60) = 1$

Approach when $\chi = -p$ or $-2p$ for p prime

For such a map M, let G be the subgroup of Aut M generated by vertex- and face-stabilizers. Then:

- For small p, we know all examples

- For large p when p divides $|G|$, we can use Sylow theory to reduce the case of a quotient G/P acting on a map of small genus

- For large p when p does not divides $|G|$, we can use the 'coprime classification'.

Recap: New theorems (to appear in J.E.M.S.)

• If M is an irreflexible (chiral) orientably-regular map of genus $p+1$ where p is prime, then
either $p \equiv 1$ mod 3 and M has type $\{6, 6\}$,
or $p \equiv 1$ mod 5 and M has type $\{5, 10\}$,
or $p \equiv 1$ mod 8 and M has type $\{8, 8\}$.

In particular, there are no such maps of genus $p+1$ whenever p is a prime such that $p-1$ is not divisible by 3, 5 or 8.

• There is no regular map M with simple underlying graph on an orientable surface of genus $p+1$ where p is a prime congruent to 1 mod 6, for $p > 13$.

Next: Regular maps of characteristic $-3p$

There are just four kinds of regular maps of characteristic $-3p$ for odd prime p [R. Nedela, J. Širáň & MC, 2009]:

- Type $\{2j, 2l\}$ where $(j-1)(l-1) = 3p+1$, for j, l odd, with $j \geq l \geq 3$, and $\gcd(j, l) \leq 3$ but $j \equiv l \not\equiv 1 \bmod 3$
- Type $\{4j, 6\}$ where $p = 4j - 3 \equiv 1 \bmod 4$ for some odd j
- Type $\{8, 7n\}$ with $p = 21n-8$ for some n coprime to 14
  [Two maps for each n in this case]
- Nine 'sporadic' examples with $p \in \{3, 5, 7, 11\}$.

Corollary: There is no regular map on a non-orientable surface of Euler characteristic $-3p$ for any prime p such that $p > 11$, $p \equiv 3 \bmod 4$, and $p \not\equiv 55 \bmod 84$.

Regular Cayley maps

Let G be a group, and let S be a generating set for G that is closed under inverses (and does not contain the identity). Then the Cayley graph Cay(G, S) has vertex-set G and edges $\{g, xg\}$ for $x \in S$. The group G acts regularly on its vertices.

If the underlying graph of the orientably regular map M is a Cayley graph Cay(G, S) for some group G — or equivalently, if Aut M has a subgroup G acting regularly on vertices of M — then M is called a *regular Cayley map* for G. Here the embedding prescribes an order on the generating set S.

Relatively little is known about regular Cayley maps, except in some specific cases, but the new census of rotary maps provides a well-spring of examples for analysis.

Curious theorem:

Let M be a regular Cayley map for a finite cyclic group A. Then M is reflexible if and only if M is anti-balanced (that is, if and only if the ordering of the generating set for A can be written in the form $\ldots x_3, x_2, x_1, {x_1}^{-1}, {x_2}^{-1}, {x_3}^{-1}, \ldots$).

Thus most regular Cayley maps for cyclic groups are chiral!
[Young Soo Kwon, Jozef Širáň & MC (February 2007)]

This is not as easy to prove as it looks. It does become easy if the generating set for A contains an element of order $|A|$.

Recent theorem [Tom Tucker & MC (August 2008)]

If M is a regular Cayley map for a finite cyclic group A of order n, then the generating set S for A can be assumed to contain an element of order n.

Sketch proof.

Let $G = \text{Aut}^{\text{o}}M = AY$ where Y is the stabilizer of a vertex v, let x be an involution in G that reverses an arc incident with v, and let D be the normal closure of x in G.

By a theorem of Marty Isaacs & MC on abelian products (2004), we know that G' is isomorphic to a subgroup of A. Next, ... [PTO]

By work of Robert Jajcay, Tom Tucker & MC (2006) on regular Cayley maps for abelian groups, we have 3 cases:
(1) $D = G' \cong A$ is an elementary abelian 2-group;
(2) $G = G'Y$ with $G' \cap Y = \{1\}$, and $G' \leq D$ with index 2;
(3) $G = DY$ with $D \cap Y = \{1\}$, and $G' \leq D$ with index 2.

In case (2), the map M is a 'balanced' regular map for $G' \cong A$, and any element c of order n in G' can be assumed to lie in the associated generating set for the map. Now if a is any element of order n in A, then $a = cy$ for some $c \in G'$ and $y \in Y$, and since $\langle c, y\rangle = \langle cy, y\rangle = \langle a, y\rangle = AY = G = G'Y$, we find that c has order n so WLOG lies in the 'balanced' generating set, and hence the given generating set S for A can be assumed to contain $a = cy$.

Case (3) is similar, and case (1) is trivial (with $|A| = 2$).

Moreover ...

These observations make other things possible.

We (TT & MC) now have a complete determination of

• all orientably-regular Cayley maps for finite cyclic groups

— or equivalently,

• all 'skew morphisms' of C_n with a generating orbit closed under inverses

— or equivalently,

• all regular embeddings of arc-transitive circulants (without needing classification of arc-transitive circulants!).

The balanced case (where $C_n \lhd G = \mathrm{Aut}^{\mathrm{o}} M$)

Here the vertex-rotation y corresponds to an automorphism of $C_n \cong \langle c \rangle$ of the form $c \mapsto c^{\lambda}$ for some unit $\lambda \in \mathbb{Z}_n$, and $\lambda^{m/2} \equiv -1 \bmod n$ where m is the valency (necessarily even).

In order for $a = cy^i$ to generate a cyclic group A of order n complementary to $\langle y \rangle$, some elementary number theory and a little Sylow theory show that
— $o(y^i)$ is odd,
— $o(y^i)$ divides both m and n, and
— $n/o(y^i)$ is divisible by every prime that divides n.

Hence the map can be represented as a *non-balanced* Cayley map for C_n iff n is divisible by the square of an odd prime.

In particular, if n is a square-free odd positive integer, then every Cayley map for C_n is balanced (and chiral).

The ‘non-balanced’ case (where $C_n = C_{2m} \ntrianglelefteq G$)

Here, theory of regular Cayley maps for abelian groups shows that the vertex-rotation y corresponds to an automorphism of the dihedral group $D_m = \langle\, u, v \mid u^2 = v^m = (uv)^2 = 1\,\rangle$ taking $u \mapsto uv$ and $v \mapsto v^\lambda$ for some unit λ in $\mathbb{Z}_m$.

When $\lambda = 1$, we get a family of anti-balanced (reflexible) examples of type $\{2m, m\}$, with $\mathrm{Aut}^{\mathrm{o}}M = D_m \rtimes C_m$.

More generally, M is reflexible if and only if $\lambda^2 \equiv 1 \bmod m$.

When $\lambda = -1$, we get a family of balanced chiral examples of type $\{2m, 2\}$ and genus 0 (covered by the previous case).

More generally, M is balanced if and only if λ is a jth root of $-1 \bmod m$ for some j, such that j is odd if m is even.

The ‘non-balanced’ case: Main theorem

For any unit λ in $\mathbb{Z}_m$, let b be the largest odd integer dividing m for which λ is a root of -1 modulo b, and let s be the smallest positive integer for which $\lambda^s \equiv -1 \bmod b$.

Then the corresponding balanced Cayley map for D_m is a regular Cayley map for C_{2m} if and only if

(1) m/b is coprime with both b and s, and

(2) $\lambda \equiv 1 \bmod p$ for every prime p dividing m/b.

Note: If $\lambda = 1$ then $b = s = 1$ and these conditions hold for all m; similarly, if $\lambda = -1$ then b is the odd part of m and $s = 1$, and again the conditions hold for all m.

A different example: $m = 9$, $\lambda = 4$ (order 3), $b = s = 1 \Rightarrow$ type $\{18, 9\}$, genus 28, neither balanced nor anti-balanced.

Enumeration: balanced case

For all odd n, the number of regular Cayley maps for C_n is simply the number of elements in the multiplicative group of units modulo n that have -1 as a root.

For $n = p^e$ where p is an odd prime, this is just the number of units mod n of even order, viz.

$$\mathrm{RCM}(p^e) = \phi(p^e) - p^{e-1}O(p-1) = (T(p-1)-1)p^{e-1}O(p-1)$$

where $T(p-1)$ and $O(p-1)$ are respectively the 2-power part and odd part of $p-1$ (so that $\phi(p^e) = p^{e-1}T(p-1)O(p-1)$)

e.g. RCM(9) $= 6 - 3 = 3$ ($1\times$ genus 0, $2\times$ genus 7)

RCM(11) $= 10 - 5 = 5$ ($1\times$ genus 0, $4\times$ genus 12)

RCM(25) $= 20 - 5 = 15$ (1×0, 2×1, 4×46, 8×101).

Enumeration: balanced case (cont.)

More generally, if λ is a root of -1 mod n (for n odd), then the 2-parts of the orders of λ modulo q must be the same for all maximal prime-powers $q = p^e$ dividing n.

For each such q, the number of roots of -1 mod q having order divisible by 2^i but not 2^{i+1} is

$$2^i p^{e-1} O(p-1) - 2^{i-1} p^{e-1} O(p-1) = 2^{i-1} p^{e-1} O(p-1).$$

It follows that if the odd integer n is divisible by s different primes, then the number of regular Cayley maps for C_n is

$$\mathrm{RCM}(n) = O(\phi(n))(t^s - 1)/(2^s - 1)$$

where t is the minimum of the 2-powers $T(p-1)$ over the s different primes p dividing n.

Regular embeddings of given families of graphs

Embeddings of the following families of graphs as orientably regular maps are now known:

- Complete graphs K_n [James & Jones (1985)]
- Cocktail party graphs [Nedela & Škoviera (1996)]
- Merged Johnson graphs [Jones (2005)]
- Some complete multipartite graphs [Du et al (2005)]
- Complete bipartite graphs $K_{n,n}$ [various authors]
- Arc-transitive graphs of specified orders (e.g. pq) [various]
- Hypercube graphs Q_n for all n [various, completed 2008]
- Hamming graphs $H(d, q)$ for all $d > 1$ [Jones (2009)]
- Generalised Paley graphs $P(q)$ for all $q = p^e$ [Jones (2009)]
- Arc-transitive circulants [Conder & Tucker (2009)]

Regular embeddings of Q_n

The automorphism group of the n-dimensional cube Q_n is the wreath product $\mathbb{Z}_2 \wr S_n$ (isomorphic to $(Z_2)^n \rtimes S_n$). Let y be the n-cycle $(1, 2, 3, \ldots, n)$ in S_n, and let e_n be the nth standard basis vector $(0, 0, \ldots, 0, 1)$ of $(Z_2)^n$.

By a theorem of Young Soo Kwon (2004) it is known that embeddings of Q_n as an orientably regular map are in one-to-one correspondence with involutions $\sigma \in S_n$ fixing n such that $e_n\sigma$ and y generate a subgroup of order $2^n n$ in $\mathbb{Z}_2 \wr S_n$.

All such σ were found for n odd by Du, Kwak & Nedela (2007) and for $n = 2m$ with m odd by Jing Xu (2007).

This left open the case of Q_n for $n \equiv 0 \bmod 4$.

Key Lemma

Let H be a permutation group of even degree $2m$ containing a regular element y (acting as a $2m$-cycle), such that the stabilizer of a point is a 2-group. Then y^m is central in H, so the m orbits of $\langle y^m \rangle$ form a system of imprimitivity.

[Proved by Steve Wilson & MC, August 2008 ... but this is likely to have been known to Wielandt (?)]

The above lemma leads to a straightforward reduction from the case of Q_n with $n = 2m$ to the case of Q_m. But making the reduction work backwards is not so easy! Still ...

Theorem [proved in December 2008]

For $n = 2m$ (even), the involution $\sigma \in S_n$ fixing n gives an orientably regular embedding of Q_n (or equivalently, $e_n\sigma$ and $y = (1, 2, \ldots, n)$ generate a subgroup of order $2^n n$ in $\mathbb{Z}_2 \wr S_n$) if and only if

(a) σ commutes with y^m, so that the m orbits of $\langle y^m \rangle$ form a system of imprimitivity for $H = \langle \sigma, y \rangle$ on $\{1, 2, \ldots, n\}$, and

(b) σ is additive mod m, or equivalently, σ induces the same permutation on the blocks $B_i = \{i, i+m\}$ for $1 \le i \le m$ as multiplication on $\{1, 2, ..., m\}$ by some square root of 1 modulo m.

[Results from joint work with Domenico Catalano, Shaofei Du, Young Soo Kwon, Roman Nedela and Steve Wilson. The full proof takes about 14 pages of a 25-page paper.]

Enumeration

For $n = 2m$ (even), every 'good' involution is additive mod m, or equivalently, induces the same permutation on blocks $B_i = \{i, i+m\}$ for $1 \le i \le m$ as multiplication on $\{1, 2, ..., m\}$ by some square root t of 1 modulo m.

This makes them easy to count.

For each integer m, let $R(m) = \{t \in Z_m : t^2 \equiv 1 \bmod m\}$. Then the number of orientably-regular embeddings of Q_n is

$$\begin{cases} |R(n)| & \text{when } n \text{ is odd, or} \\ \displaystyle\sum_{t \in R(m)} 2^{(m+\mathrm{gcd}(t-1,m)-2)/2} & \text{when } n = 2m \text{ (even).} \end{cases}$$

For the embedding to be reflexible, the involution $\sigma \in S_n$ must commute with multiplication by -1 mod n.

This happens for all σ when n is odd, while for $n = 2m$, the number of reflexible orientably-regular embeddings of Q_n is

$$\sum_{t \in R(m)} 2^{(m+\mathrm{gcd}(t-1,m)+\mathrm{gcd}(t+1,m)-\xi)/4}$$

where $\xi = 3$ when m is odd, and $\xi = 2$ when m is even.

Hence for n even, the proportion of orientably-regular embeddings of Q_n that are chiral tends to 1 as $n \to \infty$.

Open question: How prevalent is chirality?

Orientably-regular maps of small genus:
Genus 2: 6 reflexible, 0 chiral
Genus 3: 12 reflexible, 0 chiral
Genus 4: 12 reflexible, 0 chiral
Genus 5: 16 reflexible, 0 chiral
Genus 6: 13 reflexible, 0 chiral
Genus 7: 12 reflexible, 2 chiral pairs
Genus 2 to 101: 3378 reflexible, 594 chiral pairs

What about for larger genera? The proportion of orientably-regular maps of up to a given genus $g > 1$ that are chiral seems to increase as $g \to \infty$, but how quickly? and does it exceed $\frac{1}{2}$ for all but finitely many g?