Half-arc-transitive graphs of order $4p$ and accompanying combinatorial structures

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This is a joint work with D. Marušič and P. Šparl and (Ru-Ji Wang, Ming-Yao Xu).

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Different types of transitivity

- A graph is vertex-transitive if its automorphism group acts transitively on vertices.
- A graph is edge-transitive if its automorphism group acts transitively on edges.
- A graph is arc-transitive (also called symmetric) if its automorphism group acts transitively on arcs.
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Tutte

The first result linking vertex-transitivity and edge-transitivity to arc-transitivity.

Tutte, 1966

A vertex-transitive and edge-transitive graph of odd valency is arc-transitive.

Half-arc-transitive graphs are of even valency.

The first examples of HAT graphs

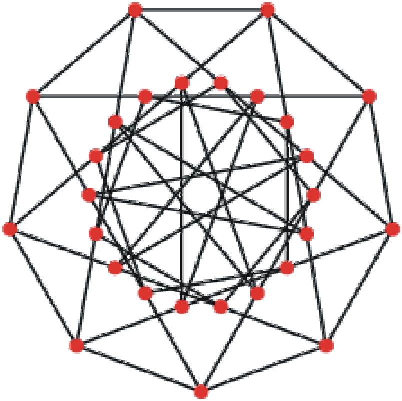
The first examples of half-arc-transitive graphs were given by Bouwer (1970), he constracted a $2k$-valent HAT graph for every $k \geq 2$.

The smallest example constraced by Bouwer had 54 vertices and was quartic.

Dolye (1976) and Holt (1981) subsequently discovered the graph on 27 vertices, now known as the Doyle-Holt graph.

The Doyle - Holt graph

The smallest HAT graph.



Current research on HAT graphs

Papers dealing with

- the constraction problem of HAT graphs,
- the classification problem of HAT graphs,

of particular order or valency (Alspach, Conder, D'Azevedo, Feng, Kwak, Li, Nedela, Malnič, Marušič, Pisanski, Praeger, Sim, Šajna, Šparl, Waller, Wang, Wilson, Zhou, Xu).

There are several approaches that are currently being taken, such as for example, investigation of (im)primitivity of half-arc-transitive group actions on graphs, and geometry related questions about HAT graphs.

HAT graphs of a particular order

Cheng, Oxley, 1987

There is no HAT graph of order $2p$, where p is a prime.

Alspach, Marušič, Nowitz, 1994

There is no HAT graph of order p or p^2, where p is a prime.

Alspach, Xu, 1994 & Wang, 1994

Classification of HAT graphs of order a product of two primes.

Metacirculants

An *(m,n)-metacirculant* has an *(m,n)-semiregular* automorphism, that is an automorphism ρ with a cycle decomposition

$$\rho = (v_0^0\, v_1^0 \ldots v_{n-1}^0)(v_0^1\, v_1^1 \ldots v_{n-1}^1)\ldots(v_0^{m-1}\, v_1^{m-1} \ldots v_{n-1}^{m-1})$$

and an automorphism τ which cyclically permutes the orbits

$$V_i = \{v_0^i, v_1^i, \ldots, v_{n-1}^i\}\ (i \in Z_m)$$

of ρ mapping according to the rule $\tau(v_j^i) = v_{rj}^{i+1}$ for all i and j, where $r \in Z_n^*$ such that $r^h \equiv 1 \pmod n$ for some multiple h of m.

HAT graphs of order $4p$, p a prime

Feng, Wang, Zhou, 2007

Classification of tetravalent HAT graphs of order $4p$, p a prime.

Feng, Wang, 2009

Classification of hexavalent HAT graphs of order $4p$, p a prime.

Our aim

A HAT graph of order $4p$, p a prime, is a metacirculant or it belongs to a "sporadic family" of such graphs.

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$p \geq 5$

In 1994 Alspach, Marušič and Nowitz proved that the smallest HAT graph has 27 vertices, and so there is no HAT graphs of order $4p$, $p \geq 5$.

Hence, we shall be assuming that $p \geq 7$.

Anaysis of graphs of order $4p$

The strategy in obtaining the classification of HAT graphs of order $4p$ is based on an analysis with respect to the (im)primitivity of the automorphism group.

(Im)primitivity of the automorphism group I

Non-trivial blocks in VTG of order $4p$ can be of size: 2, 4, p or $2p$.

With the respect to the action of the automorphism groups we divide all VTG of order $4p$, into eight classes:

Class I	$(4p)$	*Class V*	$(p:2:2)$
Class II	$(2:2p)$	*Class VI*	$(p:4)$
Class III	$(2p:2)$	*Class VII*	$(4:p)$
Class IV	$(2:p:2)$	*Class VIII*	$(2:2:p)$

(Im)primitivity of the automorphism group II

Class I	$(4p)$

Primitive graph.

Class IV	$(2 : p : 2)$

VTG of order 4p belongs to *Class IV* if $\mathrm{Aut}X$ has an imprimitivity system $\mathcal{B}$ of $2p$ blocks of size 2 and the action of $\mathrm{Aut}X/K$ on $X_{\mathcal{B}}$ has 2 blocks of size p, where K is the kernal of the action of $\mathrm{Aut}X$ on $\mathcal{B}$.

Class V	$(p : 2 : 2)$

VTG of order 4p belongs to *Class V* if $\mathrm{Aut}X$ has an imprimitivity system $\mathcal{B}$ of $2p$ blocks of size 2 and the action of $\mathrm{Aut}X/K$ on $X_{\mathcal{B}}$ has p blocks of size 2, where K is the kernal of the action of $\mathrm{Aut}X$ on $\mathcal{B}$.

We analyze the properties of vertex-transitive and edge-transitive graphs belonging to these eight classes, with special emphasis given to half-arc-transitivity.

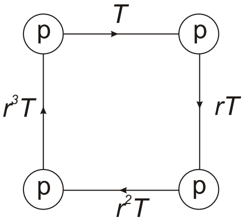
Proposition

A vertex-transitive and edge-transitive graph of order $4p$, p a prime, is arc-transitive or it belongs either to Class VI $(p : 4)$ or to Class VIII $(2 : 2 : p)$.

Class VIII $(2:2:p)$

Proposition

Let X be a HAT graph of order $4p$, p a prime, belonging to Class VIII then $p \equiv 1 \pmod 4$ and there exists a divisor $d \neq 1$ of $\frac{p-1}{4}$ such that X is isomorphic to $M(d;4,p)$.



$M(d;4,p)$ given in Frucht's notation where $r \in \mathbb{Z}_p^*$ is such that $\langle r \rangle$ is the unique subgroup of order $4d$ in $\mathbb{Z}_p^*$ and $T = \langle r^4 \rangle$ the unique subgroup of order d in $\mathbb{Z}_p^*$.

Class VI ($p : 4$)

Almost proposition

Let X be a HAT graph of order $4p$, p a prime, belonging to Class VI then $p \equiv 1 \pmod 6$ and there exists $r = 2^k$, $k > 1$, such that $p = r^2 + r + 1 > 7$ and X is isomorphic either to $X_1(r, 4p)$ or to $X_2(r, 4p)$.

To complete the classification of HAT graphs of order $4p$, p a prime, we only need to show that $p = r^2 + r + 1$. So far we managed to prove that p divides $r^2 + r + 1$.

The "sporadic family"

Let r be an integer and let n be a diviser of $r^2 + r + 1$.

$X_1(r;4n)$ is the graph with $V(X_1(r;4n)) = Z_4 \times Z_n$ and the adjacencies:

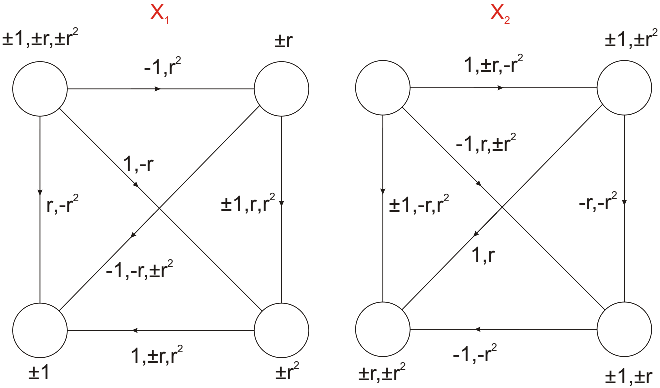
$$\begin{array}{lcl}
(0,i) \sim (0,j) & \iff & j-i \in \{\pm 1, \pm r, \pm r^2\}, \\
(0,i) \sim (1,j) & \iff & j-i \in \{-1, r^2\}, \\
(0,i) \sim (2,j) & \iff & j-i \in \{1, -r\}, \\
(0,i) \sim (3,j) & \iff & j-i \in \{r, -r^2\}, \\
(1,i) \sim (1,j) & \iff & j-i \in \{\pm r\}, \\
(1,i) \sim (2,j) & \iff & j-i \in \{\pm 1, r, r^2\}, \\
(1,i) \sim (3,j) & \iff & j-i \in \{-1, -r, \pm r^2\}, \\
(2,i) \sim (2,j) & \iff & j-i \in \{\pm r^2\}, \\
(2,i) \sim (3,j) & \iff & j-i \in \{1, \pm r, r^2\}, \\
(3,i) \sim (3,j) & \iff & j-i \in \{\pm 1\}.
\end{array}$$

$X_2(r;4n)$ is the graph with $V(X_1(r;4n)) = Z_4 \times Z_n$ and the adjacencies:

$$\begin{array}{lcl}
(0,i) \sim (1,j) & \iff & j-i \in \{1, \pm r, -r^2\}, \\
(0,i) \sim (2,j) & \iff & j-i \in \{-1, r, \pm r^2\}, \\
(0,i) \sim (3,j) & \iff & j-i \in \{\pm 1, -r, r^2\}, \\
(1,i) \sim (1,j) & \iff & j-i \in \{\pm 1, \pm r^2\}, \\
(1,i) \sim (2,j) & \iff & j-i \in \{-r, -r^2\}, \\
(1,i) \sim (3,j) & \iff & j-i \in \{1, r\}, \\
(2,i) \sim (2,j) & \iff & j-i \in \{\pm 1, \pm r\}, \\
(2,i) \sim (3,j) & \iff & j-i \in \{-1, -r^2\}, \\
(3,i) \sim (3,j) & \iff & j-i \in \{\pm r, \pm r^2\}.
\end{array}$$

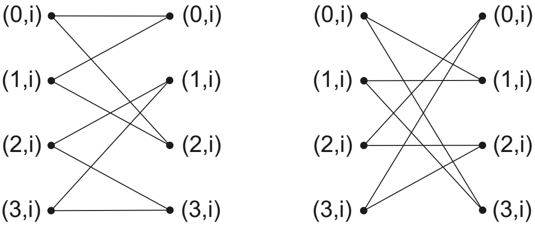
Observe that r is an element of order 3 in Z_n^*, and that the edge-disjoint union $X_1(r;4n) \cup X_2(r;4n)$ is the wreath product $Y[4K_1]$ where $Y = \mathrm{Cay}(Z_n, \{\pm 1, \pm r, \pm r^2\})$.

$X_1(r;4n)$ & $X_2(r;4n)$ given in Frucht's notation



$X_1(r;4n)$ & $X_2(r;4n)$

Bipartite graphs induced by two adjacent blocks.



The smallest HAT graphs in the "sporadic family"

The smallest HAT graphs in the "sporadic family":

$p = 73$ & $r = 2^3$

Thank you !