Regular Embeddings of Graphs of Order pq

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A map is called *regular* if its automorphism group acts regularly on the set of all flags (incident vertex-edge-face triples). An orientable map is called *orientably-regular* if the group of all orientation-preserving automorphisms is regular on the set of all arcs (incident vertex-edge pairs). If an orientably-regular map admits also orientation-reversing automorphisms, then it is regular, and called *reflexible*. A *regular embedding* and *orientably-regular embedding* of a graph $\mathcal{G}$ are respectively a 2-cell embeddings of $\mathcal{G}$ as a regular map and orientably-regular map on some closed surface.

In this talk, we shall give a classification of orientably-regular and regular embeddings of graphs of order pq, where p and q are primes.

1. Surfaces and Embeddings

2-manifold M: a topological space M which is Hausdorf and is covered by countably many open sets isomorphic to either 2-dim open ball or 2-dim half-ball;

Closed 2-manifold M: compact, boundary is empty;

Surface S: closed, connected 2-manifold;

Classification of Surfaces:

(i) Orientable Surfaces: S_g, $g = 0, 1, 2, \cdots$,
$v + f - e = 2 - 2g$

(ii) Nonorientable Surfaces: N_k, $k = 0, 1, 2, \cdots$,
$v + f - e = 2 - k$

Embeddings of a graph X in the surface is a continuous one-to-one function $i : X \to S$.

2-cell Embeddings: each region is homemorphic to an open disk.

The primitive objective of topological graph theory is to draw a graph on a surface so that no two edges cross.

Topological Map $\mathcal{M}$: a 2-cell embedding of a graph into a surface. The embedded graph X is called the *underlying graph* of the map.

Automorphism of a map $\mathcal{M}$: an automorphism of the underlying graph X which can be extended to self-homeomorphism of the surface.

Automorphism group $\mathrm{Aut}(\mathcal{M})$: all the automorphisms of the map $\mathcal{M}$.

Remark: $\mathrm{Aut}(\mathcal{M})$ acts semi-regularly on the flags of X.

Regular Map: $\mathrm{Aut}(\mathcal{M})$ acts regularly on the flags (incident vertex-edge-face triples).

Orientably-Regular Map: map is orientable and the group of all orientation-preserving automorphisms is regular on the arcs (incident vertex-edge pairs).

Reflexible Map: Orientably-regular and regular.

Chiral Map: Orientably-Regular Map but not regular.

Three main research directions:

1. Classifying regular maps by groups;
2. Classifying regular maps by underlying graphs
3. Classifying regular maps by genus

2. Combinatorial and Algebraic Map

Combinatorial Orientably-Regular Map:

connected simple graph $\mathcal{G} = \mathcal{G}(V, D)$, with vertex set $V = V(\mathcal{G})$, dart (arc) set $D = D(\mathcal{G})$.

arc-reversing involution L: interchanging the two arcs underlying every given edge.

rotation R*:* cyclically permutes the arcs initiated at v for each vertex $v \in V(\mathcal{G})$.

Map $\mathcal{M}$ *with underlying graph* $\mathcal{G}$*:*
the triple $\mathcal{M} = \mathcal{M}(\mathcal{G};\ R,\ L)$.

Remarks: Monodromy group $\mathrm{Mon}(\mathcal{M}) := \langle R, L\rangle$ acts transitively on D.

Given two maps

$$\mathcal{M}_1 = \mathcal{M}(\mathcal{G}_1;\ R_1,\ L_1),\ \ \mathcal{M}_2 = \mathcal{M}(\mathcal{G}_2;\ R_2\, L_2),$$

Map (orientation persevering) isomorphism: bijection $\phi : D(\mathcal{G}_1) \to D(\mathcal{G}_2)$ such that

$$L_1\phi = \phi L_2, R_1\phi = \phi R_2$$

Automorphism ϕ of $\mathcal{M}$: if $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$;

Automorphism group: $\mathrm{Aut}\,(\mathcal{M})$

Remarks: $\mathrm{Aut}\,(\mathcal{M}) = C_{S_D}(\mathrm{Mon}(\mathcal{M}))$;
$\mathrm{Aut}\,(\mathcal{M})$ acts semi-regularly on D,

Orientably-Regular Map: $\mathrm{Aut}\,(\mathcal{M})$ acts regularly on D.

For an orientably-regular map, we have

(i) $\mathrm{Aut}\,(\mathcal{M}) \cong \mathrm{Mon}(\mathcal{M})$;

(ii) $\mathrm{Aut}\,(\mathcal{M})$ and $\mathrm{Mon}(\mathcal{M})$ on D can be viewed as the right and the left regular representations of an abstract group $G = \mathrm{Aut}\,(\mathcal{M}) \cong \mathrm{Mon}(\mathcal{M})$

Algebraic Orientably-Regular Map:

Given an orientably-regular map $\mathcal{M}$ with $\mathrm{Aut}\,(\mathcal{M}) \cong G$, we may represent it by an *algebraic map* $\mathcal{M}(G; r, l)$, where $|l| = 2$.

(i) $\mathcal{M}(G; r_1, \ell_1) \cong \mathcal{M}(G;\ r_2, \ell_2)$ if and only if there exists an element $\sigma \in \mathrm{Aut}\,(G)$ such that $r_1^\sigma = r_2$ and $\ell_1^\sigma = \ell_2$.

(ii) $\mathcal{M}(G; r, \ell)$ is reflexible if and only if there exists an element $\sigma \in \mathrm{Aut}\,(G)$ such that $r^\sigma = r^{-1}$ and $\ell^\sigma = \ell$.

Combinatorial Regular Map:

For a given finite set F and three fixed-point free involutory permutations t, r, ℓ on F, a quadruple $\mathcal{M} = \mathcal{M}(F; t, r, \ell)$ is called a combinatorial map if they satisfy two conditions: (1) $t\ell = \ell t$; (2) the group $\langle t, r, \ell \rangle$ acts transitively on F.

F: *flag* set;

t, r, ℓ are called *longitudinal, rotary, and transverse involution,* respectively.

$\mathrm{Mon}(\mathcal{M}) = \langle t, r, \ell \rangle$: *Monodromy group* of $\mathcal{M}$,

vertices, edges and *face-boundaries* of $\mathcal{M}$ to be orbits of the subgroups $\langle t, r \rangle$, $\langle t, \ell \rangle$ and $\langle r, \ell \rangle$, respectively.

The incidence in $\mathcal{M}$ can be represented by nontrivial intersection.

The map $\mathcal{M}$ is called *unoriented*.

the even-word subgroup $\langle tr, r\ell \rangle$ of $\mathrm{Mon}(\mathcal{M})$ has the index at most 2.

orientable: if the index is 2,

nonorientable: if the index is 1

Given two maps $\mathcal{M}_1 = \mathcal{M}(F_1; t_1, r_1, \ell_1)$ and $\mathcal{M}_2 = \mathcal{M}_2(F_2; t_2, r_2, \ell_2)$,

Map isomorphism: bijection $\phi : F_1 \rightarrow F_2$ such that

$$\phi t_1 = t_2 \phi, \quad \phi r_1 = r_2 \phi, \quad \phi \ell_1 = \ell_2 \phi.$$

Automorphism of $\mathcal{M}$: if $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$;

Automorphism group: $\mathrm{Aut}\,(\mathcal{M})$

Remarks: $\mathrm{Aut}\,(\mathcal{M}) = C_{S_F}(\mathrm{Mon}(\mathcal{M}))$;
$\mathrm{Aut}\,(\mathcal{M})$ acts semi-regularly on F,

Regular Map: $\mathrm{Aut}\,(\mathcal{M})$ acts regularly on F.

Remarks: For regular map, we have
(i) $\mathrm{Aut}\,(\mathcal{M}) \cong \mathrm{Mon}(\mathcal{M})$;
(ii) $\mathrm{Aut}\,(\mathcal{M})$ and $\mathrm{Mon}(\mathcal{M})$ on F can be viewed as the right and the left regular representations of an abstract group G.

Algebraic Regular Map:

Given a regular map $\mathcal{M}$ with $\mathrm{Aut}\,(\mathcal{M}) \cong G$, we may represent it by an *algebraic regular map* $\mathcal{M}(G; t, r, \ell)$.

$\mathcal{M}(G; t_1, r_1, \ell_1) \cong \mathcal{M}(G;\, t_2, r_2, \ell_2)$ if and only if there exists an element $\sigma \in \mathrm{Aut}\,(G)$ such that $t_1^\sigma = t_2$, $r_1^\sigma = r_2$ and $\ell_1^\sigma = \ell_2$.

‘Regular Maps‘=‘Nonorientable Regular Maps‘ + ‘Reflexible Maps‘

‘Orientably-Regular Maps‘=‘Chiral Maps‘ + ‘Reflexible Maps‘

3. Classify regular embeddings of the graphs of order pq

1. S.F. Du, J.H. Kwak and R. Nedela, A Classification of regular embeddings of graphs of order a product of two primes, *J. Algeb. Combin.* **19**(2004), 123–141.

2. S.F. Du and Jinho Kwak, Nonorientable regular embeddings of graphs of order p^2, accepted by DM, 2009.

3. S.F. Du and F.R.Wang, Nonorientable regular embeddings of graphs of order a product of two distinct primes, in preparation.

Case 1: Orientably-Regular Maps:

First we define some groups and maps:

(**I**) Let $p \geq 7$, h any odd divisor of $p-1$ with $h \geq 3$, and let t be any fixed element of order $2h$ in $\mathbf{B}Z_p^*$. Define a group

$$G_1 = G_1(p,h) = \langle x, y \mid x^p = y^{2h} = 1,\ x^y = x^t \rangle.$$

$$\mathcal{M}_1 = \mathcal{M}_1(p,h,i) = \mathcal{M}(G_1; y^{2i}, xy^h).$$

where $i \in \mathbf{B}Z_h^*$.

(**II**) Let $p \geq 3$, h any even divisor of $p^2 - p$ with $h \geq 2$, and let t be any fixed element of order h in $\mathbf{B}Z_{p^2}^*$. Define a group

$$G_2 = G_2(p,h) = \langle x, y \mid x^{p^2} = y^h = 1,\ x^y = x^t \rangle.$$

$$\mathcal{M}_2 = \mathcal{M}_2(p,h,i) = \mathcal{M}(G_2; y^i, xy^{\frac{h}{2}}),$$

where $i \in \mathbf{B}Z_h^*$.

(**III**) Let $p \geq q \geq 2$, $pq > 4$ and $(t_1,t_2) \in \mathbf{B}Z_p^* \times \mathbf{B}Z_q^*$ such that $t_1 \neq t_2$ if $p = q$, and $\langle (t_1,t_2) \rangle$ contains $(-1,1)$ if $q = 2$, and contains $(-1,-1)$ if $q \geq 3$. Let $h = [|t_1|, |t_2|]$, where $h \geq 2$ is even. Define a group

$$G_3 = G_3(p,q,t_1,t_2) = \langle a, b, x \mid a^p = b^q = x^h$$

$$= [a,b] = 1, a^x = a^{t_1},\ b^x = b^{t_2}\rangle.$$

$$\mathcal{M}_3 = \mathcal{M}_3(p,q,t_1,t_2,i) = \mathcal{M}(G_3; x^i, abx^{\frac{h}{2}}),$$

where $i \in \mathbf{B}Z_h^*/(\mathbf{B}Z_h^*)^{2+}$ if $p = q$ and $h = |t_1| = |t_2|$, and $i \in \mathbf{B}Z_h^*$ otherwise.

(**IV**) Let $\mathbf{B}F_p^* = \langle\theta\rangle$ and let x be an element of order h in $\mathrm{GL}(2,p)$, where $h \geq 3$, defined as follows:

(1) If $p = 2$ and $h = 3$, then $x = ||1,1;1,0||$; or

(2) if $p \geq 3$, $h \mid (p^2-1)$ but $h \nmid (p-1)$, then $x = ||e, f\theta; f, e||$ for some fixed pair (e,f) such that $|x| = h$.

Let $T = \langle a,b\rangle$ be the translation subgroup of $\mathrm{AGL}(2,p)$ as before. Define a group

$$G_4 = G_4(p,h) = T{:}\langle x\rangle \leq \mathrm{AGL}(2,p).$$

$$\mathcal{M}_4 = \mathcal{M}_4(p,h,i) = \mathcal{M}(G_4, x^i, az),$$

where either $z = 1$ for $p = 2$ or $z = x^{\frac{h}{2}}$ for $p \geq 3$; and $i \in \mathbf{B}Z_h^*/(\mathbf{B}Z_h^*)^{2+}$.

(**V**) Define a group

$$G_5 = G_5(p) = \langle a,b,x \mid a^p = b^p = x^2 = [a,b] = 1, a^x = b.\rangle.$$

$$\mathcal{M}_5(p) = \mathcal{M}(G_5; a, x).$$

(**VI**) Let $\mathbf{B}F_p^* = \langle\theta\rangle$ and let $H = \langle x, y\rangle$ be a subgroup in $\mathrm{GL}(2,p)$ isomorphic to a Frobenius group $\mathbf{B}Z_q{:}\mathbf{B}Z_h$ with $h \geq 2$, and two elements x and y are defined as follows:

(1) If $p = 2$, $q = 3$ and $h = 2$, then $x = ||1,1;1,0||$ and $y = ||0,1;1,0||$;

(2) if $p > q \geq 3$, $q \mid (p-1)$ and $h = 2$, then $x = ||t,0;0,t^{-1}||$ where $t = \theta^{\frac{p-1}{q}}$ and $y = ||0,1;1,0||$;

(3) if $p > q \geq 3$, $q \mid (p+1)$ and $h = 2$, then $x = ||e, f\theta; f, e||$ for some fixed pair (e,f) such that $|x| = h$, and $y = ||1,0;-1,0||$; or

(4) if $p = q \geq 3$ and h is an even divisor of $p-1$, then $x = ||1,1;0,1||$ and $y = ||1,0;0,t||$, where $t = \theta^{\frac{p-1}{h}}$.

Define a group

$$G_6 = G_6(p,q,h) = T{:}H \leq \mathrm{AGL}(2,p).$$

$$\mathcal{M}_6(p,q,h,i,j) := \mathcal{M}(G_6; a'y^j, x^i y^{\frac{h}{2}}),$$

$i \in \mathbf{B}Z_q^*/(\mathbf{B}Z_q^*)^{h+}$ and $j \in \mathbf{B}Z_h^*$; and

$$a' = \begin{cases} t_{(1,0)} & \text{if } \ p = 2, \ q \mid (p+1) \ \text{ or } \ p = q, \\ t_{(1,1)} & \text{if } \ q \mid (p-1). \end{cases}$$

Theorem 0.1 *Let $\mathcal{M}$ be an orientably-regular map with a underlying graph $\mathcal{G}$ of order pq for any two primes p and q with $p \geq q$. Then $\mathcal{M}$ is isomorphic to one of the following regular maps uniquely determined by the given integer parameters:*

(1) $p = q = 2$:

$\mathcal{M} \cong \mathcal{M}_5(2)$

$\mathcal{M} \cong \mathcal{M}_4(2, 3, 1)$

(2) $p = q \geq 3$:

$\mathcal{M} \cong \mathcal{M}_2(p, h, i)$

$\mathcal{M} \cong \mathcal{M}_3(p, p, t_1, t_2, i)$

$\mathcal{M} \cong \mathcal{M}_4(p, h, i)$

$\mathcal{M} \cong \mathcal{M}_6(p, p, h, i, j)$

(3) $p \geq 3$ *and* $q = 2$:

$\mathcal{M} \cong \mathcal{M}_6(2, 3, 2, 1, 1)$ *where* $p = 3$

$\mathcal{M} \cong \mathcal{M}_1(p, h, i)$

$\mathcal{M} \cong \mathcal{M}_3(p, 2, t_1, 1, i)$

$\mathcal{M} \cong \mathcal{M}_5(p)$, *where* $p \geq 3$

(4) $p > q \geq 3$:

$\mathcal{M} \cong \mathcal{M}_3(p, q, t_1, t_2, i)$

$\mathcal{M}_6(p, q, 2, i, 1)$.

Case 2:Nonorientable Regular Maps of order p^2:

(1) $G_1 = T : \langle x, y\rangle$, where $x = \begin{pmatrix} \theta^{-\frac{p-1}{n}} & 0 \\ 0 & \theta^{\frac{p-1}{n}} \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $F_p^* = \langle\theta\rangle$, $p \geq 5$, $n \geq 4$ is even and $n \bigm| (p-1)$.
$\mathcal{M}(G_1; y, x^i y, t_{(1,1)} x^{\frac{n}{2}})$, where $i \in Z_n^* \cap \{1, \ldots, \frac{n}{2}\}$;

(2) $G_2 = T : \langle x, y\rangle$, where $x = \begin{pmatrix} e & f\theta \\ f & e \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, where $F_p^* = \langle\theta\rangle$, $F_{p^2} = F_p(\varepsilon)$, $\varepsilon^2 = \theta$ and $e + f\varepsilon$ is a given generator of the subgroup of order n of $F_{p^2}^*$, $p \geq 3$, $n \geq 4$ is even and $n \bigm| (p+1)$.
$\mathcal{M}(G_2; y, x^i y, t_{(1,0)} x^{\frac{n}{2}})$, where $i \in Z_n^* \cap \{1, \ldots, \frac{n}{2}\}$;

(3) $G_3 = T : \langle x, y\rangle$, where $x = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, with $p \geq 3$.
$\mathcal{M}(G_3; y, xy, t_{(1,0)} x^p)$, where $p \geq 3$

(4) $\mathcal{M}(S_4, (12), (13), (12)(34))$.

Theorem 0.2 *Let $\mathcal{M}$ be a nonorientable regular map with the underlying graph X of order* p^2, *for any prime p. Then $\mathcal{M}$ is isomorphic to one of the following maps*
$\mathcal{M}(G_1; y, x^i y, t_{(1,1)} x^{\frac{n}{2}})$
$\mathcal{M}(G_2; y, x^i y, t_{(1,0)} x^{\frac{n}{2}})$
$\mathcal{M}(G_3; y, xy, t_{(1,0)} x^p)$
$\mathcal{M}(S_4, (12), (13), (12)(34))$.

Case 3: Nonorienatble Case for $p \neq q$

Theorem 0.3 *Let $\mathcal{M}$ be a nonorientable regular embedding of the graph X of order pq, where $p > q$ are two primes. Then $\mathcal{M}$ is isomorphic to one of the following algebraic maps $\mathcal{M}(G; t, r, l)$*

(1) $G \cong A_5$, $t = (12)(34), r = (15)(24)$ *and* $l = (13)(24)$ *or* $(14)(23)$, *where* $X \cong K_6$.

(2) $G \cong A_5$, $t = (13)(45), r = (12)(45)$ *and* $l = (14)(35)$, *where X is Peterson graph.*

(3) $G \cong S_5$, $(t, r, l) = ((23), (12)(45), (14)(23))$; $((23), (12)(45), (14))$ $((12)(45), (23),\ (14)(25))$, *where X is the complement of Peterson graph.*

(4) $G \cong S_5$, $t = (24)$, $r = (12)(34)$ *and* $l = (15)$ *or* $(15)(24)$, *where* $pq = 15$.

(5) *Set* $G = \langle \alpha, \beta \rangle : \langle x, y \rangle \leq \mathrm{AGL}(2, p)$, *where* $\alpha = t_{(1,0)}, \beta = t_{(0,1)}$ *and* $\langle x, y \rangle \cong D_{2q}$ *and x and y are defined as follows:*

(i) $2q \bigm| (p-1)$: $x = \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}$ *and* $y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *where δ is order of $2q$ in F_p^*. Set* $t = -e_2$, $r = t_{(-\delta^i,1)} x^i y$ *and* $l = y$, *where* $i \in Z_{2q}^* \cap \in \{1, \cdots, q\}$.

(ii) $2q \bigm| (p+1)$: $x = \begin{pmatrix} e & f\theta \\ f & e \end{pmatrix}$ *and* $y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, *where* $F_p^* = \langle \theta \rangle$, $F_{p^2} = F_p(\varepsilon)$ *for* $\varepsilon^2 = \theta$, $e + f\varepsilon$ *is* *a given element of order* $2q$ *in* $F^*_{p^2}$. *Set* $t = -e_2$, $r = t_{(-f_i(e_i+1)^{-1},1)}x^i y$ *and* $l = y$, *where* $i \in Z^*_{2q} \cap \{1, \cdots, q\}$ *and* $x^i = \begin{pmatrix} e_i & f_i\theta \\ f_i & e_i \end{pmatrix}$.

In both cases, $X = C_q[\overline{K_p}]$, $p > q \geq 3$.

(6) $G = \mathrm{PGL}(2,p)$, $p \geq 7$ *and* $q = \frac{p+1}{2}$ *or* $\frac{p-1}{2}$ *is a prime.*

(i) $\frac{p+1}{2}$ *is prime:* $t = \overline{\begin{pmatrix} 0 & 1 \\ -\theta^k & 0 \end{pmatrix}}$, $r = \overline{\begin{pmatrix} 0 & 1 \\ -\theta^{i+k} & 0 \end{pmatrix}}$ *and* $l = \overline{\begin{pmatrix} 1 & \beta \\ \beta\theta^k & -1 \end{pmatrix}}$, *where* $k \in \{0, 1\}$, $i, \beta \in \{1, 2, \cdots, \frac{p-1}{2}\}$, $i \in Z^*_{p-1}$, *and if* $k = 0$ *then* $\beta^2 \neq -1$.

(ii) $\frac{p-1}{2}$ *is prime:* $t = \overline{\begin{pmatrix} 1 & \nu \\ \nu & -1 \end{pmatrix}}$, $r = t\ \overline{\begin{pmatrix} 1 & -\lambda \\ \lambda & 1 \end{pmatrix}^i}$ *and* $l = \overline{\begin{pmatrix} \nu & \beta-1 \\ -\beta-1 & -\nu \end{pmatrix}}$, *where* $F^*_{p^2} = \langle 1 + \lambda\varepsilon \rangle$ *for* $\varepsilon^2 = -1$, $\nu \in \{0, \lambda\}$, $i \in Z^*_{p+1} \cap \{1, 2, \cdots, \frac{p+1}{2}\}$, $\beta \in \{0, 1, 2, \cdots, \frac{p-1}{2}\}$ *and if* $\nu = 0$ *then* $\beta \neq 0, 1$.

(7) $G = \mathrm{PSL}(2,p)$, *where* $p \geq 11$ *and* $q = \frac{p+1}{2}$ *or* $\frac{p-1}{2}$ *is a prime.*

(i) $\frac{p+1}{2}$ *is prime:* $t = \overline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$, $r = \overline{\begin{pmatrix} 0 & \theta^{-i} \\ -\theta^i & 0 \end{pmatrix}}$ *and* $l = \overline{\begin{pmatrix} 1 & \beta \\ \beta & -1 \end{pmatrix}}$, *where* $i \in Z^*_{\frac{p-1}{2}} \cap \{1, 2, \cdots, \frac{p-1}{4}\}$, $\beta \in \{1, 2, \cdots, \frac{p-1}{2}\}$ *and* $1 + \beta^2 \in \langle \theta^2 \rangle$.

(ii) $\frac{p-1}{2}$ *is prime:* $t = \overline{\begin{pmatrix} 1 & \lambda \\ \lambda & -1 \end{pmatrix}}$, $r = t\ \overline{\begin{pmatrix} 1 & -\lambda \\ \lambda & 1 \end{pmatrix}^{2i}}$ *and*

$l = \overline{\begin{pmatrix} \lambda & \beta - 1 \\ -\beta - 1 & -\lambda \end{pmatrix}}$, *where* $F_{p^2}^* = \langle 1+\lambda\varepsilon \rangle$ *for* $\varepsilon^2 = -1$, $i \in Z_{\frac{p+1}{2}}^* \cap \{1, 2, \cdots, \frac{p+1}{4}\}$, $\beta \in \{0, 1, 2, \cdots, \frac{p-1}{2}\}$ *and* $\beta^2 - \lambda^2 - 1 \in \langle \theta^2 \rangle$.

In $(5) - (7)$, *the maps are uniquely determined by the given parameters.*