Constructions of bipartite and bipartite-regular hypermaps

Rui Duarte
rduarte@ua.pt

University of Aveiro, Portugal

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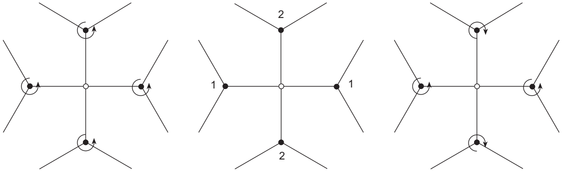
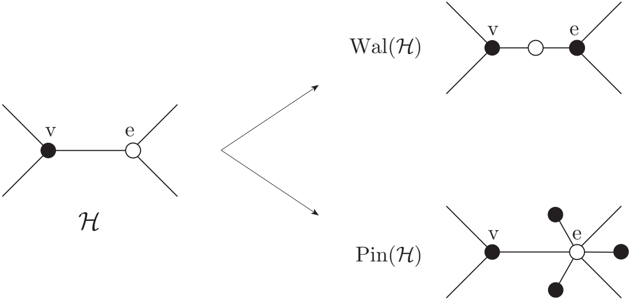


Orientable / bipartite / pseudo-orientable
Orientably-regular / bipartite-regular / pseudo-orientably-regular

Pseudo-orientable / pseudo-orientably-regular (Steve Wilson)



Walsh, 1975
Breda and Duarte, 2007

$$\varphi(\mathcal{H}) \left\{ \begin{array}{ccc} \Delta & & \\ \| 2 & & \\ \Delta^{\hat{0}} & \xrightarrow{\varphi} & \Delta \\ | & & | \\ H\varphi^{-1} & \longrightarrow & H \end{array} \right. \left. \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right\} \mathcal{H}$$

$\varphi(\mathcal{H})$ is denoted by $\mathcal{H}^{\varphi^{-1}}$ in Breda and Duarte, 2007

Lemma: Let $\varphi : \Delta^{\hat{0}} \to \Delta$ be an epimorphism, $\mathcal{B}$ a bipartite hypermap and B a hypermap subgroup for $\mathcal{B}$. Then

$$\mathcal{B} \cong \varphi(\cdot) \Leftrightarrow \ker\varphi \subseteq B^g, \text{ for some } g \in \Delta.$$

(Since $\ker\varphi \lhd \Delta^{\hat{0}}$, $\mathcal{B} \cong \varphi(\cdot) \Leftrightarrow \ker\varphi \subseteq B$ or $(\ker\varphi)^{R_0} \subseteq B$.)

Corollary: Let $\varphi : \Delta^{\hat{0}} \to \Delta$ be an epimorphism and $\mathcal{A}$ and $\mathcal{B}$ hypermaps such that $\mathcal{B}$ is bipartite.

1. $\mathcal{A} \to \mathcal{B}$ & $\mathcal{A} \cong \varphi(\cdot) \Rightarrow \mathcal{B} \cong \varphi(\cdot)$
2. $\mathcal{B}^{+} \cong \varphi(\cdot) \Rightarrow \mathcal{B} \cong \varphi(\cdot)$
3. $\mathcal{B}_{\Delta} \cong \varphi(\cdot) \Rightarrow \mathcal{B} \cong \varphi(\cdot)$
4. $\mathcal{B} \cong \varphi(\cdot) \Rightarrow \mathcal{B}^{\Delta} \cong \varphi(\cdot)$

Lemma: Let $\varphi : \Delta^{\hat{0}} \to \Delta$ be an epimorphism and $\mathcal{H}$ a hypermap.

1. $\varphi(\mathcal{H})$ has no boundary $\Rightarrow \mathcal{H}$ has no boundary.
2. $\varphi(\mathcal{H})$ is orientable $\Rightarrow \mathcal{H}$ is orientable.

Theorem: Let $\varphi : \Delta^{\hat{0}} \to \Delta$ be an epimorphism and $\mathcal{H}$, $\mathcal{N}$ and $\mathcal{O}$ be hypermaps such that $\mathcal{N}$ has no boundary and $\mathcal{O}$ is orientable. The following conditions are equivalent:

1. $R_1, R_2, {R_1}^{R_0}, {R_2}^{R_0} \notin \Delta^+\varphi^{-1}$.
2. $\Delta^+\varphi^{-1} = \Delta^+ \cap \Delta^{\hat{0}}$, that is, $\varphi(\mathcal{T}_{\Delta^+}) \cong \mathcal{T}_{\Delta^{+0\hat{0}}}$.
3. $\ker \varphi \subseteq \Delta^+$.
4. $\varphi(\mathcal{N})$ has no boundary.
5. $\varphi(\mathcal{O})$ is orientable.
6. $\varphi(\mathcal{H}^+) \cong \varphi(\mathcal{H})^+$.
7. $\varphi(\mathcal{H})^+ \cong \varphi(\cdot)$.

Corollary: Let $\varphi : \Delta^{\hat{0}} \to \Delta$ be an epimorphism such that $R_1, R_2, {R_1}^{R_0}, {R_2}^{R_0} \notin \Delta^+\varphi^{-1}$, $\mathcal{H}$ a hypermap and $\mathcal{B}$ a bipartite hypermap. Then:

1. $\varphi(\mathcal{H})$ is orientable $\Leftrightarrow$ $\mathcal{H}$ is orientable.
2. $\varphi(\mathcal{H})$ has no boundary $\Leftrightarrow$ $\mathcal{H}$ has no boundary.
3. $\varphi(\mathcal{H}^+) \cong \varphi(\mathcal{H})^+$.
4. $\mathcal{B} \cong \varphi(\cdot) \Leftrightarrow \mathcal{B}^+ \cong \varphi(\cdot)$.

Corollary: $\mathcal{B}$ non-orientable & $\mathcal{B} \cong \varphi(\mathcal{H}) \Rightarrow \mathcal{B}^+ \cong \varphi(\mathcal{H}^+)$.

g	$g\varphi_1$	$g\varphi_2$	$g\varphi_3$	$g\varphi_4$	$g\varphi_5$
R_1	R_1	R_1	R_1	R_1	R_1
R_2	R_2	R_2	R_2	R_2	R_2
${R_1}^{R_0}$	R_0	R_0	R_2	R_0	${R_1}^{R_0}$
${R_2}^{R_0}$	R_2	R_0	R_0	R_1	R_0

$\varphi_1(\mathcal{H})$ and $\varphi_2(\mathcal{H})$ are denoted by Wal$(\mathcal{H})$ and Pin$(\mathcal{H})$ in Breda and Duarte, 2007

Surfaces with bipartite-regular hypermaps

Theorem: All surfaces have bipartite-regular hypermaps.

Orientable case: Breda and Duarte, 2007
Non-orientable case:
Let $\mathcal{PP}_{2k}$ the regular hypermap on the projective plane of type $(2, 2, 2k)$. Then:

1. $\varphi_4(\mathcal{PP}_{2k})$ is non-orientable and has genus $2k$

$$\chi(\varphi_4(\mathcal{PP}_{2k})) = 2 - 2k$$

2. $\varphi_5(\mathcal{PP}_{2k})$ is non-orientable and has genus $2k - 1$

$$\chi(\varphi_5(\mathcal{PP}_{2k})) = 2 - (2k - 1)$$

- Antonio Breda d'Azevedo and Rui Duarte, *Bipartite-uniform hypermaps on the sphere*, Electron. J. Combin. **14** (2007), 1–20.
- Rui Duarte, *2-restrictedly-regular hypermaps of small genus*, Ph.D. thesis, University of Aveiro, 2007.
- T.R.S. Walsh, *Hypermaps versus bipartite maps*, J. Comb. Theory, Ser. B **18** (1975), 155–163.