Goal-minimally k-diametric graphs from finite fields

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1. k-GMD graphs
   - Definition and examples
   - Basic properties and known results
2. A new construction
   - Motivation
   - Basic Idea
   - Results
3. Open problems
   - Question
   - The present situation

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Definition

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A graph G is said to be **k-GMD**, if the diameter of G is equal to k, and for every edge $uv \in E(G)$

$$d_{G-uv}(x, y) > k \iff \{u, v\} = \{x, y\}.$$

k-GMD graphs were introduced by Kyš (1980) and studied by Gliviak and Plesník.

Examples

Example

The complete graph K_n is 1-GMD for any $n \geq 3$.

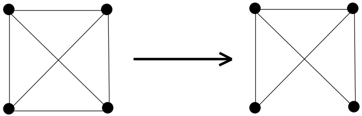

Figure: The graph K_4 before and after removing an edge.

Examples

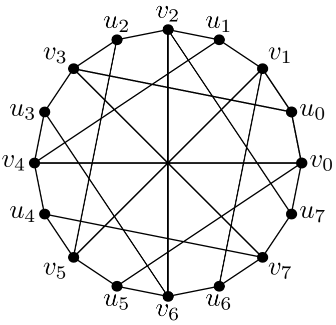

Figure: A 3-GMD graph on 16 vertices.

Basic properties

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3. *for any two non-adjacent vertices u and v there are at least two internally disjoint $u-v$ paths of length not exceeding k.*

Conversely, if 1 and 3 hold for a graph G with diameter k, then G is k-GMD.

Kyš's conjecture

Conjecture [Kyš, 1980]

For every positive integer k exists a k-GMD graph.

Known results

Table: The situation before 2009.

Author (Year)	Values of k: Sporadic examples	Values of k: Infinite families
Kyš (1980)	1,2,3,4,6	1,2,4
Plesník (2006)	5,7,8,10,12,14	6
Š. Gy. (2008)	9,13	5

Known results

Theorem (Š. Gy.)

The lift of graph G_n in cyclic group $\mathbb{Z}_{2n}$ gives a 5-GMD graph for every $n \geq 3$.

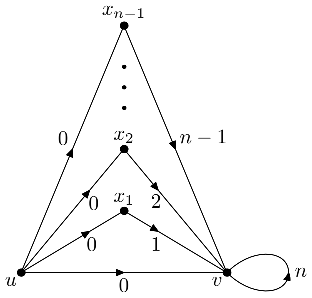

Figure: The voltage graph G_n with voltages in cyclic group $\mathbb{Z}_{2n}$.

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A graph is called a **symmetric cubic graph** if it is cubic, and its group of automorphisms acts transitively on the set of its vertices, and on the set of edges too.

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Many *k*-GMD graphs one can found in the following families of graphs:

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Definition

A graph is called a **symmetric cubic graph** if it is cubic, and its group of automorphisms acts transitively on the set of its vertices, and on the set of edges too.

Those graphs are collected on the website of M. Conder on up to 2048 veritces. Among them we have found 38 *k*-GMD graphs for various values of *k*.

Cages

Definition

A δ-regular graph ($\delta \geq 2$) with girth $g \geq 3$ is called a (δ, g)**-graph**. A (δ, g)**-cage** is a (δ, g)-graph with the least possible number of vertices. Actually the smallest known (δ, g)-graph is called as **near-cage**, or **best-current cage**.

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Conjecture [Plesník, 2007]

If a (δ, g)-cage has diameter $g - 2$, then it is a $(g - 2)$-GMD graph.

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- Conder has constructed many best-current cages as cubic Cayley graphs.

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- Conder has constructed many best-current cages as cubic Cayley graphs.
- Among them we have found a 16-GMD and a 20-GMD graph.

Finite fields

Let GF(q) be the finite field on q elements, where q is a power of a prime. Let ω be a primitive element of GF(q), so

$$\mathrm{GF}^*(q) = \langle \omega \rangle.$$

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$$\mathrm{GF}^*(q) = \langle \omega \rangle.$$

Let $T = \mathrm{GF}(q) \cup \{\infty\}$ and $g \in \mathrm{GF}(q) \setminus \{0, 1\}$. Define $\varphi_g : T \to T$ by

$$\varphi_g(x) = \frac{x-1}{gx-1}$$

$\varphi_g(\infty) = g^{-1}$ and $\varphi_g(g^{-1}) = \infty$.

Finite fields

For every integer $1 \leq n \leq q-2$ define $\Phi_{g,n} : T \to T$ as

$$\Phi_{g,n}(x) = \omega^n \cdot \varphi_g\left(\frac{x}{\omega^n}\right).$$

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- φ_g and $\Phi_{g,n}$ are permutations of T.
- they are mutually conjugated involutions, i.e. $\varphi_g = \varphi_g^{-1}$ and $\Phi_{g,n} = \Phi_{g,n}^{-1}$.

Our construction

Conder's Idea

Conder's generating set was $X = \{\varphi_g, \Phi_{g,m}, \Phi_{g,2m}\}$ in $\langle X \rangle$ with mappings over the field $GF(q)$ for $q = 2^{2p}$ and $m = (q-1)/3$.

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Results

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- They have diameter $k = 12, 16, 18, 19, 20, 21, 22, 23, 24$ and 26. So we have found k-GMD graphs for 9 new values of k.

Open problem

Question

How can we choose $g \in \mathrm{GF}(q)$ and $a, b \in \mathbb{N}$ to get a k-GMD graph for some k?

Table: The present situation.

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Š. Gy. (2008)	9,13	5
Š. Gy. (2009)	16, 18–24, 26	
Missing	11, 15, 17, 25, ≥ 27	3, ≥ 7

Thank You

THANK YOU FOR YOUR ATTENTION.