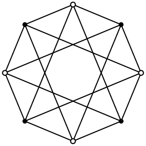


On tetravalent bicirculants

Boštjan Kuzman

Joint work with I. Kovacs, A. Malnič and S. Wilson

July 2nd, 2009
GEMS 2009, Tale, Slovakia.

Definition

Let X be a connected finite simple graph on $2n$ vertices. Then X is a *bicirculant* $\iff$ $\exists \alpha \in \mathrm{Aut}(X)$ with 2 orbits of size n.

Notation: $X \in \mathrm{BC}_n[L, M, R]$, where

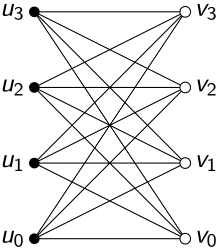
$$V(X) = \{u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}\}, i \in \mathbb{Z}_n$$

$$E(X) = \bigcup_{i \in \mathbb{Z}_n} \begin{cases} (u_i, u_{i+t}), & t \in L, \\ (u_i, v_{i+t}), & t \in M, \\ (v_i, v_{i+t}), & t \in R, \end{cases}$$

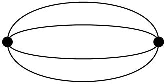
with $L, M, R \subseteq \mathbb{Z}_n$, $M \neq \emptyset, 0 \notin L \cup R$.

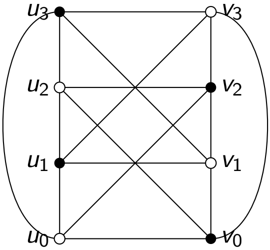
and α: $\begin{cases} u_i \mapsto u_{i+1} \\ v_i \mapsto v_{i+1} \end{cases} \in \mathrm{Aut}(X)$ implies $L = -L, R = -R$.

Example ($K_{4,4}$)

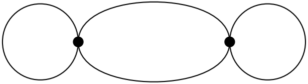


$\mathrm{BC}_4[\emptyset, \{0,1,2,3\}, \emptyset]$

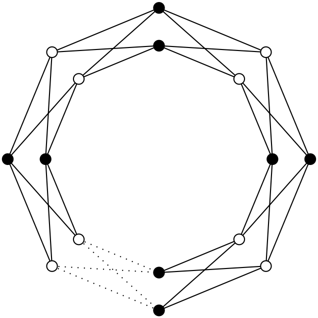




$\mathrm{BC}_4[\{\pm 1\}, \{0,2\}, \{\pm 1\}]$



Example (Wreath graphs)



For $n \geq 3$, $C_n[2K_1] \cong \begin{cases} BC_n[\{\pm 1\}, \{0, 2\}, \{\pm 1\}] & \forall n, \\ BC_n[\emptyset, \{0, 1, \frac{n}{2}, \frac{n}{2} + 1\}, \emptyset] & n \text{ even.} \end{cases}$

Example (Dihedrants)

Any $X = \mathrm{Cay}(D_n, S)$ is a bicirculant. The reverse is not true.

However, if $L = R$, then

$$\mathrm{BC}_n[L, M, R] \cong \mathrm{Cay}(D_n, \{a^i, ba^j \mid i \in L, j \in M\}).$$

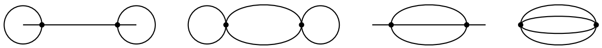
Problem: Classify all 4-valent edge transitive bicirculants. Thus, describe such $BC_n[L, M, R]$ in terms of n, L, M, R.

Motivation:

- Frucht, Graver and Watkins have classified all ET generalized Petersen graphs.
- The results of Pisanski and Marušič later completed the classification of all cubic ET bicirculants.

General strategy: deal with cases $|M| = 1, 2, 3, 4$ separately.

Note that we may treat these graphs as regular $\mathbb{Z}_n$-covers of one of the following *pregraphs* on two vertices:

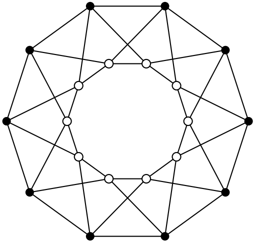


However, for $|M| = 1, 3$, no 4-valent $BC_n[L, M, R]$ is edge transitive.

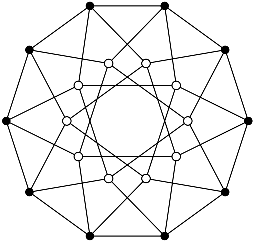
Case $|M| = 2$: Rose Windows

At GEMS '97, S. Wilson presented some examples of regular maps and their underlying graphs. Inspired by three of those, he defined a family of *the Rose Window graphs*:

$$RW_n(r, m) = BC_n[\{\pm 1\}, \{0, m\}, \{\pm r\}].$$



$RW_{10}(2, 1)$



$RW_{10}(2, 3)$

Theorem (Kovacs, Kutnar, Marušič)

Let X be an edge transitive Rose Window graph. Then X is isomorphic to $BC_n[L, M, R]$, $n \geq 3$, where $L = \{\pm 1\}$, $M = \{0, m\}$, $R = \{\pm r\}$, and one of the following applies for n, m, r:

#	n	m	r	*Conditions*
I.	*any*	2	1	/
II.	$2k$	$k+2$	$k+1$	/
III.	$2k$	$2y$	$\begin{cases} 1 \mod 2 \\ 1 \mod k \end{cases}$	$y^2 \equiv \pm 1 \mod k$
IV.	$12k$	$2+3y$	$1+9y$	$y = \pm k$

Proof.

$\mathrm{Core}_H\, G \equiv$ the largest normal subgroup of G, contained in $H \leq G$.
Take $G = \mathrm{Aut}(X)$, $H =< (u_0 \dots u_{n-1})(v_0 \dots v_{n-1}) >$ and study cases $|\,\mathrm{Core}\,| = 1$, $1 < |\,\mathrm{Core}\,| < n/2$, $|\,\mathrm{Core}\,| = n/2$. $\square$

Now define the *generalized Rose Window graphs* as

$$GRW_n(l, m, r) = BC_n[\{\pm l\}, \{0, m\}, \{\pm r\}].$$

Theorem (Kovacs, K., Malnič, Wilson)

Every ET generalized Rose Window graph is isomorphic to some Rose Window graph.

Thus, case $|M| = 2$ is solved.

Case $|M| = 4$: Bipartite dihedrants

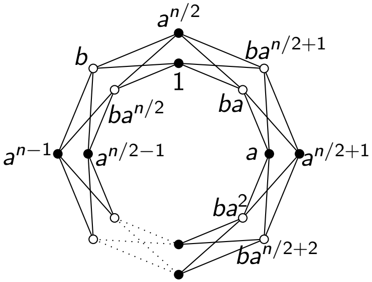
Observe that

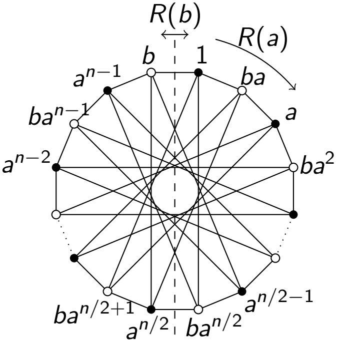
$$BC_n[\emptyset, \{0, x, y, z\}, \emptyset] \cong \mathrm{Cay}(D_n, \{b, ba^x, ba^y, ba^z\},$$

so these graphs are bipartite dihedrants with bipartition sets $\{1, a, \ldots, a_{n-1}\}$ and $\{b, ba, \ldots, ba_{n-1}\}$, where $D_n =< a, b \mid a^n = b^2 = (ab)^2 = 1 >$.

A Cayley graph $X = \mathrm{Cay}(G, S)$ is called *normal*, if the right regular representation $R(G)$ is normal in $\mathrm{Aut}(X)$.

Example (Wreath graph)





For $n \geq 4$ even, $C_n[2K_1] \cong \mathrm{Cay}(D_n, \{b, ba, ba^{n/2}, ba^{n/2+1}\})$ is an AT *non-normal bipartite dihedrant*.

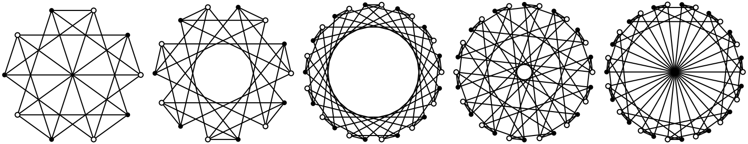
Note that $\tau = (1a^{n/2}) \in \mathrm{Aut}(X)$, but $\tau R(a) \tau^{-1} \notin R(G)$.

For $n \geq 3$ odd, $C_n[2K_1] \cong \mathrm{Cay}(D_n, \{b, ba^2, a, a^{n-1}\})$ is an also AT *non-normal dihedrant*, but not bipartite.

Theorem (Kovacs, K., Malnič)

Let $X = \mathrm{Cay}(D_n, S)$ be a connected 4-valent arc transitive dihedrant. If X is non-normal and bipartite (in the above sense), then X is isomorphic to $C_n[2K_1]$ with $n \geq 4$ even, or to one of the five sporadic graphs:

- $n = 5$, $S = \{b, ba, ba^2, ba^3\}$,
- $n = 7$, $S = \{b, ba, ba^2, ba^4\}$,
- $n = 13$, $S = \{b, ba, ba^3, ba^9\}$,
- $n = 14$, $S = \{b, ba, ba^4, ba^6\}$,
- $n = 15$, $S = \{b, ba, ba^3, ba^7\}$.



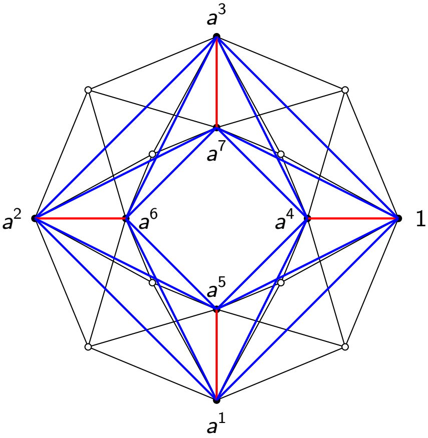
Sketch of proof

Sketch of proof:

- Non-normal 4V AT BP Dih are not 1-regular.
- For X non-normal 4V AT BP Dih we define *the auxiliary circulant*

$$AuxCirc(X) = \frac{1}{2}X^2 \approx \mathrm{Cay}(Z, T),$$

  where $T = \{a^{\pm x}, a^{\pm y}, a^{\pm z}, a^{\pm(x-y)}, a^{\pm(y-z)}, a^{\pm(z-x)}\}$.
- $AuxCirc(X) = Y$ or $Y_1 + Y_2$ for some orbital graphs, defined via the action of G_1 on T. According to their properties we assign to each X *an uniformity index* (r_1, r_2) and analyze possible cases for $1 \leq r_1, r_2 \leq 4$.



$$AuxCirc(C_8[2K_1]) = C_4[2K_1] + 4K_2.$$

$C_n[2K_1]$ graphs have uniformity index $(4,2)$ for $n \geq 6$ even.

Case analysis for uniformity index (r_1, r_2):

- $r_1 \neq r_2$:
  - $r_1 = 4 \neq r_2 \implies X \approx C_n[2K_1], n \geq 6$ even.
  - $r_1 = 3 \neq r_2$ and $r_1 = 2 \neq r_2$ are excluded by algebraic arguments.
  - $r_1 = 1 \neq r_2$ is excluded by application of the result of Baik, Feng, Sim and Xu on non-normal Cayley graphs of abelian groups.
- $r_1 = r_2 \neq 1$:
  - $r_1 = r_2 = 4 \implies X \approx K_{4,4} = C_4[2K_1]$.
  - $r_1 = r_2 = 3 \implies X \approx K_{5,5} - 5K_2$.
  - $r_1 = r_2 = 2 \implies X \approx NonincidenceGraph(PG(2,2))$.
- $r_1 = r_2 = 1$:
  The classification of AT circulants by I. Kovacs gives a description of connection sets for the orbital circulants in $AuxCirc(X)$. We use this to show that X/J for some $J < Z$ increases uniformity index. Further analisys gives 3 more possible X.

Case $|M| = 4$: Normal case

If ET $X = \mathrm{BC}_n[\emptyset, M, \emptyset]$ is normal, then G_1 acts transitively as affine simmetries on $M = \{0, x, y, z\}$ as either $\mathbb{Z}_4$ or $\mathbb{Z}_2^2$.

Algebraicaly, this yields 2 sets of equations, describing x, y, z.

- $\mathbb{Z}_4$ One infinite family of solutions.
- $\mathbb{Z}_2^2$ Two infinite families of solutions (some/none of x, y, z are invertible).

Theorem (Kovacs, K., Malnič, Wilson)

Let X be an edge transitive tetravalent bicirculant. Then X is isomorphic to $BC_n[L, M, R]$, $n \geq 3$, where

1. $L = \{\pm 1\}$, $M = \{0, m\}$, $R = \{\pm r\}$, *and one of the following applies for* n, m, r:

#	n	m	r	*Conditions*
I.	*any*	2	1	/
II.	$2k$	$k+2$	$k+1$	/
III.	$2k$	$2y$	$\begin{cases} 1 \mod 2 \\ 1 \mod k \end{cases}$	$y^2 \equiv \pm 1 \mod k$
IV.	$12k$	$2+3y$	$1+9y$	$y = \pm k$

2. $L = R = \emptyset$, $M = \{0, x, y, z\}$, *and one of the following applies for* n, x, y, z:

#	n	x, y, z	*Conditions*
I.	5	$1, 2, 3$	/
II.	7	$1, 2, 4$	/
III.	13	$1, 3, 9$	/
IV.	14	$1, 4, 6$	/
V.	15	$1, 3, 7$	/
VI.	$2k$	$1, k, k+1$	/
VII.	*any*	$1, k+1, k^2+k+1$	$(k+1)(k^2+1) \equiv 0 \mod n$
VIII.	*any*	$1, k, 1-k$	$2k(1-k) \equiv 0 \mod n$
IX.	$krst$	$r, (rs'+t)s, (rt'+s+rs)t$	$1 < r, s, t$ *pairwise coprime, etc.*

To do:

- Determine the overlap of different families.
- Identify dihedrants within case $|M| = 2$.
- Identify the isomorphic graphs within the same family.
- Enumerate 4V ET BC class (up to graph isomorphism).

Thank You for attention!