Notes on Balanced Regular Covers of Regular Cayley Maps

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Orientable embeddings, and orientation preserving automorphisms of maps.

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Definition

A **regular** (sometimes called **rotary**) **map** is a map whose group of orientation preserving automorphisms acts regularly on the arcs of the map.

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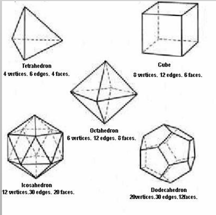
- all regular embeddings of complete graphs K_n are balanced Cayley maps (James, Jones, 1985)
- four of the five (regular) Platonic solids are Cayley maps



Definition

Given a group G, and a generating set $X = \{x_1, x_2, \ldots, x_d\}$, $\langle X \rangle = G$, that is closed under taking inverses and does not contain 1_G, the vertices of the Cayley graph $Cay(G, X)$ are the elements of the group G, and each vertex $g \in G$ is connected to all the vertices $gx_1, gx_2, \ldots, gx_d$.

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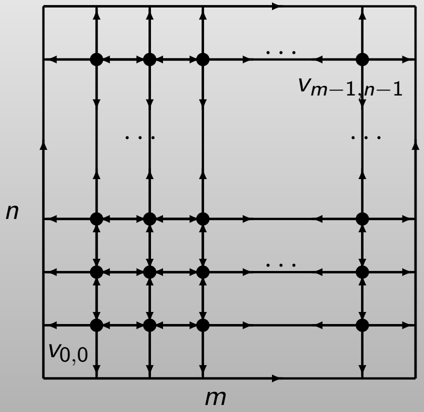
Definition

A *Cayley map* $CM(G, X, p)$ is an embedding of a connected *Cayley graph*, $C(G, X)$, that has the same local orientation p at each vertex.

Cayley Graph - Example

$$G = \mathbb{Z}_n \times \mathbb{Z}_n,\ X = \{(1,0),(0,1),(n-1,0),(0,n-1)\}$$



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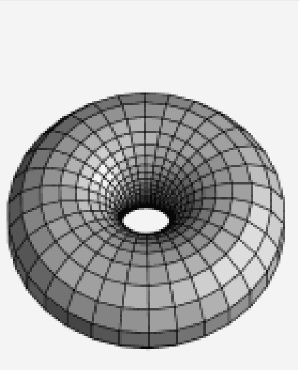

Fact:

The underlying group of a Cayley map, G, always acts transitively on the vertices of the map as a subgroup of the orientation preserving automorphism group via left multiplication:

$$G_L \leq Aut(CM(G, X, p))$$

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Cayley maps are the *only* embeddings of Cayley graphs with this property (Richter et al.) $\Longrightarrow$ Cayley maps are automatically endowed with a rich automorphism group, and are the best candidates to become regular maps.

Regular Cayley Maps

$$|Aut(CM(G,X,p))| \leq |G| \cdot |X|$$

and

$CM(G,X,p)$ is regular iff $|Aut(CM(G,X,p))| = |G| \cdot |X|$

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$$\Longrightarrow$$

$CM(G,X,p)$ is regular
iff
there exists a $\varphi \in Aut(CM(G,X,p))$ such that
$\varphi(1_G) = 1_G$ and $\varphi((1_G,x)) = (1_G,p(x))$

Regular Cayley Maps vs. Regular Maps

$$\lim_{n\to\infty} \frac{\#\text{ of regular Cayley maps with } n \text{ vertices}}{\#\text{ of regular maps with } n \text{ vertices}} = \quad ?$$

Regular Cayley Maps vs. Regular Embeddings of Cayley Graphs

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Related to a conjecture about the automorphism groups of Cayley graphs.

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We say that a balanced Cayley map $CM(G, X, p)$ is

- **Type I** if the elements in X are non-involutions
- **Type II** if the elements in X are involutions

Theorem (Conder, RJ, Tucker)

Let $A = \langle x, y \rangle$ *where* x *is an involution and the normal closure of* $\langle x \rangle$ *is* A. *Then* A *has a (Type II) balanced regular Cayley map. In particular, every nonabelian finite simple group and every symmetric group* S_n *has a balanced regular Cayley map, as does any group generated by such elements where the orders of* y *and* xy *are relatively prime.*

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Every abelian group is an index two subgroup of at least one group that admits a balanced regular Cayley map of Type II.

Normal Quotient and Cover

Definition (Li)

If $\Gamma = (V, E)$ is a graph with a vertex-transitive group of automorphisms $H \leq Aut(\Gamma)$, and N is a normal subgroup of H acting intransitively of V, then the **normal quotient** of Γ by N is the "orbit graph" Γ_N whose vertices are the orbits of N on V with two orbits joined by an edge if and only if there is at least one edge between them in Γ.

We say that Γ is a **normal cover** of Γ_N.

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$CM(N, S, p)$ is a balanced regular Cayley map of Type II

q.e.d.

Balanced Regular Cayley Maps vs. Regular Cayley Maps

$$\lim_{n\to\infty} \frac{\#\text{ of balanced regular Cayley maps with } n \text{ vertices}}{\#\text{ of regular Cayley maps with } n \text{ vertices}} = \quad ?$$

Theorem (RJ and Širáň)

If $\mathcal{M} = CM(G, X, p)$ is a regular Cayley map, then there exists a **skew-morphism** φ *of* G*, with a* **power function** π*, that equals* p *on* X*.*

Automorphism Group of a Cayley Map

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$$Aut(\mathcal{M}) \cong G \odot_{rot} \langle \varphi \rangle$$

$$(g, \varphi^i) \cdot (f, \varphi^j) = (g\varphi(f), (\varphi^i(f))^{-1} \cdot \varphi^i(f \cdot \varphi^j)).$$

Automorphism Group of a Cayley Map

Equivalently,

$$(g, \varphi^i) \cdot (f, \varphi^j) = (g\varphi(f), \varphi^{\pi(i,f)+j}),$$

where

$$\pi(i, f) = \sum_{j=0}^{i-1} \pi(\varphi^j(f)).$$

Automorphism Group of a Cayley Map Cont.

$$G \odot_{rot} \langle \varphi \rangle \cong G \times \mathbb{Z}_k,$$
$$(g, i) \cdot (f, j) = (g \cdot \varphi^i(f), \pi(i, f) + j)$$

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$$\langle \rho, \lambda \rangle \cong G \odot_{rot} \langle \varphi \rangle$$

$$\rho \mapsto (1_G, 1)$$

$$\lambda \mapsto (x, \chi(x)),$$

$x \in X$, $p^{\chi(x)} = x^{-1}$.

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$$S = \{(x, \chi(x)), (\varphi(x), \pi(1, x) + \chi(x) - 1), \dots,$$
$$(\varphi^{k-1}(x), \pi(k-1, x) + \chi(x) - (k-1))\}$$

Lemma

Let $\mathcal{M} = CM(G, X, p)$ *be a balanced Cayley map of valency* k*. Then one of the following must be true:*

1. $\mathcal{M}$ *is balanced of Type II,*
2. $[G : \langle X^2 \rangle] = 1$*, the semidirect product* $G : \langle \varphi^{k/2} \rangle$ *can be embedded as a regular balanced Cayley map* $\tilde{\mathcal{M}}$ *of Type II, and* $\mathcal{M}$ *is isomorphic to a normal quotient of* $\tilde{\mathcal{M}}$ *by a subgroup of order* $2k$*;*
3. $[G : \langle X^2 \rangle] = 2$ *and* $\mathcal{M}$ *is isomorphic to a balanced Cayley map of a* $\mathbb{Z}_2$ *extension of* $\langle X^2 \rangle$ *of Type II.*

Notes:

If G is abelian:

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- if $\mathcal{M}$ is balanced of Type II, *and G is abelian*, then G is an elementary abelian 2-group (of rank at most k),
- "the majority" of abelian balanced regular Cayley maps are of Type I, and $\varphi^{k/2}(a) = a^{-1}$, for all $a \in G$,
- if G is a finite group of *odd order*, then $\mathcal{M}$ is a normal quotient of a Type II balanced regular Cayley map of $G : \langle\psi\rangle$, $\psi(a) = a^{-1}$, by a normal subgroup of order $2k$.

t-Balanced Cayley Maps, $t > 1$

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Theorem (Feng,RJ)

Let $CM(G, X, p)$ be a t-balanced, $0 < t < t - 1$, regular Cayley map of an abelian group G with skew-morphism φ, then φ^{t+1} is a group automorphism of G.

t-balanced regular Cayley maps

Lemma

Let $CM(G, X, p)$ be a t-balanced regular Cayley map of valency d, $1 < t < d - 1$, or $t = d - 1$ and d is even, and let φ be its corresponding skew-morphism. Then there exists a proper *subgroup N of $G \odot_{rot} \langle \varphi \rangle$ that admits a balanced regular Cayley map $CM(N, \tilde{X}, \tilde{p})$ of Type II such that $Cay(N, \tilde{X})$ is a normal cover of $Cay(G, X)$.*

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Corollary

Let $CM(G, X, p)$ be a t-balanced regular Cayley map of valency d, $0 < t$, let φ be its corresponding skew-morphism. If there exists an $x \in X$ such that $\chi(x) \equiv 0 \pmod{t-1}$, then there exists a subgroup of $N = G \odot_{rot} \langle\varphi^{t-1}\rangle$ that admits a balanced regular Cayley map $CM(N, \tilde{X}, \tilde{p})$ of Type II such that $Cay(N, \tilde{X})$ is a normal cover of $Cay(G, X)$.

Balanced Regular Cayley Maps vs. t-Balanced Regular Maps

$$\lim_{n\to\infty} \frac{\#\text{ of balanced regular Cayley maps with } n \text{ vertices}}{\#\text{ of } t\text{-balanced regular Cayley maps with } n \text{ vertices}} = \quad ?$$

t-Balanced Regular Cayley Maps vs. Non-t-Balanced Regular Maps

$$\lim_{n\to\infty} \frac{\#\text{ of } t\text{-balanced regular Cayley maps with } n \text{ vertices}}{\#\text{ of non-}t\text{-balanced regular Cayley maps with } n \text{ vertices}} = \quad ?$$