Equivalence in terms of paths between inputs and outputs of a family of planar graphs

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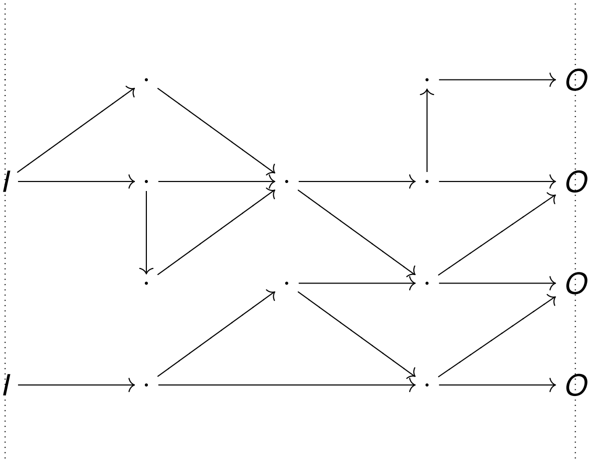
Rectangular Graphs

We will consider finite directed graphs embedded in a rectangle, with the following properties:

- A vertex is on the left edge if and only if it is not the terminal vertex of any edge.
- A vertex is on the right edge if and only if it is not the initial vertex of any edge.
- G is acyclic (i.e. it has no directed cycles, so the relation "There is a path from u to v." is a partial order).

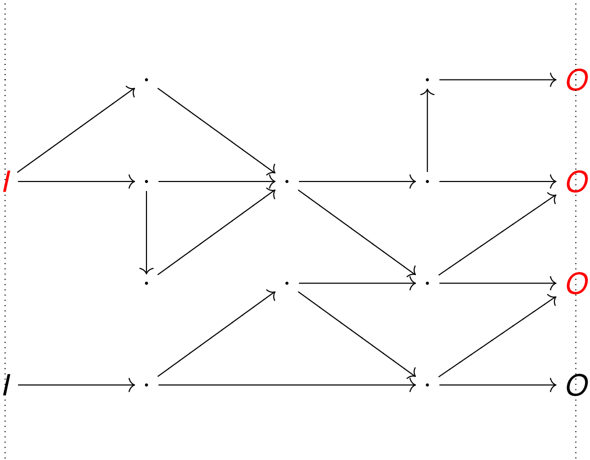
Example



The Path Relation

We will be interested in the relation between the vertices on the left edge and the vertices on the right edge, that relates u to v if and only if there is a directed path from u to v.

The Path Relation on the Example



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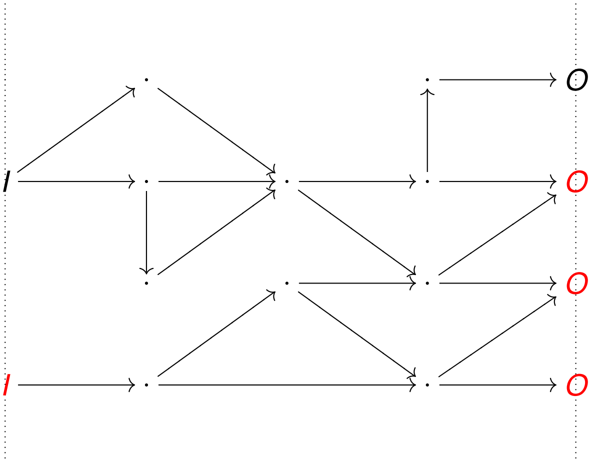
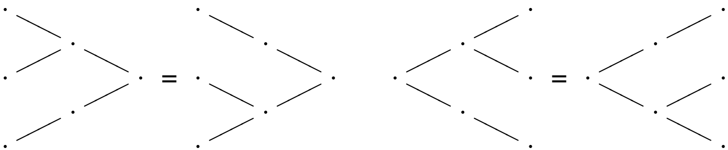


Simplifications

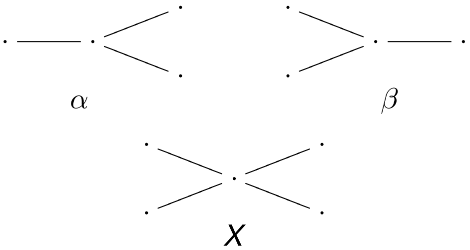
- Rearrange so that all edges are directed from left to right.
- Split vertices, so that every vertex has at most 2 inputs and 2 outputs.

We see that all ways of splitting a vertex to be equivalent, i.e.



Types of Vertices

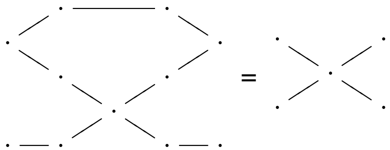
After these simplifications, there will only be 3 types of vertices in the graph:

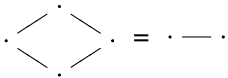


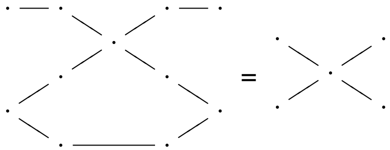
We can combine a β followed by an α into a single X, or we can split an X into a β followed by an α.

Axioms

We will show that two graphs are equivalent if and only if we can get from one to the other using the following axioms (in addition to the axioms that the different ways of splitting a vertex are the same):

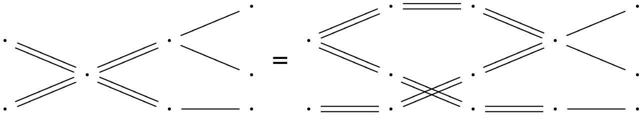






α, X, β-factorisation

Consider the equation:



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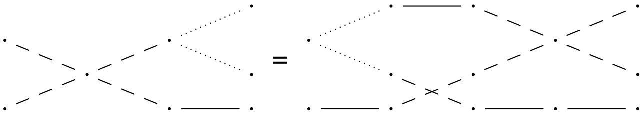

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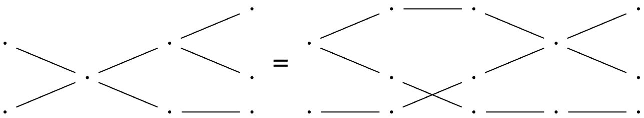
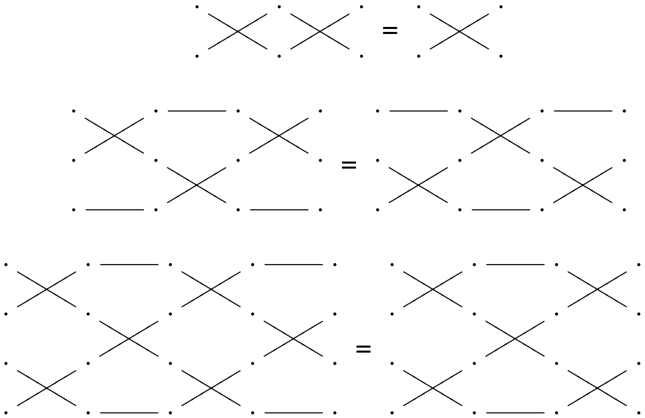


Lemma

Any graph is equivalent to a graph consisting of a collection of α, then a collection of X, then a collection of β.

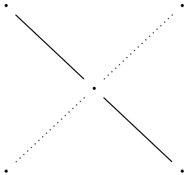
Axioms for X



Natural Paths

Definition

When we have the X-component in a graph, we will say that the natural paths are the ones that go from the lower input to the higher output, or from the higher input to the lower output.



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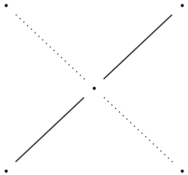
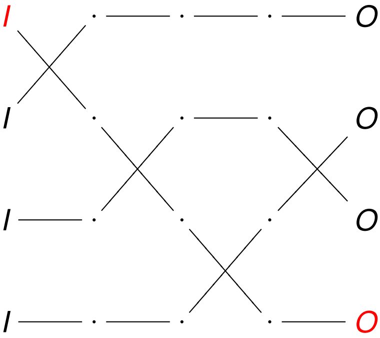


Permutations

For an X-graph, the relation that relates an input to an output if there is a natural path between them, is a permutation.



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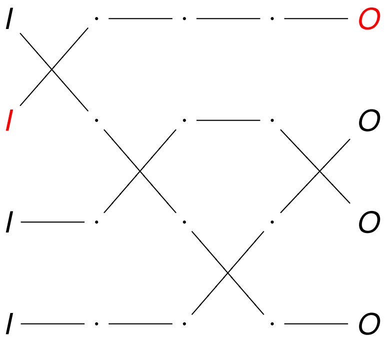

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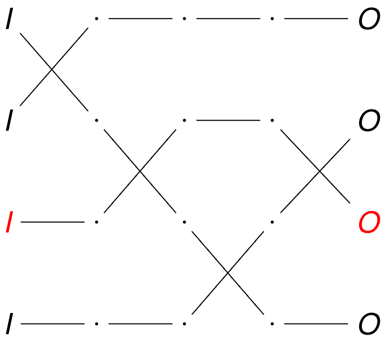

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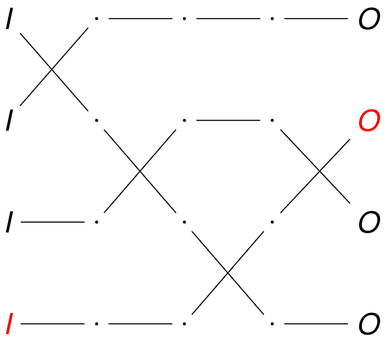
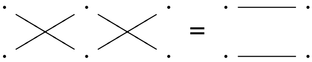
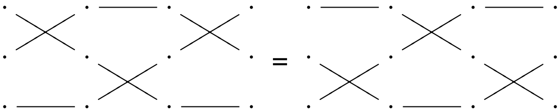


Axioms for S_n

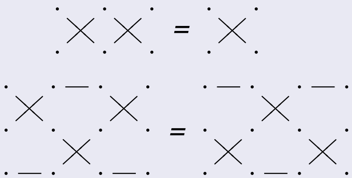
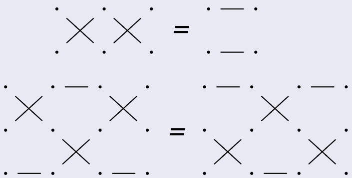




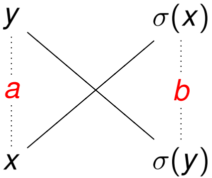
The Bijection Between M_n and S_n

Fact

The monoid generated by the first 2 axioms for an X-graph on n elements, is in bijection with the group S_n of permutations of an n-element set.

X-graph monoid	*Permutation group*
	

The Function from Permutations to Relations



$$\sigma \mapsto \{(a,b) | (\exists x \leqslant a)(\sigma(x) \geqslant b) \wedge (\exists y \geqslant a)(\sigma(y) \leqslant b)\}$$

Order-reversing Quadruples

Lemma

If there are $a \leqslant b \leqslant c \leqslant d$ with $\sigma(d) \leqslant \sigma(c) \leqslant \sigma(b) \leqslant \sigma(a)$, then defining

$$\phi(x) = \begin{cases} \sigma(c) & \text{if } x = b \\ \sigma(b) & \text{if } x = c \\ \sigma(x) & \text{otherwise} \end{cases}$$

The graphs corresponding to σ and ϕ are equivalent using the X-axioms.

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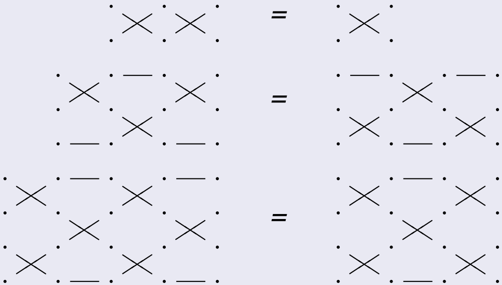
Corollary

Any permutation is equivalent (under the X-axioms) to one with no order-reversing quadruples.

For X-graphs

Theorem

If two X-graphs give the same path relation, then they are equivalent under the X-axioms.



For General Graphs

Theorem

If two graphs give the same path relation, then they are equivalent under the axioms.