Looseness of Plane Graphs

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joint work with
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- Let $G = (V, E, F)$ be a connected plane graph with the vertex set V, the edge set E and the face set F.
- A k-colouring of a graph G is a mapping $\varphi : V(G) \to \{1, \ldots, k\}$.
- Let $\varphi(f)$ denote the set of colours used on the vertices incident with the face f.
- A face $f \in F$ is called loose if $|\varphi(f)| \geq 3$.
- A k-colouring of a graph G is called the loose k-colouring if G contains a loose face.

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Question

What is the minimum number of colours $ls(G)$ that any surjective vertex colouring of a connected plane graph G with $ls(G)$ colours enforces a loose face?

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Theorem 1

Let $G = (V, E, F)$ be a connected plane graph such that the dual G^* of G has t vertex disjoint cycles. Then

$$ls(G) \geq t + 2 \quad .$$

Proof:

- $\mathcal{C} = \{C_1, \dots, C_t\}$ disjoint cycles in G^*.
- We write $C_i \preceq C_j$ if C_i is inside C_j.
- We can assume that the cycles in $\mathcal{C}$ are indexed in such a way that $C_i \preceq C_j$ implies $i \leq j$.
- Let D_i be the inner part of C_i.

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- Let D_i be the inner part of C_i.

- Faces of G^* contained in $D_i \setminus \bigcup_{s=1}^{i-1} D_s$ receive the colour i; faces not contained in any D_i receive the colour $t+1$.
- The faces inside C_i (the faces forming D_i) received the colours at most i.
- Let us suppose for a contradiction that the graph G contains a face f which is incident with three vertices v_i, v_j, v_k having three different colours $1 \leq i < j < k \leq t+1$.
- These correspond to three faces v_i^*, v_j^*, v_k^* of G^* coloured i, j, k, all incident with a vertex f^*.

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- Since the face v_i^* has colour i, $v_i^* \subseteq D_i$. On the other hand, the face v_j^* has colour $j > i$, therefore v_j^* is not inside C_i.
- The vertex f^* is incident both with a face inside and outside C_i, hence $f^* \in C_i$.
- Analogously, v_j^* is inside C_j, v_k^* is outside C_j, hence $f^* \in C_j$. The vertex f^* lies on two vertex disjoint cycles, a contradiction.

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Theorem 2

Let $G = (V, E, F)$ be a connected plane graph and let G^* be its dual. Then there are t_0 vertex disjoint cycles in G^* such that

$$ls(G) = t_0 + 2 \quad .$$

Proof:

- Let φ be a nonloose k-colouring of the graph G, such that $k = ls(G) - 1$.
- E_{ij} ...the set of two coloured edges of G, $i \neq j$, E^*_{ij} corresponding edges of G^*.
- $G^*[E^*_{ij}]$ has minimum degree at least two, hence it contains at least one cycle, say C_{ij}.

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- $G^*[E^*_{ij}] \cap G^*[E^*_{lk}] = \emptyset$ for $\{i, j\} \neq \{k, l\}$.
- C_{ij} and C_{kl} are vertex disjoint for $\{i, j\} \neq \{k, l\}$.
- Let t_0 denote the number of such cycles. We know that $ls(G) \geq t_0 + 2$.
- φ uses k colours, therefore the graph G has at least $k - 1$ types of heterochromatic edges. Hence, $t_0 \geq k - 1 = ls(G) - 2$.

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The looseness of a connected plane graph G equals 2 plus the maximum number of vertex disjoint cycles in the dual graph G^*.

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The looseness of a connected plane graph G equals 2 plus the maximum number of vertex disjoint cycles in the dual graph G^*.

The girth of a graph G is the length of its shortest cycle.

Theorem

Let $G = (V, E, F)$ be a connected plane graph, let g be the girth of the dual graph G^* of G. Then

$$ls(G) \leq \frac{1}{g}|F(G)| + 2 \ .$$

Moreover, the bound is sharp.

The dual graph G^* of a graph G contains t vertex disjoint cycles such that $ls(G) = t + 2$. Clearly, each cycle contains at least g vertices. Hence, we get $t \leq \frac{|F(G)|}{g}$.

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Each minimum edge-cut of size g in G corresponds to a cycle in G^* and vice versa, therefore, the edge connectivity of a graph G is equal to the girth of the dual graph G^*.

Theorem

Let $G = (V, E, F)$ be a connected plane graph with the edge connectivity κ'. Then

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Observation

Let G be a plane graph on n vertices which contains a face incident with at least three vertices. Then $ls(G) \leq n$.

Theorem

Let G be a connected simple plane graph on n vertices. Then

$$ls(G) \leq \frac{2n+2}{3}.$$

Theorem

For any integer $t \geq 1$ and any $k \in \{1, 2, 3\}$ there exists a simple k-connected plane graph G on n vertices, $n \geq t$, such that

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THANK YOU !

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