On local properties of 1-planar graphs with specified minimum degree and girth

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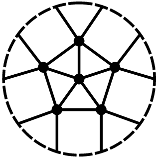

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The family of planar (or plane) graphs is one of the oldest graph families being studied (it dates back to 19th century in the connection with the Four Colour Problem) and nowadays is well explored.

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- the thickness of a graph
- embeddings into plane which have constant number of crossings per edge (k-planarity)

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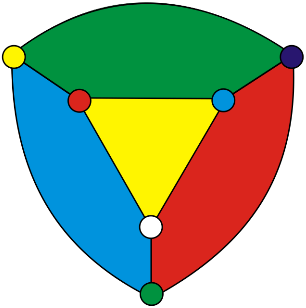
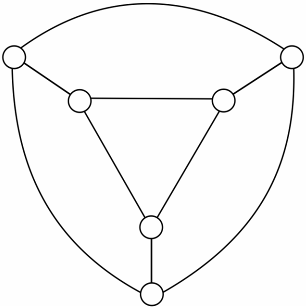
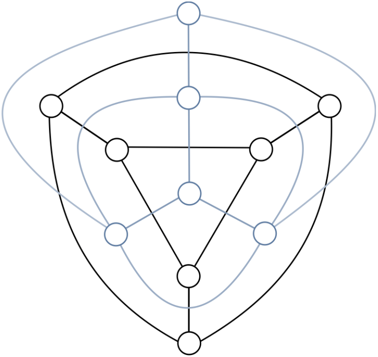
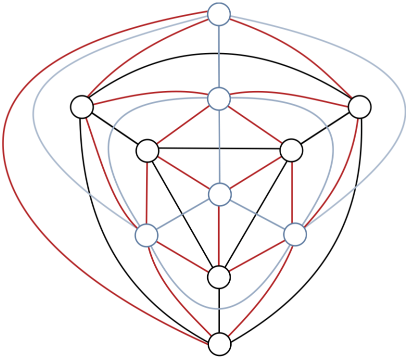
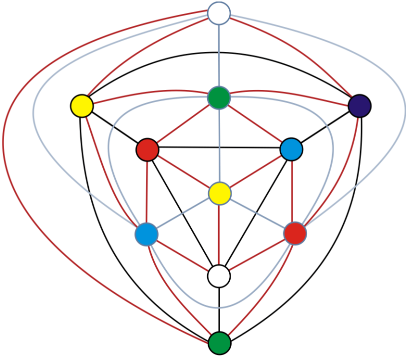
1-planar graphs were first introduced by G. Ringel in 1965 in the connection with simultaneous colouring of vertices and faces of plane graphs (which corresponds to vertex colouring of the incidence/adjacency graph of vertices and faces of a plane graph; these graphs are 1-planar and maximal with this property):

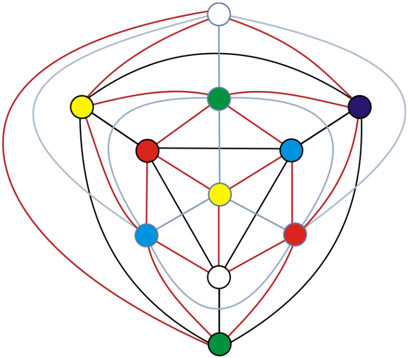












In contrast to the family of planar graphs, the family of 1-planar graphs is still little explored (there are max. 30 papers).

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- their recognition is NP-complete problem (Korzhik, Mohar 2008)
- maximal 1-planar graphs with the same number of vertices need not to have the same number of edges (Hudák, T.M. 2009)
- maximal 1-planar drawings need not to correspond to maximal 1-planar graphs (Fabrici, T.M. 2009)

On the other hand, concerning local structure, 1-planar graphs are similar to planar ones (linear number of edges, small degree vertices, subgraphs of small weight), which motivates us to look for analogues of structural results on planar graphs.

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Notation:

$\overline{\mathcal{P}}$ ... the family of all 1-planar graphs

$\overline{\mathcal{P}}_\delta$ ... the family of all 1-planar graphs with minimum degree at least δ

Theorem (Ringel 1965)

Each graph $G \in \overline{\mathcal{P}}$ contains a vertex of degree at most 7; the bound 7 is best possible.

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Theorem (I. Fabrici, T.M. 2007)

Each 3-connected graph $G \in \overline{\mathcal{P}}$ contains an edge such that degrees of its endvertices are at most 20. The bound 20 is best possible.

Theorem (I. Fabrici, T.M. 2007; D. Hudák, T.M. 2009)

Each graph $G \in \overline{\mathcal{P}}_6$ contains

- *a 3-cycle with all vertices of degree at most 10; the bound 10 is sharp,*
- *a 4-cycle with all vertices of degree at most 71,*
- *a 3-star with all vertices of degree at most 15,*
- *a 4-star with all vertices of degree at most 23.*

Theorem (I. Fabrici, T.M. 2007)

Each graph $G \in \overline{\mathcal{P}}_7$ contains

- *a 5-star with all vertices of degree at most 11,*
- *a 6-star with all vertices of degree at most 15.*

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Theorem (D. Hudák, T.M. 2009)

Each graph $G \in \overline{\mathcal{P}}_7$ contains

- *a $(7,7)$-edge,*
- *a graph K_4 with all vertices of degree at most 13,*
- *a graph $K_{2,3}^*$ ($K_{2,3}$ with extra edge in smaller bipartition) with all vertices of degree at most 13,*
- *a graph $K_{2,3}^{\ddagger}$ (4-sided pyramid) with all vertices of degree at most 11,*
- *a 5-cycle with all vertices of degree at most 9.*

Theorem (D. Hudák, T.M. 2008)

Each graph $G \in \overline{\mathcal{P}}_5$ of girth 4 contains

- *a* $(5, \leq 6)$*-edge,*
- *a 4-cycle with all vertices of degree at most 9,*
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Here, the assumption on girth 4 is essential – if $G \in \overline{\mathcal{P}}_5$, then there is no finite bound for degrees of vertices of $C_4 \subseteq G$ or $K_{1,4} \subseteq G$ which is independent on G. In other words, for any m there exists a graph $G_m \in \overline{\mathcal{P}}_5$ such that each 4-cycle $C_4 \subseteq G_m$ contains a vertex of degree at least m (similarly for 4-star).

Definition

Let $G = (V, E)$ be 1-planar graph and let D be its 1-planar drawing. The asociated plane graph $D^{\times}$ is the plane graph that is obtained from D by turning all crossings of D into new 4-valent vertices. These new vertices are called false, the original vertices of D are true vertices of $D^{\times}$.

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The local structure results on 1-planar graphs are proved by contradiction - we assume the existence of a hypothetical counterexample; then, we use the discharging method applied at its asociated plane graph.

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- local redistribution of charges according to certain rules such that the total sum of charges remains the same
- the analysis of new charges of vertices/faces and proving that they are **nonnegative**; hence their sum is nonnegative, a contradiction

Open problems

1. Determine the maximum possible girth of an 1-planar graph with minimum degree at least 3 (or 4).
2. Find an analogy of Kotzig theorem for light edge weight in 1-planar graphs in general, and with specified girth.
3. Find an analogy of Lebesgue result concerning incidence of small degree vertices with short cycles in 1-planar graphs.

Thanks for your attention :-)