A new family of distance regular covers of complete graphs

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Outline

Outline

Introduction

- In 2008 we (i.e. K. and Pech), using a computer, discovered two new antipodal distance regular graphs on 108 and 135 vertices, respectively (with new parameters).
- After long efforts it became possible to embed the example on 108 vertices to a potentially wide infinite class of distance regular graphs.
- Also progress with the understanding of the example on 135 vertices was achieved.

Goals of the presentation

- To provide first acquaintance with distance regular graphs, and in particular antipodal graphs.
- To outline promising lines between algebraic and topological group theory based on the use of voltage groups.
- To present our new results.

- We were using the methodology of coherent configurations and association schemes.
- It however will remain almost invisible in the course of the lecture.
- We use computer packages:
  - COCO (Faradžev, K. 1991),
  - GAP,
  - COCO II (Reichard, in development).

Outline

Equitable partitions

An equitable partition of the vertex set V of a graph $\Gamma = (V, E)$ is a partition

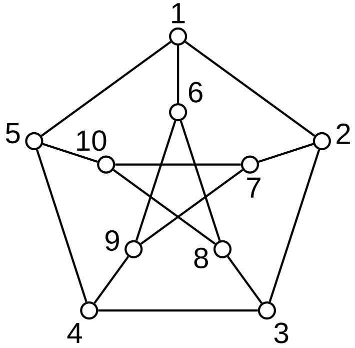
$$\pi = \{C_1, \ldots, C_l\}$$

such that the number of neighbors in C_j of a vertex u in C_i is a constant $b_{i,j}$, independent of the selection of $u \in C_i$.

Equitable partitions (cont.)

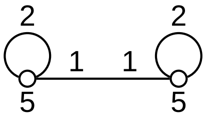
- A significant origin of equitable partitions is orbits of an arbitrary subgroup of the group $\mathrm{Aut}(\Gamma)$.
- Sometimes such partitions are called automorphic equitable partitions.
- However, not each equitable partition is automorphic.

Example 1: Petersen graph P



P

"Internal" and "external" cycles form two cells of an equitable partition.



Intersection diagram

Metric Decompositions

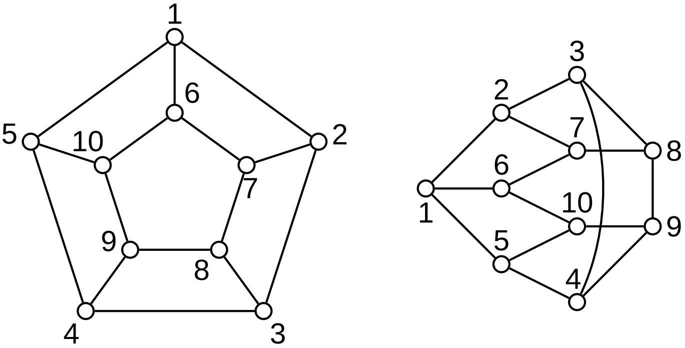
Given:

- A graph Γ,
- a vertex u of Γ.

Metric Decomposition:

- Cells of the metric partition of Γ with respect to u are the vertices on the same distance i from u.
- If the diameter $d = d(\Gamma)$ of Γ is finite, we have $d + 1$ cells.
- We denote by $\Gamma_i(u)$ the subgraph of Γ induced by the vertices on distance i from u.

Example 2: Pentagonal Prism

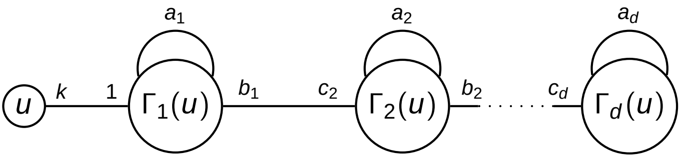


The metric decomposition (for $u = 1$) is not equitable:
e.g. $\Gamma_2(1)$ is not regular.

A connected regular graph Γ of valency k and diameter d is called distance regular (briefly DRG) if for each vertex u the metric partition

$$\{\{u\}, \Gamma_1(u), \ldots, \Gamma_d(u)\}$$

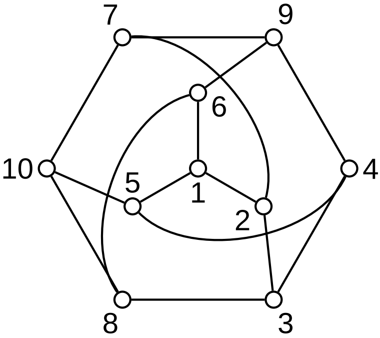
is equitable with the set of intersection numbers which does not depend on the selection of u.



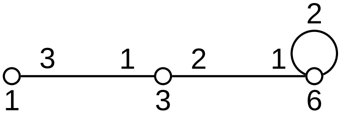
Intersection diagram of a DRG

- A DRG of diameter $d = 2$ is called a strongly regular graph (briefly SRG).
- A DRG Γ is called primitive if all distance i graphs Γ_i for $1 \leq i \leq d$ are connected. Otherwise Γ is called imprimitive.
- Note that $\{x, y\}$ is an edge in Γ_i if and only if $d(x, y) = i$ in the graph Γ.

Example 1 (cont.)



Metric decomposition of the Petersen graph and its intersection diagram:

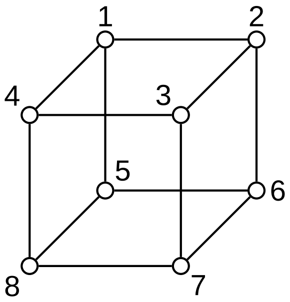


- An imprimitive DRG Γ of diameter d is called antipodal if its distance graph Γ_d is disconnected.
- In this case Γ_d is a disjoint union of n copies of the complete graph K_r.
- The partition formed by the vertices of these n copies is called the antipodal partition of Γ.

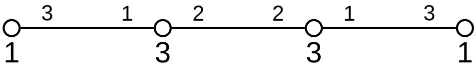
Theorem (D.H.Smith, A.Gardiner)

An imprimitive DRG is bipartite or antipodal (here "or" is not exclusive).

Example 3: the 3-dimensional cube Q_3



$n = 4, r = 2$

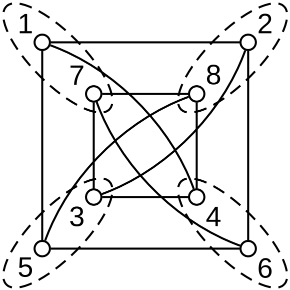


Q_3 is bipartite and antipodal.

Outline

Example 3 (cont.)

Another glance onto Q_3:

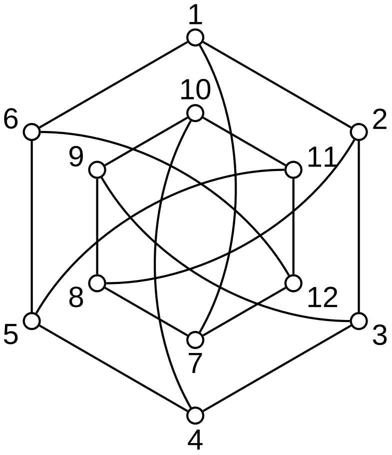


- Antipodal cells are “metavertices”.
- The quotient graph is K_4.
- Each edge of K_4 is represented by 1-factor between two metavertices.

- A graph Γ is called a cover of another graph Δ if there is a surjection $h : V(\Gamma) \to V(\Delta)$ that maps edges of Γ to edges of Δ which is locally an isomorphism.
- The function h is called a covering of Δ.
- Preimages of vertices from Δ are called the fibres of the covering.

- Each fibre induces an empty subgraph.
- Between two fibres there are either no edges, or the edges between the two fibres form a perfect matching.
- For the covers of a connected graph all fibres have the same size r.
- Let $\ker h$ be the equivalence relation defined by the fibres.
- Clearly, $\Gamma / \ker h$ is isomorphic to Δ.

Example 4

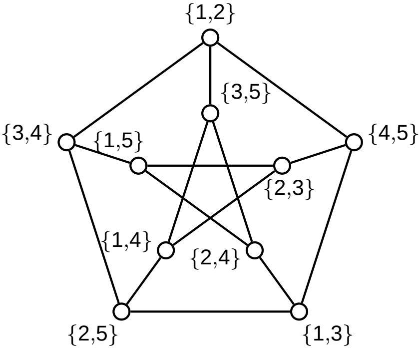


- Γ is an antipodal cover of $\Delta = K_{3,3}$.
- The quotient graph $K_{3,3}$ is a DRG.
- However, Γ is not a DRG.

- If the cover Γ of the graph Δ is distance regular, then Γ is called antipodal distance regular cover of Δ.
- Note that in this case Δ is also a DRG.

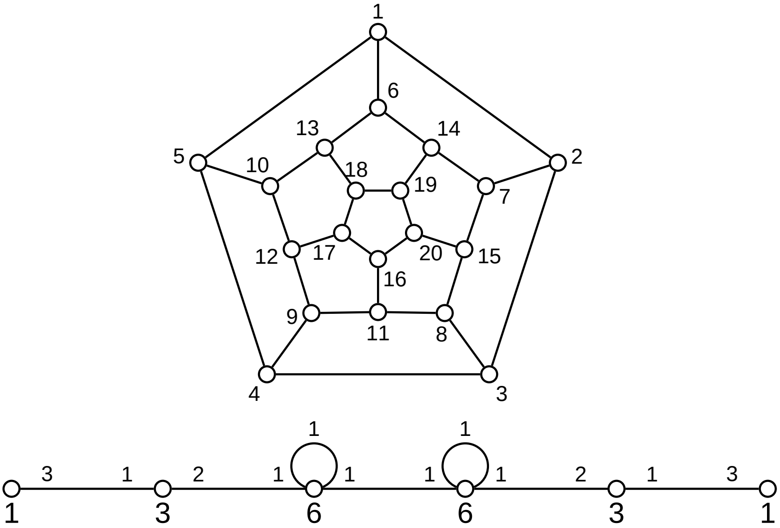
Example 3 (cont.)

- We try in the role of Δ the Petersen graph.
- It is now convenient to recall its formal definition:



- Vertices are 2-subsets of a 5-set.
- Adjacency means subsets are disjoint.

Example 5: Dodecahedron



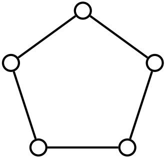
We wish to observe that the dodecahedron $\mathcal{D}$ is an antipodal cover of the Petersen graph P.

A “geometrical vision” may help.

- We try to provide a constructive approach to $\mathcal{D}$.
- We know in advance that

$$\mathrm{Aut}(\mathcal{D}) \cong A_5 \times Z_2.$$

- Let us start from A_5.
- We wish to present A_5 as the automorphism group of an auxiliary structure.

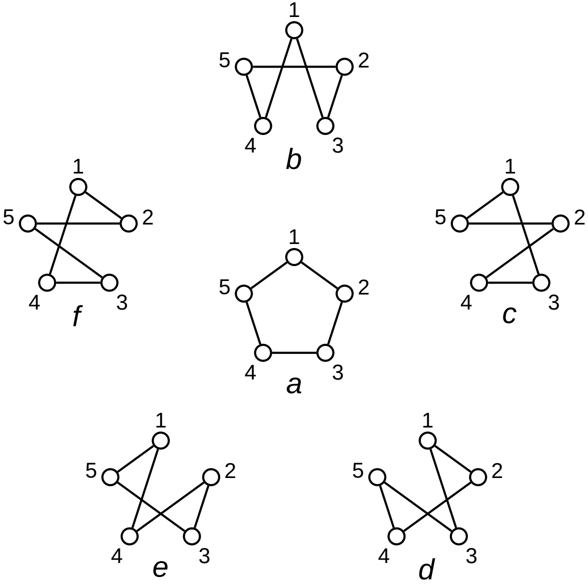


Pentagon C_5

$\mathrm{Aut}(C_5) = D_5$ consists only of even permutations.

Thus the orbit O of A_5 on C_5 has length $60/10 = 6$.

The orbit O of 6 pentagons:



- Now we construct a design $\mathfrak{S} = (\mathcal{P}, \mathcal{B})$.
- The set $\mathcal{P} = \{a, b, c, d, e, f\}$ of points is our orbit O of six pentagons.
- Blocks are labeled by the edges of the complete graph K_5 on the set $[1, 5]$.
- Incidence is inclusion.

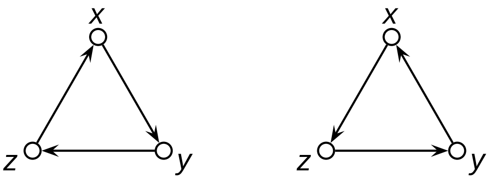
Blocks of $\mathfrak{S}$

{1,2}	{a,d,f}
{1,3}	{b,c,d}
{1,4}	{b,e,f}
{1,5}	{a,c,e}
{2,3}	{a,b,e}
{2,4}	{c,d,e}
{2,5}	{b,c,f}
{3,4}	{a,c,f}
{3,5}	{d,e,f}
{4,5}	{a,b,d}

We get a BIBD:

$$v = 6, \quad b = 10, \quad k = 3, \quad r = 5, \quad \lambda = 2$$

- Now we construct a new graph $\mathcal{D}'$ with the aid of the auxiliary structure $\mathfrak{S}$ as follows:
- $\mathcal{D}' = (V, E)$.
- V is the set of directed triangles on the block set of $\mathfrak{S}$.
- E.g. $\{x, y, z\}$ gives



- Two triangles from V form an edge in $E \iff$ they share one arc.

Easy observations

- $\mathcal{D}'$ is an antipodal graph, each fibre consists of opposite triangles.
- $\mathcal{D}'$ is a DRG with the intersection diagram of $\mathcal{D}$.
- The quotient graph of $\mathcal{D}'$ is isomorphic to P.
- $\mathrm{Aut}(\mathcal{D}') \cong A_5 \times \mathbb{Z}_2$.
- $\mathcal{D}' \cong \mathcal{D}$.

One more significant message:

- The group $\mathbb{Z}_2$ of order 2 transposes each pair of opposite triangles.
- In other words, $\mathbb{Z}_2$ acts semiregularly on the set of fibres.
- Such property will be crucial for us later on.

Outline

- From now on and onwards the complete graph will serve as the quotient graph Δ.
- The antipodal distance regular covers in this case have diameter d equal to 3.

More examples!

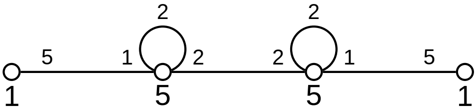
Example 6: Icosahedron

To construct the icosahedron $\mathfrak{I}$ we use the same spirit as for $\mathcal{D}$ and consider 12 directed pentagons from the same orbit of A_5.

Adjacency:

two directed pentagons share a common arc.

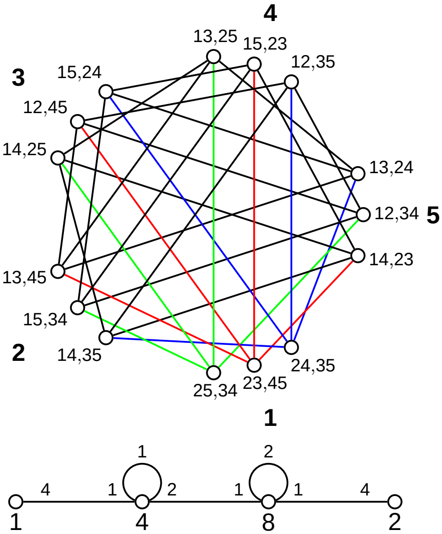
We get a DRG:



$\mathrm{Aut}(\mathfrak{I}) \cong A_5 \times \mathbb{Z}_2$.

$\mathfrak{I}$ is a 2-fold cover of K_6.

Example 7: Line graph of the Petersen graph P



3-fold cover of K_5

Example 8: Johnson graph $J(6, 3)$ on 20 vertices

Points:

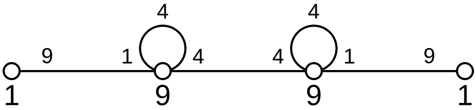
3-subsets of 6-set.

Adjacency:

two 3-subsets share two common points.

Fibres:

subset and its complement.



2-fold cover of K_{10}

Outline

Brouwer theorem

- Suppose that Σ is an SRG with the parameters of a point graph of a generalized quadrangle $\mathrm{GQ}(s,t)$.
- Assume that Σ allows a spread, that is a partition into cliques of size $(s+1)$.
- Remove all edges in the spread from Σ.
- Then the remaining graph Γ is an antipodal distance regular cover of K_{st+1}.
- Conversely, each cover with such parameters arises in this way.

- A number of DRGs appear in this manner from the spreads in known GQs.
- Example 7 is the smallest case.
- Godsil & Hensel (1992) were asking for an example of a pseudogeometric SRG which provides an antipodal DRG via the deletion of a spread.

- First such example was constructed by K. (1999) as a Cayley graph over a suitable group of order 96.
- Nowadays many examples are available via the theory developed by Wallis - Fon Der Flaass - Muzychuk.

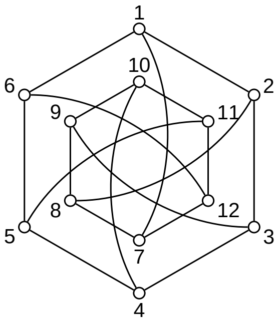
Outline

Lemma

An antipodal r-fold cover of K_n is antipodal distance regular if and only if there exists a constant c_2 such that any two non-adjacent vertices from different fibres of the cover have exactly c_2 common neighbors.

Thus we will call an antipodal distance regular cover of K_n an (n, r, c_2)-cover.

Example 4 (cont.)



Vertices 1 and 11 have 0 common neighbors, while vertices 1 and 3 have 1 common neighbor.

Thus the constant c_2 does not exist, therefore we have no DRG cover.

The basic feasibility conditions for triples (n, r, c_2) of parameters:

(F1) (n, r, c_2) are integers with $1 \leq (r-1)c_2 \leq n-2$.

(F2) If n is even, then c_2 is even.

(F3) The multiplicities of the eigenvalues of the cover are integers.

Denote:

$$\delta = n - 2 - rc_2,$$
$$\Delta = \delta^2 + 4(n-1).$$

Eigenvalues and multiplicities are:

Eigenvalue	Multiplicity
$n-1$	1
-1	$n-1$
$\theta = \dfrac{\delta + \sqrt{\Delta}}{2}$	$m_\theta = \dfrac{n(r-1)\tau}{\tau - \theta}$
$\tau = \dfrac{\delta - \sqrt{\Delta}}{2}$	$m_\tau = \dfrac{n(r-1)\theta}{\theta - \tau}$

Many more additional conditions on the triples (n, r, c_2) are known from diverse sources.
A key to the classification is:

Theorem

For fixed r and δ there are only finitely many feasible parameter triples, unless $\delta \in \{-2, 0, 2\}$.

Thus four classes of parameters are distinguished:

I $\delta = -2$,

II $\delta = 0$,

III $\delta = 2$,

IV $\delta \notin \{-2, 0, 2\}$.

Corollary

If $\delta \in \{-2, 2\}$, *then* Δ *must be a square.*

Starting examples repeated:

#	Name	n	r	c_2	δ	Δ
3	3-cube	4	2	2	-2	16
6	Icosahedron	6	2	2	0	20
7	$L(P)$	5	3	1	0	16
8	$J(6,3)$	10	2	4	0	36

Outline

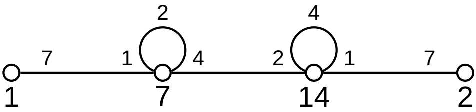
Main known infinite series

Construction	Parameters	Conditions	Class
Mathon	$(q+1, r, c)$	$q = rc + 1$ is a prime power	II
Bondy	$(n, n-2, 1)$	Projective plane of order $n-1$ exists	II
Thas-Somma	(q^{2j}, q, q^{2j-1})	q is a prime power	I
Brouwer	$(st+1, s+1, t-1)$	spread in pseudo GQ	all
Godsil-Hensel	$(p^{2i}, p^{i-k}, p^{i+k})$	p is prime, $0 \leq k < i$	I
de Caen-Mathon-Moorhouse	$(2^{2t}, 2^{2t-1}, 2)$		I
de Caen-Fon Der Flaass	(q^{d+1}, q^d, q)	$q = 2^t$	I

There are some possible intersections between families, combinations of constructions.

There is also a possibility that other series appear in an unpublished paper of Brouwer, Godsil, Wilbrink.

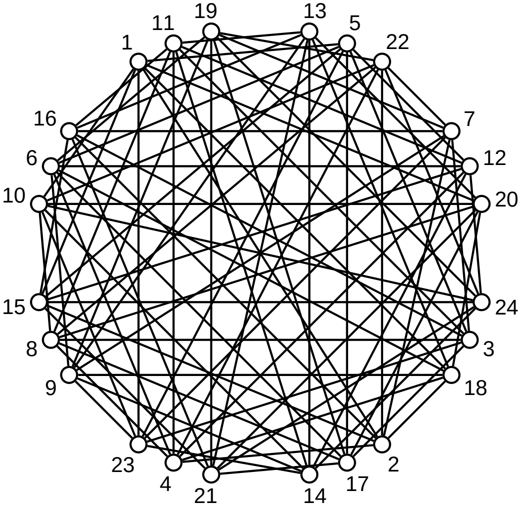
Example 9: Mathon, $q = 7$, an $(8, 3, 2)$-cover



- Take Aut (Fano plane).
- It has cycle of length 7.
- Consider all cycles as undirected regular graph C_7 of valency 2.
- Get 24 such cycles.
- Two cycles are adjacent $\iff$ they share one edge.

Dual of the famous Klein map on a surface of genus 3.

Example 9: Mathon, $q = 7$, an $(8, 3, 2)$-cover (cont.)



$$\delta = 0, \Delta = 28$$

Example 10: Exceptional $(7, 6, 1)$-cover

It appears as the subgraph of non-neighbors in the Moore graph of valency 7 — that is, in the Hoffman-Singleton graph.

- Start from the auxiliary structure $\mathfrak{S}$ as it appears in Example 5.
- Let $\overline{\mathfrak{S}}$ be the design with block set consisting of the 10 remaining 3-element subsets of a 6-set.
- Observe that $\overline{\mathfrak{S}} \cong \mathfrak{S}$, and that $\mathrm{Aut}(\{\mathfrak{S}, \overline{\mathfrak{S}}\}) \cong S_5$.
- Thus there are 6 different isomorphic copies of the systems $\{\mathfrak{S}, \overline{\mathfrak{S}}\}$ sharing the same 6-element point set.
- Consider now a 7-element set, and remove from it an arbitrary point and get 6 systems.
- Altogether we obtain $7 \cdot 6 = 42$ different copies of systems of the form $\{\mathfrak{S}, \overline{\mathfrak{S}}\}$.

Example 10: Exceptional $(7, 6, 1)$-cover (cont.)

- This is the point set of our forthcoming graph Γ.
- Define edges of Γ as follows:
  systems $\{\mathfrak{S}, \overline{\mathfrak{S}}\}$ and $\{\mathfrak{S}', \overline{\mathfrak{S}'}\}$ are adjacent $\iff$ they have distinct isolated points and the design $\mathfrak{S}$ shares with design $\mathfrak{S}'$ 0 or 5 common blocks.
- Here antipodal fibres are formed by systems, having the same isolated point.

$$\delta = 5 - 6 = -1; \Delta = 1 + 4 \cdot 6 = 25.$$

Outline

First new example on 108 vertices

Originally was obtained with the aid of a computer.

- We start from the incidence graph Σ of a resolvable transversal design $\mathrm{RT}(6, 2; 3)$.
- Σ is a bipartite and antipodal DRG with diagram:

- The automorphism group of Σ is related to the famous Schur multiplier of the group S_6 (cf. lecture of Cai-Heng Li at Ljubljana).

$$G = \mathrm{Aut}(\Sigma) \cong \mathbb{Z}_3.S_6.\mathbb{Z}_2$$

  is a group of order $3 \cdot 6! \cdot 2 = 4320$.
- A point stabilizer is isomorphic to the symmetric group S_5.
- We consider a new transitive action (G, Ω) of the group G on the set Ω of cardinality 108 (cosets of a suitable subgroup of order 40, aka flags of the incidence structure $\mathrm{RT}(6, 2; 3)$).

The computer package COCO tells us:

- $\mathrm{rank}(G, \Omega) = 8$;
- there is a number of merging association schemes formed by the 2-orbits of (G, Ω);
- one of these schemes is so-called non-Schurian association scheme $\mathfrak{M}$ with 3 classes.

In addition, the computer package GAP tells us:

- $\mathfrak{M}$ is a metric association scheme;
- in other words, $\mathfrak{M}$ is generated in a canonical manner by an antipodal DRG Γ with the parameters $(36, 3, 12)$. Here $\delta = -2$, $\Delta = 144$.

The antipodal DRG Γ turns out to be new!

We wish to get a computer free interpretation of Γ.

This implies the real beginning of all our current story.

Outline

- Let Γ be a connected antipodal distance regular cover of K_n with index r.
- $G = \mathrm{Aut}(\Gamma)$.
- Let us consider the subgroup $T \leq G$ which stabilizes each of the fibres of Γ (that is, T preserves each fibre as a set).

Lemma

Every element $\sigma \in T$, $\sigma \neq e$ is fixed point free.

Corollary

- $|T| \leq r$,
- *T acts semiregularly on the fibres.*

If the group T has order r and thus acts regularly on each fibre, then we say that Γ is a regular cover.

If in addition T is abelian or cyclic, then Γ is called abelian or cyclic cover, respectively.

- The group T will be called the voltage group.
- Classical techniques from topological graph theory may and should be applied to investigate regular covers.

(Godsil and Hensel developed elements of such techniques, probably independently of topological graph theory.)

- A significant new input of them was to use group representation theory in order to get additional feasibility conditions for regular covers.
- An example of their result:

  Let Γ be a cyclic r-fold cover of K_n with $r > 2$. Then r divides n.

Outline

Godsil-Hensel matrices

We slightly modify the original language of Godsil-Hensel

"Matrix-representation of symmetric arc functions":

Let T be a voltage group, $A = (a_{i,j})$ be a square matrix of order n, where

- $a_{i,j} \in \overline{T}$,
- $\overline{T} = T \cup \{0\}$, where 0 is an additional element distinct from any element of T.

A will be called a formal matrix over T.

Godsil-Hensel matrices (cont.)

We call $A = (a_{i,j})$ a covering matrix if

- $a_{i,j} = (a_{j,i})^{-1}$ for all $i, j \in \{1, \ldots, n\}$,
- $a_{i,i} = 0$ for all $i \in \{1, \ldots, n\}$,

Godsil-Hensel matrices (cont.)

We associate to the covering matrix A two graphs:

- the underlying graph $\Delta = \Delta_A$ with the vertex set

$$V(\Delta_A) = \{1, 2, \dots, n\},$$

  and the edge set

$$E(\Delta_A) = \{\{i, j\} \mid a_{i,j} \neq 0\};$$

- the cover of $\Gamma = \Gamma^A$ with the vertex set

$$V(\Gamma^A) = \{1, 2, \dots, n\} \times T,$$

  and the edge set

$$E(\Gamma^A) = \{\{(i, g), (j, h)\} \mid a_{i,j} \neq 0, g \cdot a_{i,j} = h\}.$$

Godsil-Hensel matrices (cont.)

It is easy to observe that the function

$$h : V(\Gamma^A) \to V(\Delta_A) \text{ defined as } (i,g) \mapsto i$$

is a covering function, thus the graph Γ^A is a cover of the graph Δ_A.

Moreover, each regular cover of Δ_A (up to isomorphism) with the voltage group T can be obtained in this way.

If Γ^A is an antipodal distance regular cover of Δ_A, then we call A the Godsil-Hensel matrix of this cover (briefly GH-matrix).

Theorem (Godsil-Hensel, 1992)

Let T be a voltage group and let A be a covering matrix of order n over T.
Then A is a GH-matrix of a regular antipodal (n, r, c_2)-cover of K_n with the voltage group T if and only if

$$A^2 = (n-1)I + \delta A + c_2 \underline{T}(J - I), \qquad (*)$$

where as before $\delta = n - 2 - rc_2$.

(Here I and J are natural modifications of classical notations to formal matrices. Moreover, matrix multiplication is performed in the matrix-ring over the group-ring of T, and $\underline{T}$ denotes the sum of all elements of T in the group-ring of T.)

- The class of all such matrices which satisfy $(*)$ will be denoted by
$$\mathrm{GHM}(T, n, r, c_2).$$
- It is now a nice exercise to derive again a number of known constructions in a unified manner with the aid of suitable GH-matrices over suitable voltage groups T.

Outline

We extend the concept of a conjugate transpose matrix A^* onto formal matrices.

Definition

Let T be a finite group, and let $A = (a_{i,j})$ be a formal matrix of order n over T such that A does not contain the entry 0. Then we call A a generalized Hadamard matrix if for $c = {}^{n}/_{|T|}$ we have:

$$AA^* = A^*A = nI + c\underline{T}(J - I).$$

We denote by $\mathrm{gH}(T, n)$ the set of all gH-matrices of order n over T.

Remarks

(1) There are a few known ways to generalize the concept of a Hadamard-matrix. We follow the way of Drake (1979).

(2) Our (additional) condition $AA^* = A^*A$ is essential, because we do not require from T to be abelian.

Lemma

Let T be a finite group and let A be a covering matrix over T with $\Delta_A = K_n$. Then the graph Γ^A is a regular (n, r, c_2)-cover if and only if

$$(A+I)^2 = nI + (n - rc_2)A + c_2\underline{T}(J - I).$$

Proof.

$$\begin{aligned}(A+I)^2 &= A^2 + 2A + I \\ &= (n-1)I + (n-2-rc_2)A + c_2\underline{T}(J-I) + 2A + I \\ &= nI + (n - rc_2)A + c_2\underline{T}(J-I).\end{aligned}$$

□

This slight modification turns out to be helpful for the case $\delta = -2$ (that is $n - rc_2 = 0$).

Corollary

Let T be a finite group with neutral element e and let A be a covering matrix over T with $\Delta_A = K_n$.
Then the graph Γ^A is a regular (n, r, c_2) cover with $\delta = -2$ if and only if $A + I$ is a self-adjoint $\mathrm{gH}(T, n)$-matrix (note that this generalized Hadamard matrix has everywhere on its diagonal the element e).

Remarks

(3) It is convenient to call self-adjoint gH-matrices with identity diagonal skew gH-matrices.

(4) If T is a cyclic group of order 2, then we obtain the equivalence of distance regular double covers of K_n to regular two-graphs and in turn to classical skew Hadamard matrices.

Outline

The following breakthrough reached by Pech leads to new infinite series of DRGs.

Theorem

Let T be a finite group, let $H = (h_{i,j})$ be any $\mathrm{gH}(T,n)$*. Let* $\psi : \{1,2,\ldots,n\}^2 \to \{1,2,\ldots,n^2\}$ *be any bijection.*
Define

$$R_H = (r_{(i,j)^\psi,(k,l)^\psi})$$

according to

$$r_{(i,j)^\psi,(k,l)^\psi} = h_{k,j} \cdot h_{i,l}^{-1}.$$

Then R_H is a skew $\mathrm{gH}(T,n^2)$*.*

Corollary

If there exists a $\mathrm{gH}(T, n)$ *over a finite group* T*, then for all* $t \in \mathbb{N} \setminus \{0\}$ *there exists a skew* $\mathrm{gH}(T, n^{2^t})$*.*

Therefore, starting from any $\mathrm{gH}(T, n)$-matrix we obtain an infinite series of regular covers of complete graphs.

Outline

As illustration, we provide a computer free interpretation of our new example on 108 vertices:

- Consider the following gH-matrix A of order 6 over $\mathbb{Z}_3$:

$$A = \begin{pmatrix} 0_3 & 0_3 & 0_3 & 0_3 & 0_3 & 0_3 \\ 0_3 & 0_3 & 1_3 & -1_3 & -1_3 & 1_3 \\ 0_3 & 1_3 & 0_3 & 1_3 & -1_3 & -1_3 \\ 0_3 & -1_3 & 1_3 & 0_3 & 1_3 & -1_3 \\ 0_3 & -1_3 & -1_3 & 1_3 & 0_3 & 1_3 \\ 0_3 & 1_3 & -1_3 & -1_3 & 1_3 & 0_3 \end{pmatrix}$$

- Consider, for example, the function $\psi : (i, j) \mapsto 6(i-1) + j$

Get a skew gH-matrix B of order 36:

$$
B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 \\
0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 1 & 1 & 0 & -1 & -1 & 0 & -1 & -1 & 1 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & -1 & -1 & 0 \\
0 & 0 & -1 & 1 & 1 & -1 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 1 & 1 & 0 & -1 & -1 & 0 & -1 & -1 & 1 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 1 \\
0 & 0 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 1 & 1 & 0 & -1 & -1 & 0 & -1 & -1 & 1 & 0 & 0 & 1 \\
0 & 0 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 1 & 1 & 0 & -1 & -1 & 0 \\
0 & 0 & -1 & 1 & 1 & -1 & 1 & 1 & 0 & -1 & -1 & 0 & -1 & -1 & 1 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\
0 & -1 & 0 & -1 & 1 & 1 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & -1 & 0 & -1 & 1 & 1 \\
0 & -1 & 0 & -1 & 1 & 1 & 0 & -1 & 0 & -1 & 1 & 1 & 1 & 0 & 1 & 0 & -1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 \\
0 & -1 & 0 & -1 & 1 & 1 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 1 & 1 & 1 & 0 & 1 & 0 & -1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\
0 & -1 & 0 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 1 & 1 & 1 & 0 & 1 & 0 & -1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 \\
0 & -1 & 0 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 1 & 1 & 1 & 0 & 1 & 0 & -1 & -1 \\
0 & -1 & 0 & -1 & 1 & 1 & 1 & 0 & 1 & 0 & -1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 1 & 1 \\
0 & 1 & -1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & -1 & 1 \\
0 & 1 & -1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 1 & -1 & 0 & 1 & 0 & -1 \\
0 & 1 & -1 & 0 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & -1 & 0 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\
0 & 1 & -1 & 0 & -1 & 1 & -1 & 0 & 1 & -1 & 1 & 0 & 1 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & -1 & 0 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & -1 & 1 & 0 \\
0 & 1 & -1 & 0 & -1 & 1 & -1 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 1 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & -1 & 0 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 \\
0 & 1 & -1 & 0 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 1 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & -1 & 0 & -1 & 1 \\
0 & 1 & 1 & -1 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & -1 \\
0 & 1 & 1 & -1 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & -1 & 1 & -1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & 0 & 1 & 0 \\
0 & 1 & 1 & -1 & 0 & -1 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & -1 & 0 & -1 & 1 & -1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 1 & -1 & 1 \\
0 & 1 & 1 & -1 & 0 & -1 & -1 & 0 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & -1 & 0 & -1 & 1 & -1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & 1 \\
0 & 1 & 1 & -1 & 0 & -1 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & -1 & 0 & -1 & 1 & -1 & -1 & 0 & 1 & 0 \\
0 & 1 & 1 & -1 & 0 & -1 & 1 & -1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & -1 & 0 & -1 \\
0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 \\
0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & -1 & 0 & 1 \\
0 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 1 & -1 \\
0 & -1 & 1 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & -1 \\
0 & -1 & 1 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \\
0 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & 1 & 1 & -1 & 0
\end{pmatrix}
$$

- Matrix B is a skew gH$(\mathbb{Z}_3, 36)$.
- Hence $B - I$ is a $GHM(\mathbb{Z}_3, 36, 3, 12)$,
- We obtain a regular $(36, 3, 12)$-cover.
- To the best of our knowledge, the series of DRGs on $3 \cdot 6^{2^k}$ vertices is new.

Outline

One more new example on 135 vertices

This graph Γ originally was also obtained with the aid of the computer package COCO.

$$\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_3.S_6.\mathbb{Z}_2$$

(It is isomorphic to the automorphism group of the DRG on 108 vertices that was described above.)
We were trying to get a similar interpretation of Γ in terms of formal matrices.

We start from a very famous incidence structure $\mathrm{GQ}(2,2) = W_2$, that is the generalized quadrangle of order 2. It has many alternative names:

- Sylvester system of duads and synthemes;
- Cremona-Richmond configuration;
- Tutte's 8-cage graph.

- The Levi (incidence) graph $L(W_2)$ is bipartite, therefore instead of traditional use of adjacency matrices we use incidence matrices.
- The well-known Foster graph F on 90 vertices is a 3-fold regular cover of $L(W_2)$.
- It is an antipodal bipartite DRG of diameter 8.
- We consider also the graph F_3, that is the distance-3 graph of the DRG F.
- The graph F_3 turns out to be a 3-fold cover of the complement of $\overline{L(W_2)}$ to $L(W_2)$.
- We present both F and F_3 with the aid of formal matrices and combine these matrices.

	12 34 56	14 23 56	13 24 56	16 25 34	15 26 34	12 36 45	12 35 46	14 26 35	14 25 36	16 23 45	15 23 46	16 24 35	15 24 36	13 26 45	13 25 46
56	×	×	×												
34	×			×	×										
12	×					×	×								
14		×						×	×						
23		×								×	×				
24			×									×	×		
13			×											×	×
25				×					×						×
16				×						×		×			
15					×						×		×		
26					×			×						×	
36						×			×				×		
45						×				×				×	
46							×				×				×
35							×	×				×			

The incidence table of W_2

	12 34 56	14 23 56	13 24 56	16 25 34	15 26 34	12 36 45	12 35 46	14 26 35	14 25 36	16 23 45	15 23 46	16 24 35	15 24 36	13 26 45	13 25 46
56	0	0	0												
34	0			0	0										
12	0					0	0								
14		0						0	0						
23		0								0	0				
24			0									0	0		
13			0											0	0
25				0					1						-1
16				0						-1		1			
15					0						1		-1		
26					0			-1						1	
36						0			-1				1		
45						0				1				-1	
46							0				-1				1
35							0	1				-1			

The collapsed incidence matrix of F

	12 34 56	14 23 56	13 24 56	16 25 34	15 26 34	12 36 45	12 35 46	14 26 35	14 25 36	16 23 45	15 23 46	16 24 35	15 24 36	13 26 45	13 25 46
56				0	0	0	0	0	0	0	0	0	0	0	0
34		0	0			0	0	-1	1	-1	1	1	-1	1	-1
12		0	0	0	0			1	-1	1	-1	-1	1	-1	1
14	0		0	-1	1	1	-1			0	0	1	-1	-1	1
23	0		0	1	-1	-1	1	0	0			-1	1	1	-1
24	0	0		-1	1	-1	1	-1	1	1	-1			0	0
13	0	0		1	-1	1	-1	1	-1	-1	1	0	0		
25	0	1	-1		0	-1	1	1		-1	0	1	0	-1	
16	0	-1	1		0	1	-1	0	1		-1		1	0	-1
15	0	1	-1	0		1	-1	-1	0	1		-1		1	0
26	0	-1	1	0		-1	1		-1	0	1	0	-1		1
36	0	-1	1	1	-1		0	-1		1	0	1		-1	0
45	0	1	-1	-1	1		0	0	-1		1	0	1		-1
46	0	-1	1	-1	1	0		1	0	-1		-1	0	1	
35	0	1	-1	1	-1	0			1	0	-1		-1	0	1

The collapsed incidence matrix of F_3

	12 34 56	14 23 56	13 24 56	16 25 34	15 26 34	12 36 45	12 35 46	14 26 35	14 25 36	16 23 45	15 23 46	16 24 35	15 24 36	13 26 45	13 25 46
56	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	−1	1	−1	1	1	−1	1	−1
12	0	0	0	0	0	0	0	1	−1	1	−1	−1	1	−1	1
14	0	0	0	−1	1	1	−1	0	0	0	0	1	−1	−1	1
23	0	0	0	1	−1	−1	1	0	0	0	0	−1	1	1	−1
24	0	0	0	−1	1	−1	1	−1	1	1	−1	0	0	0	0
13	0	0	0	1	−1	1	−1	1	−1	−1	1	0	0	0	0
25	0	1	−1	0	0	−1	1	1	1	−1	0	1	0	−1	−1
16	0	−1	1	0	0	1	−1	0	1	−1	−1	1	1	0	−1
15	0	1	−1	0	0	1	−1	−1	0	1	1	−1	−1	1	0
26	0	−1	1	0	0	−1	1	−1	−1	0	1	0	−1	1	1
36	0	−1	1	1	−1	0	0	−1	−1	1	0	1	1	−1	0
45	0	1	−1	−1	1	0	0	0	−1	1	1	0	1	−1	−1
46	0	−1	1	−1	1	0	0	1	0	−1	−1	−1	0	1	1
35	0	1	−1	1	−1	0	0	1	1	0	−1	−1	−1	0	1

The combined collapsed incidence matrices form a formal matrix M.

- The matrix M is a formal $|\mathcal{P}| \times |\mathcal{B}|$-matrix, where $\mathcal{P}$ and $\mathcal{B}$ are the point- and the line-set of W_2, respectively.
- Let us for simplicity refer to the entries by pairs (P, l) where $P \in \mathcal{P}$ and $l \in \mathcal{B}$.
- Thus
$$M = (m_{P,l})_{P \in \mathcal{P}, l \in \mathcal{B}}.$$
Now we define a formal 45×45 matrix A from W. the rows and columns will be indexed by the 45 flags of W_2.
- In particular
$$A = (a_{(P_1,l_1),(P_2,l_2)}), \text{ where } a_{(P_1,l_1),(P_2,l_2)} := m_{(P_2,l_1)} \cdot m^{-1}_{(P_1,l_2)}.$$

Theorem

- *The formal matrix A is a* $\mathrm{GHM}(\mathbb{Z}_3; 45, 3, 12)$.
- *Therefore this matrix gives rise to a regular* $(45, 3, 12)$*-cover of* K_{45} *with the voltage group* $\mathbb{Z}_3$.

Remark:

Still the computer was asked for the proof.

Outline

Further perspectives

- Inspection of known gH-matrices with the goal to get new DRGs.
- Careful analysis and comparison of known constructions on antipodal covers of K_n.
- More links with topological graph theory.
- Move example on 135 vertices to a possible generic construction.

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