On the enumeration of skew-morphisms of cyclic groups

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Definition and examples

After Jajcay and Širáň, we say that a permutation $\sigma : G \to G$ is a **skew-morphism** of a group G with a **power function** $\pi\colon G \to \{0, 1, \ldots, \text{ord}(\sigma) - 1\}$ if

- $\sigma(1_G) = 1_G$,
- $\sigma(xy) = \sigma(x)\sigma^{\pi(x)}(y)$ for all $x, y \in G$.

EXM: every automorphism of G is a SM of G with $\pi(x) = 1$ for all $x \in \mathbb{Z}_n$.

EXM: $G = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$.

The permutation

$$\sigma(x) = \begin{cases} x & \text{if } x \text{ is even} \\ x \oplus 2 & \text{if } x \text{ is odd} \end{cases}$$

is a SM of $\mathbb{Z}_6$ with power function

$$\pi(x) = \begin{cases} 1 & \text{if } x \text{ is even} \\ 2 & \text{if } x \text{ is odd} \end{cases}$$

$$\sigma(x \oplus y) = \sigma(x) \oplus \sigma^{\pm}(y).$$

Motivation

• A group G admits a regular Cayley map iff G has a skew-morphism σ with an orbit T such that $\langle T\rangle = G$, and $T = T^{-1}$ (Jajcay and Širáň, 2002)

• A classification of regular embeddings of $K_{n,n}$ is equivalent to find the set of SM's σ of $\mathbb{Z}_n$ s. t. ord(σ) $\mid n$, and $\pi(x) = -\sigma^{-x}(-1)$ for all $x \in \mathbb{Z}_n$ (Feng, Kwak, Nedela).

(Du, Jones, Kwak, Kwon, Nedela, Škoviera, Zlatoš 2002-2007)

Skew product group

Prop. (i) Let σ be a SM of G, G_L is the left regular repr. of G. Then $P = \langle G_L, \sigma \rangle$ is a permutation group in Sym(G) such that $P_{1_G} = \langle \sigma \rangle$.

(ii) Let $P \leq$ Sym(G) such that $G_L \leq P$, and P_{1_G} is a cyclic group. Then any generator of P_{1_G} is a SM of G.

After Conder, Jajcay and Tucker, we call $\langle G_L, \sigma \rangle$ the **skew product group** over G induced by σ.

Action on a generating orbit

Prop. If σ is a SM of G, and T is an orbit of σ, $\langle T \rangle = G$, then σ acts regularly on T.

In particular, for the orbit T in the above proposition, $\mathrm{ord}(T) = |T|$.

S-rings induced by SM's

S-rings over G are certain subalgebras of the group algebra $\mathbb{Q}G$.

Some notations: for $S \subset G$, denote $\underline{S} = \sum_{x \in G} a_x x \in \mathbb{Q}G$ such that $a_x = 1$ if $x \in S$ and $x = 0$ if $x \notin S$ (such elements are called simple quantities). The transpose of an element $\eta = \sum_{x \in G} a_x x$ in $\mathbb{Q}G$ is the element $\eta^t = \sum_{x \in G} a_x x^{-1}$.

An **S-ring** over G is a subalgebra $\mathcal{A}$ of $\mathbb{Q}G$ which satisfies:

- $\mathcal{A}$ has a basis of elements $\underline{T}_1, \ldots, \underline{T}_r$ for subsets T_i of G,
- $T_1 = \{1_G\}$, $T_i \cap T_j = \emptyset$ for all i, j, and $T_1 \cup \cdots \cup T_r = G$,
- for every $i \in \{1, \ldots, r\}$ there exists $j \in \{1, \ldots, r\}$ such that $\underline{T}_i^t = \underline{T}_j$.

T_i: basic sets, r: the rank.

S-rings arise from permutation groups as follows: Let P be of a rank r permutation group in Sym(G). Suppose $G_L \leq P$, and let P_{1_G} be the stabilizer of 1_G in P. Denote by $T_{1_G} = \{1_G\}, T_2 \ldots, T_r$ the the orbits of P_{1_G}.

Then $T_1, \ldots, T_r$ are the basic sets of a rank r S-ring over G (Schur, 1933) – also called the transitivity module of G induced by P_{1_G}.

Every SM σ of G induces an S-ring over G, we denote this by $\mathcal{A}_\sigma$.

S-rings over cyclic groups have been developed much (Klin, Pöschel, Muzychuk, Leung, Ma, Man, Evdokimov and Ponomarenko).

The following related question seems to be natural.

Question. What are the S-rings over cyclic groups which are induced by skew-morphisms?

Theorem. Let $n = p_1^{e_1} \cdots p_r^{e_r}$, n is odd, $(p_i, p_j - 1) = 1$ for all i, j. Then for every SM σ of $\mathbb{Z}_n$ there exists an automorphism α of $\mathbb{Z}_n$ s. t. $\mathcal{A}_\sigma = \mathcal{A}_\alpha, \alpha \in \mathrm{Aut}(\mathbb{Z}_n)$.

SM's with trivial radicals

Motivated by Evdokimov and Ponomarenko, we define the **radical** of a SM σ of $\mathbb{Z}_n$ as

$$\mathrm{rad}(\sigma) = \{x \in \mathbb{Z}_n \mid T + x = T\},$$

where T is an orbit of $\sigma, \langle T \rangle = \mathbb{Z}_n$.

$\mathrm{rad}(\sigma) \leq \mathbb{Z}_n$ (does not depend on the orbit T).

In fact, $\mathrm{rad}(\sigma)$ is the same as $\mathrm{rad}(\mathcal{A}_\sigma)$.

Prop. If σ is a SM of $\mathbb{Z}_n$. If $\mathrm{rad}(\sigma) = 1$, then σ is in $\mathrm{Aut}(\mathbb{Z}_n)$.

This follows directly from a structure theorem of S-rings over cyclic groups with trivial radical (Evdokimov and Ponomarenko, 2002).

As corollaries, we obtain that

- $(\mathrm{ord}(\sigma), n) = 1 \Rightarrow \sigma \in \mathrm{Aut}(\mathbb{Z}_n)$,
- $\mathrm{ord}(\sigma) \mid n\varphi(n)$.

SM's of prime order

Prop. Let σ be a SM of $\mathbb{Z}_n$, ord$(\sigma) = p$, p is a prime, σ is not in Aut$(\mathbb{Z}_n)$, and let π be the power function of σ. Then

- $n = pm$, $d = (m, p-1) > 1$, and
- there are $a, b \in \mathbb{Z}_p$, $a \neq 0$, $b \neq 1$, $b^d = 1$, such that

$$\sigma(xm + y) = \Big(x + a(1 + b + \cdots + b^{y-1})\Big)m + y,\ x \in \mathbb{Z}_p,\ y \in \mathbb{Z}_m.$$

and $\pi(z) = b^z$, $z \in \mathbb{Z}_n$.

EXM: Let $n = pq$, $(n, \varphi(n)) = 1$, σ be a SM of $\mathbb{Z}_{pq}$.

Suppose that rad$(\sigma) = \langle q \rangle$. Then ord$(\sigma) = pl$, and the $\langle q \rangle$-cosets form a block system of $G = \langle 1_L, \sigma \rangle$. The kernel of G permuting the blocks is the group $K = \langle q_L, \sigma^l \rangle$. Thus the stabilizer of 0 in $K\mathbb{Z}_L$ is $\langle \sigma^l \rangle$, so σ^l is a SM of order p, σ^l is not in Aut$(\mathbb{Z}_n)$, a contradiction to the previous Prop.

Therefore rad$(\sigma) = 1$, σ is in Aut$(\mathbb{Z}_{pq})$.

Letting $\mathbb{Z}_{pq} = \mathbb{Z}_p \times \mathbb{Z}_q$, Aut$(\mathbb{Z}_{pq})$ = Aut$(\mathbb{Z}_p) \times$ Aut$(\mathbb{Z}_q)$, we have $\sigma = \sigma_1\sigma_2$, $\sigma_1 \in$ Aut$(\mathbb{Z}_p)$ and $\sigma_2 \in$ Aut$(\mathbb{Z}_q)$.

Decomposition Theorem

Let $n = n_1 n_2$, $(n_1, n_2) = 1$, $\mathbb{Z}_n = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$. Let σ_1 be a SM of $\mathbb{Z}_{n_1}$. The the mapping

$$\widehat{\sigma}_1 \colon (x_1, x_2) \mapsto (\sigma_1(x_1), x_2)$$

is a SM of $\mathbb{Z}_n$, this we call the **extension** of σ_1 to $\mathbb{Z}_n$.

Extension $\widehat{\sigma}_2$ is defined analogously.

We say that two natural numbers n_1 and n_2 are **disjoint** if

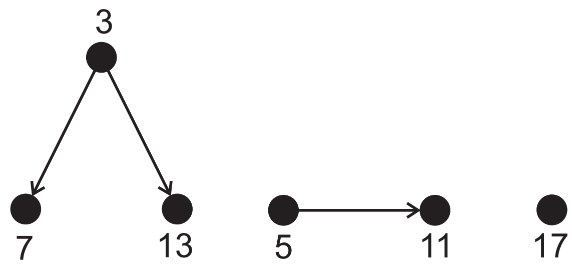
$$(n_1 \varphi(n_1), n_2 \varphi(n_2)) = 1.$$

Note that, if n_1 and n_2 are disjoint, then $\widehat{\sigma}_1 \widehat{\sigma}_2$ is a SM of $\mathbb{Z}_n$.

Theorem. Let $n = n_1 n_2$, n_1 and n_2 are disjoint. Then every SM of $\mathbb{Z}_n$ is of the form $\widehat{\sigma}_1\,\widehat{\sigma}_2$, where σ_i is a SM of $\mathbb{Z}_{n_i}$.

We can interpret the theorem in terms of a digraph (which also occurs in the formula of Jones, 2007, enumerating regular embeddings of $K_{n,n}$.)

EXM: $n = 3 \times 5 \times 7 \times 11 \times 13 \times 17$.



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$\varphi_s(n) = \varphi_s(3 \times 7 \times 13) \times \varphi_s(5 \times 11) \times \varphi_s(17)$

$n = p^m$, p is a prime

Lem. Let σ be a SM of $\mathbb{Z}_{p^n}$, p is a prime. If $\mathrm{ord}(\sigma) = p^v$, then σ^p is also a SM of $\mathbb{Z}_{p^n}$.

Therefore, if G is a skew product p-group, then every element in G_0 is a SM. Enumeration in this case is reduced to finding the poset of skew product p-groups.

Note that, if $p = 2$, then all SM's are found this way.

Lem. Let σ be a SM of $\mathbb{Z}_{p^n}$ of order $p^v d$, $d > 1$, $d \mid (p-1)$ $(p > 2)$. Then
(1) the Sylow-p-subgroup P of $\langle \tau, \sigma \rangle$ is a skew product p-group.
(2) $P \triangleleft \langle \tau, \sigma \rangle$, hence σ^{p^v} acts by conjugation on P as an automorphism of order d.

Note that,

- Skew product p-groups are metacyclic if $p > 2$ (Huppert, 1953).

- A nonsplit metacyclic p-group is also a p-group (Menegazzo, 1993), hence the group P in the above lemma must be a split metacyclic p-group.

- The automorphism groups of split metacyclic p-groups are explicitely described by Bidwell and Curran, 2006.

We have the following strategy of enumerating SM's of $\mathbb{Z}_{p^n}$:

- Determine the poset of skew product p-groups.

- If $p > 2$, then determine those groups which are split meta-cyclic p-groups.

- If $p > 2$ and P is a split metacyclic skew product p-group, then describe a correspondence between skew product groups G with $P = \mathrm{Syl}_p(G)$ and $\mathrm{Aut}(P)$.

- Derive formula for $\varphi_s(p^n)$.

Skew product p-groups, $p > 2$

Let α be the automorphism of $\mathbb{Z}_{p^n}$ defined by $\alpha(x) = (p+1) \odot x$.

For $i, j \in \{0, 1, \ldots, p^{n-1}-1\}$, define the permutations $\beta_j, \sigma_{i,j}$ in Sym($\mathbb{Z}_{p^n}$) as $\beta_j(0) = 0$, and if $x \neq 0$,

$\beta_j(x) = 1 \oplus (p+1)^j \oplus (p+1)^{2j} \oplus \cdots \oplus (p+1)^{(x-1)j}$, and

$\sigma_{i,j} = \beta_j{}^{-1} \alpha^i \beta_j,\ i \in \{0, 1, \ldots, p^{n-1} - 1\}$.

Lem. Every permutation $\sigma_{i,j}$ is a SM of $\mathbb{Z}_{p^n}$.

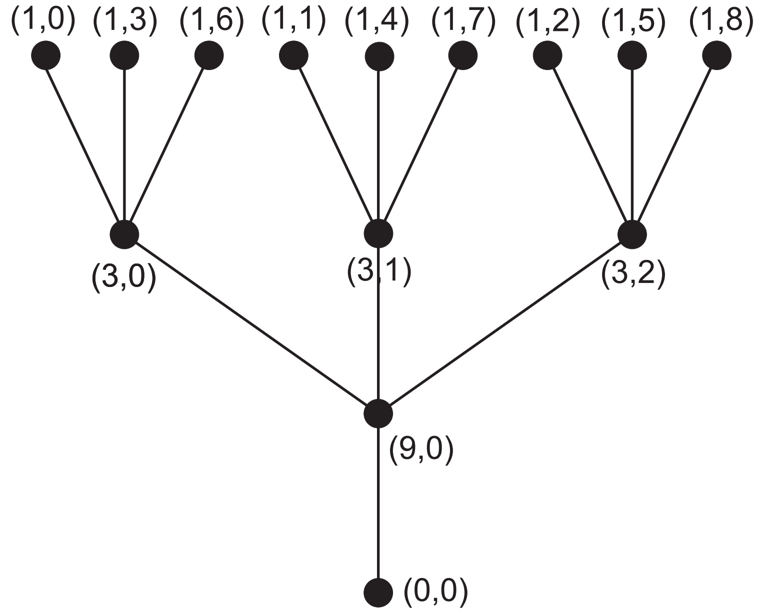
Prop. If σ is a SM of $\mathbb{Z}_n$ of p-power order and $p > 2$, then $\sigma = \sigma_{i,j}$ for some $i, j \in \mathbb{Z}_{p^n}$.

Proof First, let $\mathrm{ord}(\sigma) = p^{n-1}$. Let $G = \langle 1_L, \sigma \rangle$. There exists a cyclic subgroup $N \triangleleft G$, G/N is cyclic. Thus $|G/N| = \max\{\mathrm{ord}(N1_L), \mathrm{ord}(N\sigma)\} = \max\{\mathrm{ord}(N1_L), p^{n-1}\} = p^{n-1}$, as $\langle p^{n-1} \rangle \leq \mathbf{Z}(G) \cap N$. Also, $|N| = p^n$, hence N is a regular normal subgroup. Then $N^\beta = \mathbb{Z}_L$, $\sigma^\beta = \alpha^i$. From these $\beta = \beta_j$, $\sigma = \sigma_{i,j}$.

The proof is completed by showing that every skew product p-group of order less then p^{2n-1} is properly contained in a skew product p-group.

Prop. $\sigma_{i,j} = \sigma_{i',j'}$ iff $i = i'$ and $j \equiv j' \pmod{p^{n-2}/(p^{n-i}, i)}$.

EXM: The poset of Skew product 3-groups over $\mathbb{Z}_{81}$, label (i, j) identifies group $\langle 1_L, \sigma_{i,j} \rangle$:



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The number of SM's of $\mathbb{Z}_{p^n}$ of p-power order is $\frac{p^{2n-2}+p}{p+1}$.

Thank you!