Enumeration of Vertex-Valency Restricted Planar Maps

Valery A. Liskovets

Institute of Mathematics, National Academy of Sciences of Belarus, Minsk

liskov@im.bas-net.by

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OUTLINE

1. General problem formulation:

- Valency-specified versus valency-permitted planar maps
- Eulerian/non-eulerian dichotomy

2. Sketchy survey (less significant – in blue):

- Counting rooted maps: arbitrary, eulerian and unicursal
- Counting unrooted maps: – " – " –

3. Recent results:

- Counting regular odd-valent maps, rooted and unrooted
- Some open questions

Maps

Planar maps only: 2-cell embeddings of planar connected graphs (loops and multiple edges allowed) in an oriented sphere.

The vertex valency = the degree of the vertex.

Dually: face degrees.

Rooting (after Tutte): distinguishing one edge-end as the root (accordingly: the root-edge and the root-vertex). Eliminates all non-trivial symmetries.

$$n = \# \text{ edges} \qquad p = \# \text{ vertices}$$

Unrooted maps: maps with undistinguishable vertices and edges considered up to (orientation-preserving) homeomorphism.

Problem settings: valency-specified vs. valency-permitted maps

1. Valency-specified: exactly r_i vertices of valency i, $i = 1, 2, \ldots$
Vertex partition $\pi = (1^{r_1}2^{r_2}3^{r_3}\ldots)$: a partition of $2n$:

$$|\pi| := \sum_i ir_i = 2n \qquad \sum_i r_i = p \qquad (\pi \in \Pi \ldots)$$

As usual, the superscripts $r_i = 1$ and terms i^{r_i} with $r_i = 0$ are dropped.

Arbitrary maps: no structure restrictions. Few results for other types.

$A'(n;\, 1^{r_1}2^{r_2}3^{r_3}\ldots)$ = the number of rooted maps,
$A^+(n; 1^{r_1}2^{r_2}3^{r_3}\ldots)$ = the number of unrooted maps
with the given vertex partition π (n may be dropped).

2. Valency-permitted: the list of admissible valencies $\mathbf{D} \subseteq \mathbf{N}$
with any repetitions: arbitrary r_i for $i \in \mathbf{D}$ and $r_i = 0$ for $i \notin \mathbf{D}$.

Enumerators: $A'_{\mathbf{D}}(n)$ and $A^+_{\mathbf{D}}(n)$, resp. ($\mathbf{D} = \mathbf{N}$ is dropped usually)

$A'_{\mathbf{D}}(n) = \sum_{\pi \in \Pi_{\mathbf{D}}} A'(n;\pi)$ ($\Pi_{\mathbf{D}}$: $\mathbf{D}$-part partitions)

Sometimes, the root-vertex is subject to separate restrictions.

Eulerian/non-eulerian dichotomy

Are ODD valencies FORBIDDEN or NOT?

Accordingly, w.r.t. enumeration, maps are: TAME or WILD.

Eulerian: $r_i = 0$ for $i = 1, 3, 5, \ldots;$ $\mathbf{D} \subseteq 2\mathbf{N}$

- Eulerian maps are, in general, enumerated efficiently and effectively.
- Counting maps with odd-valent vertices is typically untractable!

(I don't know fully convincing and exhausting explanations.)

There exist significant exceptions.

Tractable exceptional classes of non-eulerian maps:

- arbitrary: valency permission with $\mathbf{D} = \mathbf{N}$
- unicursal (in both settings): exactly two odd-valent vertices
- cubic (= regular trivalent): valency permission with $\mathbf{D} = \{3\}$
- regular odd-valent counted by valency with a fixed p

Enumerated by closed formulae: often sum-free for rooted maps (sums of simple terms over divisors of n for unrooted maps).

Rooted eulerian valency-specified maps. Tutte's formula:

$$A'(n;\ 2^{r_2}4^{r_4}6^{r_6}\dots) = \frac{2\cdot n!}{(n-p+2)!\prod_{k\geq 1}(r_{2k}!)}\prod_{k\geq 1}\left(\frac{(2k)!}{2\cdot k!^2}\right)^{r_{2k}} \qquad \text{(E)}$$

Discovered by Tutte in 1962.
Fundamental for map enumeration theory.

In particular, for regular $2d$-valent maps with p vertices ($n = dp$):

$$A'((2d)^p) = \frac{2}{\big((d-1)p+1\big)\big((d-1)p+2\big)}\left[\frac{1}{2}\binom{2d}{d}\right]^p\binom{d\,p}{p}$$

Formula (E) can be represented in terms of the generating function via

$$\sum_{k=0}^{\infty}\frac{(2k)!}{k!^2}x^k = 1\Big/\sqrt{(1-4x)}$$

Rooted unrestricted maps

Arbitrary n-edged maps. Tutte'63:

$$A'(n) = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$$

An easy corollary of (E) via the ordinary map quadrangulations.

$a(z) := \sum_n A'(n) z^n$ is a rational function over $\sqrt{1-12z}$.

$A'(n,p)$ (enumeration by the numbers of edges and vertices):
can be extracted and expressed in gen. functions.
But there is no convenient closed explicit formula.

Rooted eulerian unrestricted maps

Mullin; Walsh:

$$E'(n) = A'_{2\mathbf{N}}(n) = \frac{3 \cdot 2^n}{2(n+1)(n+2)} \binom{2n}{n}$$

By Tutte's fundamental formula (E) and Lagrange's inversion.

Equivalently

$$e(x) := \sum_{n=0}^{\infty} E'(n) x^n = \frac{8x^2 + 12x - 1 + (\sqrt{1-8x})^3}{32x^2}$$

– a rational function over $\sqrt{1-8z}$. Valency-permitted: $e_{\mathbf{D}}(x)$.
Solvable similarly for arbitrary $\mathbf{D} \subseteq \mathbf{N}$ (not necessary resulting in a sum-formula for $E'_{\mathbf{D}}(n)$). Asymptotics, etc.

Tutte's formula for unicursal valency-specified maps

Tutte's original formulation (1962, "A census of slicings"), not in terms of rooted maps but equivalent.

$C(2d_1+1, 2d_2+1, 2d_3, \ldots, 2d_p)$ = the number of (unicursal) maps with $n = 1 + \sum_i d_i$ edges and p labelled vertices $v_1, v_2, v_3, \ldots, v_p$ of the specified valencies, each vertex being "rooted" by distinguishing one of its edge-ends.

$$C(2d_1+1, 2d_2+1, 2d_3, \ldots, 2d_p) = \frac{(n-1)!}{(n-p+2)!} \frac{(2d_1+1)!(2d_2+1)!}{d_1!^2 d_2!^2} \prod_{i=3}^{p} \frac{(2d_i)!}{d_i!(d_i-1)!} \qquad \text{(U)}$$

No further generalizations to more than 2 odd-valent vertices!

For a long time, formula (U) was considered merely as a curious, useless generalization of (E). The class stayed unnamed till 2004.

Rooted unicursal unrestricted maps

Unicursal maps = maps containing eulerian paths but not cycles
= maps with exactly two odd-valent vertices.
Generally under-estimated and under-investigated.
Unicursal maps are not reduced to eulerian maps by adding
an edge connecting the odd-valent vertices
(unless these vertices lie on a common face).

L – Walsh'04:

$$U'(n) = 2^{n-2}\binom{2n}{n}$$

Unexpected but easy by formula (U) and Lagrange's inversion.
Is there a direct combinatorial proof?

$$\Longrightarrow \quad U'(n) \sim \frac{1}{2\sqrt{\pi}} n^{-1/2} 8^n \qquad \text{as} \quad n \to \infty$$

Rooted unicursal partially restricted maps

Combined restrictions:
two odd valencies are specified, the others are restricted to be even.

$U_i'(n)$ = the number of rooted unicursal maps with $i = 0, 1, 2$ endpoints (= vertices of valency 1). Similarly (for $n \geq 2$):

$$U_0'(n) = 2^{n-2} \frac{n-2}{n} \binom{2n-2}{n-1}$$

$$U_1'(n) = 2^{n-1} \binom{2n-2}{n-1}$$

$$U_2'(n) = 2^{n-2} \binom{2n-2}{n-1}$$

$\widetilde{U}'(n)$ = the number of unicursal maps rooted at an odd-val. vertex:

$$\widetilde{U}'(n) = \frac{2^{n-2}}{n+2} \binom{2n+2}{n+1} = 2^{n-2} C_{n+1} \text{ – the Catalan number}$$

Rooted unicursal partially restricted maps (cont.)

$U'(2d_1+1, 2d_2+1; n)$ = the number of rooted unicursal maps with the specified odd valencies. $d := d_1 + d_2$ ($n \geq d+2$):

$$U'(2d_1+1, 2d_2+1; n) = \frac{h_d(n)\, 2^{n-2d-2}}{(n-2)\dots(n-d)} \binom{2d_1}{d_1}\binom{2d_2}{d_2}\binom{2n-2d-2}{n-d-1} \times \begin{cases} 2 \text{ if } d_2 \neq d_1 \\ 1 \text{ if } d_2 = d_1 \end{cases}$$

where $h_d(n)$ is a polynomial in n of degree $d-1$ (for $d \geq 1$).

Explicitly:

$$h_d(n) = \frac{1}{n-1} \sum_{j=0}^{d} \binom{d}{j} 2^{d-j} (n-d-2)(n-d-3)\dots(n-d-1-j) \cdot n(n-1)\dots(n-d+j+1).$$

$h_1(n) = 3, \quad h_2(n) = 9n - 20, \quad h_3(n) = 3(3n-7)(3n-10),$
$h_4(n) = 3(27n^3 - 279n^2 + 934n - 1008), \dots$

Unrooted unrestricted maps

L'81: a method of reduction to rooted quotient maps.

Quotient map = the induced map in the orbit space w.r.t. the cyclic group generated by a (rotational) automorphism.
The orbit space (orbifold): a sphere with 2 distinguished points.

$$A^{+}(n) = \frac{1}{2n}\left[A'(n) + \sum_{\substack{t<n \\ t|n}} \phi\left(\frac{n}{t}\right)\binom{t+2}{2} A'(t)\right] + \begin{cases} \frac{n+3}{4} A'\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd} \\ \frac{n-1}{4} A'\left(\frac{n-2}{2}\right) & \text{if } n \text{ is even} \end{cases}$$

Straightforward: the quotient maps are arbitrary maps (eventually, up to easily tractable cases of the poles in the middles of one or two edges)! A typical necklace-like form: a sum over divisors of n.

Unrooted eulerian unrestricted maps

L − Walsh'04:

$$E^{+}(n) = \frac{1}{2n}\left[\frac{3\cdot 2^{n}}{2(n+1)(n+2)}\binom{2n}{n} + 3\sum_{t<n,\, t|n}\phi\left(\frac{n}{t}\right)2^{t-2}\binom{2t}{t}\right]$$

$$+\begin{cases} \dfrac{2^{(n-1)/2}}{n+1}\dbinom{n-1}{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\[2ex] \dfrac{1}{n}\displaystyle\sum_{t|\frac{n}{2}}\phi\left(\frac{n}{t}\right)2^{t-3}\binom{2t}{t} + \dfrac{2^{(n-4)/2}}{n+2}\dbinom{n}{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}$$

Luckily: the quotient maps are eulerian or unicursal
(the two distinguished axial vertices may become odd-valent)!

The generating functions for the summands contain both $\sqrt{1-4z}$ and $\sqrt{1-8z}$ (unusually).

Unrooted unicursal unrestricted maps

$$U^{+}(n) = \frac{1}{2n} \sum_{\substack{t|n \\ n/t \text{ odd}}} \phi\left(\frac{n}{t}\right) 2^{t-2}\binom{2t}{t} + \begin{cases} 2^{(n-3)/2}\binom{n-1}{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ 2^{(n-6)/2}\binom{n}{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}$$

Easier: the quotient maps are unicursal.

Counting unrooted maps

Generally, quotient maps can have a subtler and considerably modified structure (say, a loopless map can become looped) not easy for the rooted enumeration. An art of applying the method.

Fusy'07: a recent alternative enumeration of $A^{+}(n)$. An effective analytical method based on a special tree-like decomposition. The generating functions of the summands (up to the factor $\phi(n/k)$) in the standard "Burnside's" (orbit counting) formula for $A^{+}(n)$ is proved to be a rational function w.r.t. the generating function $a(z) = \sum_n A'(n) z^n$.
The same holds for diverse types of maps (including polyhedral).
However, seems to be bad applicable to valency-restricted maps.

Mednykh – Nedela'06: a remarkable generalization to unrooted non-planar maps (of any fixed genus). A similar reduction to counting rooted maps in general orbifolds w.r.t. cyclic automorphism groups... What about valency-restricted maps?

Regular maps. Trivalent (= cubic) maps

$A'(3^{4m}) = A'(6m; 3^{4m})$ = the number of rooted cubic maps with $p = 4m$ vertices (thus, with $n = 6m$ edges). Mullin'66:

$$A'(3^{4m}) = \frac{2^{6m}}{(2m+1)(m+1)}\binom{3m}{m}$$

A similar sum-free formula is valid for $A'(3^{2m})$ where m is odd.

L – Walsh'87: a simple formula for $A^{+}(3^{2m})$.
Easily: quotient maps are again cubic, or reducible to cubic, maps.
(Recall: these maps may contain multiple edges and loops,
i.e. be separable.)

Counting rooted valency-permitted maps: the dichotomy revisited

Bender – Canfield'94 (loosely speaking):
given $\mathbf{D} \subseteq \mathbf{N}$, the generating function for valency-permitted maps

$$a_{\mathbf{D}}(z) = \sum_n A'_{\mathbf{D}}(n) z^n$$

satisfies a pair of fairly complicated algebraic equations...

In general, no explicit formula for their unique solution can be extracted, unless the maps are restricted to be eulerian ($\mathbf{D} \subseteq 2\mathbf{N}$). For eulerian maps (and in some different particular cases), the equations are simplified considerably and admit a simple closed expression for the solution (with hypergeometric coefficients).

Strengthened significantly and fruitfully by physicists Bouttier – Di Francesco – Guitter'02 and by Bousquet-Mélou – Jehanne'06.

Counting rooted regular maps of odd valencies

Incomparably more difficult than all previous tractable cases!

Gao – Rahman'97:

a closed formula for $A'(s^p) = A'(n; s^p)$ as functions of odd s for every fixed (even) p (where $n = sp/2$).

A special analytic technique (hypergeometric functions, truncations of power series, etc.).

Gao–Rahman's theorem for rooted regular odd-valent maps:

If $s = 2d + 1$ and $p = 2m$, then

$$A'(s^p) = (2d + 1)d^2 \binom{2d}{d}^{2m} \times Q_m(d)$$

where $Q_m(d)$ is a polynomial in d of degree $2m - 5$ (for $m \geq 3$).

In particular, for $m = 4$,

$$Q_4(d) = (2048d^3 - 387d^2 + 20d - 1)/315$$

cannot be factorized into linear factors over $\mathbb{R}$. Thus,
it is not a hypergeometric term over $\mathbb{R}$ unlike eulerian $A'((2d)^p)$.

No simple explicit formulae for the polynomials $Q_m(d)$ are known (they are calculated recursively).

A similar more cumbersome result is valid for the number of rooted almost regular maps with one exceptional vertex $A'(s^{p-1}u)$.

Counting unrooted regular odd-valent maps

The reductive enumeration method requires to count rooted maps that are regular OR regular excepting one or TWO vertices.
The latter class is most challenging (in fact, only particular cases with the exceptional valencies like $(1,1),(1,u),(u,u),\ u|s$, arise).

Fairly unpleasant formulae by strengthened G–R's technique (together with Bender–Canfield's and B–DiF–G's equations) after very long tedious calculations and transformations.

As a result:
there exist closed formulae for the numbers of unrooted regular odd-valent maps $A^{+}(s^p)$.
These formulae become more cumbersome as p increases but have (for $p > 2$) a bounded number of terms independent of s.

Theorem (Gao − L − Wormald'09):

Let $p > 2$ be fixed, s odd and let d denote $(s-1)/2$. Then $A^+(s^p)$ is expressible as a polynomial in $\binom{2d}{d}$ and the quantities $\delta_{i|s}\binom{2\lfloor d/i\rfloor}{\lfloor d/i\rfloor}$ (for all odd $i > 2$ dividing $p-1$ or $p-2$) with coefficients being polynomials in d, where $\delta_{i|s} = 1$ if $i|s$ and $\delta_{i|s} = 0$ otherwise.

Explicit formulae (by Maple up to $p = 10$ vertices; $s = 2d + 1$):

$$A^+(s^2) = \frac{1}{2}\binom{2d}{d} + \frac{1}{2(2d+1)} \sum_{1 \le i | 2d+1} \phi(i)\binom{2\lfloor d/i\rfloor}{\lfloor d/i\rfloor}^2$$

$$A^+(s^4) = \frac{1}{6}d\binom{2d}{d}^4 + \frac{1}{2}d\binom{2d}{d}^2 + \frac{2}{3}\delta_{3|r}\binom{2\lfloor d/3\rfloor}{\lfloor d/3\rfloor}\binom{2d}{d}$$

$$A^+(s^6) = \frac{36d-1}{120}d^2\binom{2d}{d}^6 + \frac{3}{4}d^2\binom{2d}{d}^3 + \frac{1}{3}d\binom{2d}{d}^2 + \frac{4}{5}\delta_{5|r}\binom{2\lfloor d/5\rfloor}{\lfloor d/5\rfloor}\binom{2d}{d}$$

The first, exceptional, formula (with the sum over divisors of s and Euler's totient ϕ) belongs to Bousquet − Labelle − Leroux†'02.

Some open questions

1. Counting rooted unicursal maps. Curious identities:

$$U'_1(n) = 2U'_2(n) \qquad \text{[however, } U_1^+(n) \neq 2U_2^+(n)\text{]}$$

$$U'(n) = \frac{1}{6}(n+1)(n+2)E'(n)$$

Are there direct bijective proofs?

2. Counting rooted maps with combined restrictions:

Some valencies are prescribed, and the others are restricted to belong to a set $\mathbb{D}$. Promising for eulerian maps.

3. Counting maps with restricted degrees of vertices AND faces:

??

References (chronol.)

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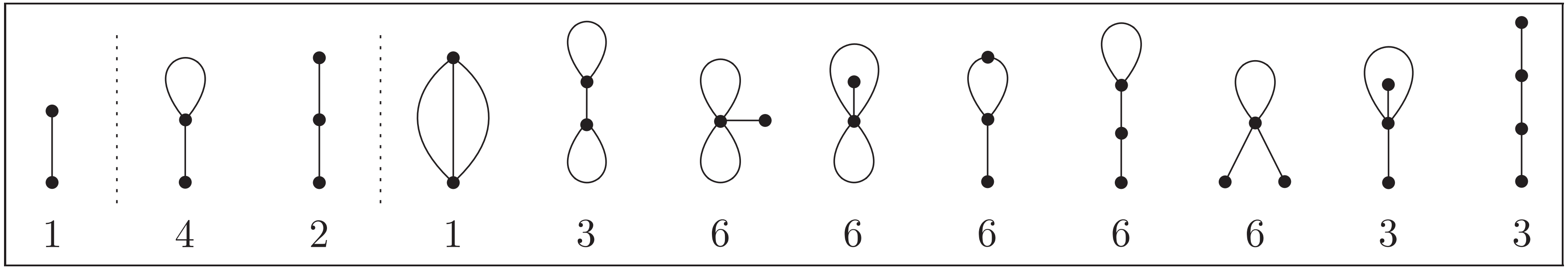
Unicursal planar maps with $n = 1, 2, 3, 4$ edges

$\boxed{U'(n) = 2^{n-2}\binom{2n}{n}}$ =1,6,40,280,... $\boxed{U^{+}(n)}$ =1,2,9,38,...



#map rootings = $2n/|\mathrm{Aut}|$

Rooted: 1, $4 + 2 = 6$, $1 + 3 + 6 + 6 + \ldots = 40$