Looking back and ahead: three open problems in vertex-transitive graphs

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- *"I'd like to say Hi to my parents, especially my mother and my father."*
- *"Nema budale dok ne doktorira."*
  (You don't recognize the fool before he gets the PhD.)

Outline

Strongly regular bicirculants and primitive groups of degree $2p$

Hamilton cycles in Cayley graphs

Semiregular automorphisms in vertex-transitive graphs

Outcome

Open problem 1: Classify strongly regular bicirculants and consequently obtain a CFSG-free proof of nonexistence of simply primitive groups of degree $2p$.

Open problem 2: Hamilton cycles in Cayley graphs of groups with a $(2, s, t)$-presentation (interim report).

Open problem 3: Existence of semiregular automorphisms in vertex-transitive graphs admitting a transitive solvable group.

Strongly regular bicirculants and primitive groups of degree $2p$

VTG of order p, where p is a prime

Then the graph is a p-circulant, that is, a Cayley graph of a cyclic group of order p.

VTG of order $2p$, where p is a prime

Then the graph is a p-bicirculant, that is, a graph with a $(2, p)$-semiregular automorphism.

An element of a permutation group is *semiregular*, more precisely (m, n)-semiregular, if it has m orbits of size n and no other orbit.

VTG of order $2p$, where p is a prime

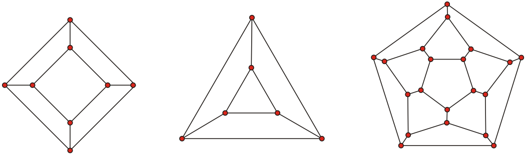
Let ρ be a $(2, n)$-semiregular automorphism of an n-bicirculant X, let U and W be the two orbits of ρ, and let $u \in U$ and $w \in W$.

Let $S = \{s \in Z_n \setminus \{0\} \mid u \sim \rho^s(u)\}$ be the symbol of the n-circulant induced on U and, let R be the symbol of the n-circulant induced on W (relative to ρ). Moreover, let $T = \{t \in Z_n \mid u \sim \rho^t(w)\}$. The ordered triple $[S, R, T]$ is the symbol of X relative to (ρ, u, w). Note that $S = -S$ and $R = -R$ are symmetric, that is, inverse-closed subsets of $Z_n \setminus \{0\}$ and are independent of the choice of vertices u and w.

VTG of order $2p$, where p is a prime

In VT n-bicirculant either

- $\exists$ a swap, an automorphism interchanging the two orbits;
- $\exists$ a mixer, an automorphism mixing the two orbits;
- $\exists$ a swap and a mixer.



both swap mixer

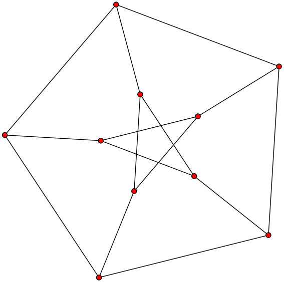
VTG of order $2p$, where p is a prime

X VTG of order $2p$, $G \leq \mathrm{Aut}X$ transitive

- G-imprimitive blocks of size p;
- G-imprimitive blocks of size 2,
- G is primitive.

G-imprimitive blocks of size p

iff $\exists\ S = -S \subseteq \mathbb{Z}_p^*$, $T \subseteq \mathbb{Z}_p$ and $a \in \mathbb{Z}_p^*$ such that $[S, aS, T]$ with $a^2 S = S$ and $aT = T$ is a symbol of X.



The Petersen graph
$S = \{\pm 1\}$, $T = \{0\}$ and $a = 2$.

G-imprimitive blocks of size 2

Theorem (DM, '81)

There exists a transitive subgroup $H \leq G$ with blocks of size p.

Consequently the same type of algebraic description for X with symbol $[S, aS, T]$ exists for appropriate S, T, a.

G is primitive

Wielandt's theorem, '56

Let $G \leq Sym(V)$ be a primitive group of degree $2p$. Then:

- The stabilizer G_v of $v \in V$ has at most three orbits.
- If the number of orbits is exactly three, then $2p = (2s+1)^2 + 1$ for some natural number s. The lengths of the orbits of G_v are 1, $s(2s+1)$, $(s+1)(2s+1)$.

Example:

$p = 5$, A_5, S_5 acting on pairs from $\{1, 2, 3, 4, 5\}$.
Associated graphs: the Petersen graph and its complement.

G is primitive

By the classification of finite simple groups (CFSG) any primitive group of degree $2p$, $p > 5$, is 2-transitive.

How about a CFSG-free proof of this fact?

Generalization of Wielandt's theorem

Kovács, DM, Muzychuk, to appear in Transactions AMS

Let $G \leq Sym(V)$ be a primitive group of degree $2p^e$. Then:

- The stabilizer G_v of $v \in V$ has at most three orbits.
- If the number of orbits is exactly three, then $2p^e = (2s+1)^2 + 1$ for some natural number s. The lengths of the orbits of G_v are 1, $s(2s+1)$, $(s+1)(2s+1)$.

In fact, above is a corollary of a more general result for association schemes.

Since every rank 3 group gives rise to a strongly regular graph, such graphs are a natural framework for the study of primitive bicirculants.

Strongly regular graphs

A regular graph X with v vertices and of valency k is (v, k, λ, μ)-strongly regular if each pair of adjacent vertices has λ common neighbors and each pair of nonadjacent vertices has μ common neighbors.

A strongly regular graph X is trivial if either X or its complement X^c is disconnected.

X is disconnected if and only if $\mu = 0$, in which case X is a disjoint union of isomorphic copies of complete graphs.

Strongly regular p-bicirculants

Necessary arithmetic conditions for the existence of $(2p, k, \lambda, \mu)$-strongly regular p-bicirculants with symbol $[S, R, T]$ (DM, '88):

- $p = 2s^2 + 2s + 1$ for some positive integer s;
- $k = s(2s+1)$ or $k = (s+1)(2s+1)$;
- $|S| = |R| = (p-1)/2 = s(s+1)$;
- $|T| = \mu = \lambda + 1$ equals s^2 or $(s+1)^2$;
- S and R are complements of each other relative to $\mathbb{Z}_p^*$.

Strongly regular p-bicirculants

This result was extended to strongly regular n-bicirculants for n odd (de Resmini, Jungnickel, '92) and for n even (Leung, Ma, '93)

Existence of strongly regular p-bicirculants

n	S	T	Is VT	Is ET
5	$\{\pm 1\}$	$\{0\}$	Yes	Yes
8	$\{\pm 1\}$ $(R = \{\pm 3\})$	$\{0, \pm 1, 4\}$	Yes	Yes/No*
13	$\{\pm 1, \pm 3, \pm 4\}$	$\{0, 1, 3, 9\}$	No	No
25	$\{\pm 1, \pm 2, \pm 4, \pm 6, \pm 9, \pm 10\}$	$\{0, \pm 1, \pm 2, \pm 7, \pm 11\}$	Yes	No
41	$\{\pm 1, \pm 4, \pm 6, \pm 10, \pm 14, \pm 15, \pm 16, \pm 17, \pm 18, \pm 19\}$	$\{0, \pm 1, \pm 4, \pm 10, 11, 12, \pm 16, \pm 18, 28, 34, 38\}$	No	No
41	$\{\pm 1, \pm 6, \pm 8, \pm 9, \pm 10, \pm 12, \pm 13, \pm 15, \pm 16, \pm 20\}$	$\{\pm 1, \pm 3, \pm 4, \pm 5, \pm 9, \pm 12, \pm 14, \pm 15, \}$	Yes	No
61	$\{\pm 1, \pm 2, \pm 4, \pm 6, \pm 7, \pm 8, \pm 10, \pm 13, \pm 18, \pm 19, \pm 20, \pm 23, \pm 25, \pm 28, \pm 29\}$	$\{0, \pm 1, \pm 2, \pm 4, \pm 10, \pm 11, \pm 12 \pm 15, \pm 17, \pm 18, \pm 19, \pm 22, \pm 26\}$	Yes	No

*The graph with smaller valency is edge-transitive, but its complement is not.

Strongly regular p-bicirculants

Further study of strongly regular bicirculants in order to

- understand their structure, so as to
- obtain a CFSG-free proof of non-existence of simply primitive groups of degree $2p$.

Hamilton cycles in Cayley graphs

Tying together two seemingly unrelated concepts: traversability and symmetry

Lovász question, '69

Does every connected vertex-transitive graph have a Hamilton path?

Lovász problem is, somewhat misleadingly, usually referred to as the Lovász conjecture, presumably in view of the fact that, after all these years, a connected vertex-transitive graph without a Hamilton path is yet to be produced.

VT graphs without Hamilton cycle

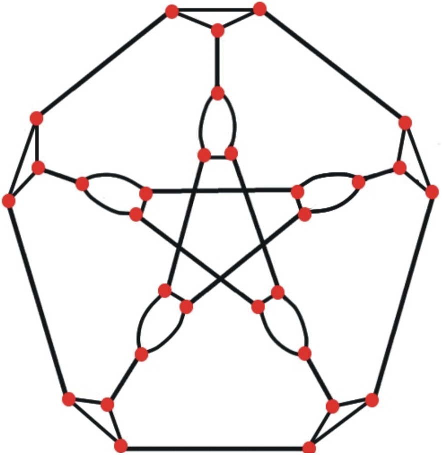
Only four connected VTG (having at least three vertices) not having a Hamilton cycle are known to exist:

- the Petersen graph,
- the Coxeter graph,
- and the two graphs obtained from them by truncation.

All of these are cubic graphs, suggesting that no attempt to resolve the problem can bypass a thorough analysis of cubic VTG.

None of these four graphs is a Cayley graph, leading to the conjecture that every connected Cayley graph has a Hamilton cycle.

The truncation of the Petersen graph



Conjectures/counterconjectures

Babai, '79

There exist infinitely many connected vertex-transitive graphs without a Hamilton cycle.

Thomassen, '91

There exist only finitely many such graphs.

Current situation

Hamilton cycles (paths) are known to exist for various families of Cayley graphs.

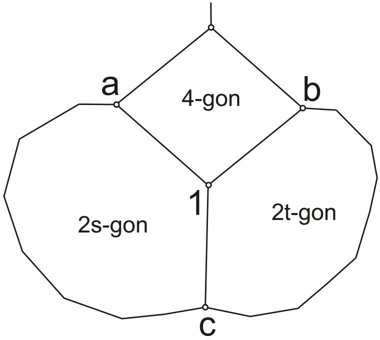
But not known whether they exist, e.g., for Cayley graphs of dihedral groups of order $2 \pmod 4$.

Cubic Cayley graphs

Hamilton cycles are known to exist for

- cubic CG of dihedral groups (Alspach, Zhang,'89)
- cubic CG $\mathrm{Cay}(G, S)$, where $S = \{a, b, c\}$ and $a^2 = b^2 = c^2 = 1$ and $ab = ba$ (Cherkassoff, Sjerve).
- cubic CG $\mathrm{Cay}(G, S)$ of order $2 \pmod 4$, where $S = \{a, x\}$ and $x^s = 1$, $a^2 = 1$ and $(ax)^3 = 1$ (Glover, DM, '07).
- and for some other cases.

Hamilton cycles in Cay($G, \{a, b, c\}$), where $G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^s = (bc)^t = 1, ab = ba \rangle$



Hamilton cycles in (2,s,3)-Cayley graphs

Glover, DM, '07

Let $s \geq 3$ and $G = \langle a, x | a^2 = 1, x^s = 1, (ax)^3 = 1, \ldots \rangle$ a group with a $(2, s, 3)$-presentation. Then $\mathrm{Cay}(G, \{a, x, x^{-1}\})$ has

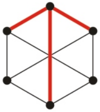
- a Hamilton cycle when $|G|$ is congruent to 2 modulo 4, and
- a cycle of length $|G| - 2$, and also a Hamilton path, when $|G|$ is congruent to 0 modulo 4.

Proof strategy

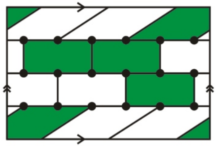
Based on an embedding of $X = \mathrm{Cay}(G, \{a, x, x^{-1}\})$, onto a corresponding orientable surface with s-gonal and hexagonal faces, in which one then looks for a long tree of faces – a tree of faces whose boundary is either a Hamilton cycle in X or a cycle missing two adjacent vertices.

Example: $|G| \equiv 2 (\mathrm{mod}\ 4)$

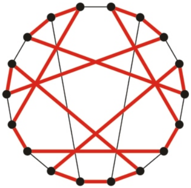
$G = S_3 \times \mathbb{Z}_3$ with a $(2, 6, 3)$-presentation
$\langle a, x \mid a^2 = x^6 = (ax)^3 = 1, \ldots \rangle$,
where $a = ((12), 0)$ and $x = ((13), 1)$.



The corresponding hexagon graph.



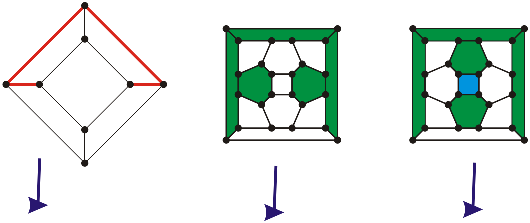
A Hamilton tree of hexagons.



The corresponding Hamilton cycle in X.

Example: $|G| \equiv 0 (\mathrm{mod}\ 4)$

$G = S_4$ with a $(2,4,3)$-presentation $\langle a, x \mid a^2 = x^4 = (ax)^3 = 1 \rangle$, where $a = (12)$ and $x = (1234)$.



The same tree in the corresponding hexagon graph

A tree of hexagons, whose boundary is a cycle missing only two vertices in the spherical Cayley map of X.

A modified tree of faces (including also a square).

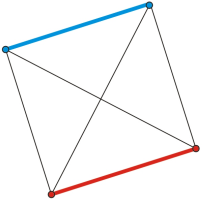
Payan, Sakarovitch

Payan, Sakarovitch, '75

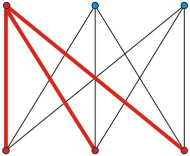
Let X be a cyclically 4-edge-connected cubic graph of order n, and let S be a maximum cyclically stable subset of $V(X)$. Then $|S| = \lfloor (3n-2)/2 \rfloor$ and more precisely, the following hold.

- If $n \equiv 2 \,(mod\, 4)$ then $|S| = (3n-2)/4$, and $X[S]$ is a tree and $V(X) \setminus S$ is an independent set of vertices;
- If $n \equiv 0 \,(mod\, 4)$ then $|S| = (3n-4)/4$, and either $X[S]$ is a tree and $V(X) \setminus S$ induces a graph with a single edge, or $X[S]$ has two components and $V(X) \setminus S$ is an independent set of vertices.

Cyclically stable subsets



F4



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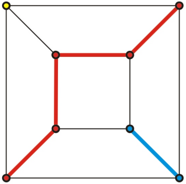

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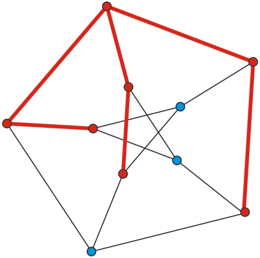


F10

$(2, s, 3)$-Cayley graphs of order $0 \pmod 4$

To go from a Hamilton path to a Hamilton cycle in a $(2, s, 3)$-Cayley graph of order $0 \pmod 4$ three cases can occur:

- $s \equiv 0 \pmod 4$.
- $s \equiv 2 \pmod 4$.
- s odd.

Hamiltonicity of $(2, s, 3)$-Cayley graphs, $s \equiv 0 \pmod 4$

Glover, Kutnar, DM, J. Alg. Combin., in press

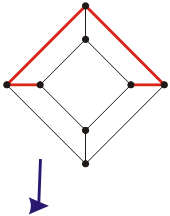
Let $s \equiv 0 \pmod 4 \geq 4$ be an integer. Then a $(2, s, 3)$-Cayley graph has a Hamilton cycle.

Essential ingredients in the proof

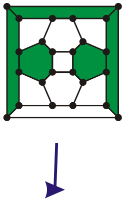
- Method used in the proof of the first result.
- Classification of cubic ATG of girth 6.
- Results on cubic ATG admitting a 1-regular subgroup.

Example: $|G| \equiv 0 (\mathrm{mod}\ 4)$

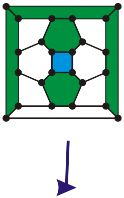
$G = S_4$ with a $(2,4,3)$-presentation $\langle a, x \mid a^2 = x^4 = (ax)^3 = 1 \rangle$, where $a = (12)$ and $x = (1234)$.



The same tree in the corresponding hexagon graph



A tree of hexagons, whose boundary is a cycle missing only two vertices in the spherical Cayley map of X.



A modified tree of faces (including also a square).

Hamiltonicity of $(2, s, 3)$-Cayley graphs, s odd

Glover, Kutnar, DM

Let s be an odd integer. Then a $(2, s, 3)$-Cayley graph has a Hamilton cycle.

Essential ingredients in the proof

- $\langle x \rangle$ is corefree in $G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \ldots \rangle$:
  A method similar to the method used in $s \equiv 0 \pmod 4$ case gives us a Hamilton cycle as a boundary of a Hamilton tree of faces consisting of hexagons and two s-gons.
- $\langle x \rangle$ is not corefree in $G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \ldots \rangle$:
  Results about lifts of Hamilton cycles in covers of graphs are needed.

Hamiltonicity of $(2, s, 3)$-Cayley graphs, s odd

Hamiltonicity of $(2, s, 3)$-Cayley graphs, $s \equiv 2 \pmod 4$

$s \equiv 2 \pmod 4$ requires a different approach, work in progress.

Three diffrent cases need to be consider

- $\langle x \rangle$ is not corefree, normal part of $\langle x \rangle$ of even order DONE
- $\langle x \rangle$ is not corefree, normal part of $\langle x \rangle$ of odd order
- $\langle x \rangle$ is corefree.

Hamiltonicity of $(2, s, t)$-Cayley graphs, $t \geq 4$

With similar methods one can obtain existence of large cycles in $(2, s, 4)$-Cayley graphs (not necessarily Hamilton cycles).

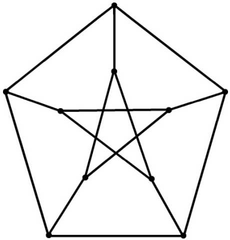
Semiregular automorphisms in vertex-transitive graphs

Semiregular automorphisms in VTG

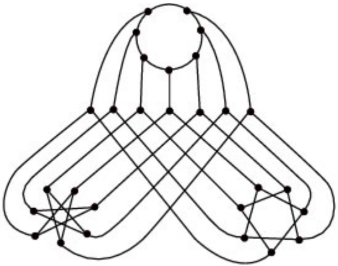
Does every vertex-transitive graph have a semiregular automorphism (DM, 1981; for transitive 2-closed groups, Klin, 1996)?

An element of a permutation group is semiregular, more precisely (m, n)-semiregular, if it has m orbits of size n and no other orbit.

Examples



Has a (2,5)-semiregular automorphism



Has a (4,7)-semiregular automorphism

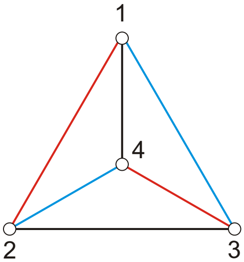
Semiregular elements

- (A) Automorphism groups of vertex-transitive (di)graphs;
- (B) 2-closed transitive permutation groups;
- (C) Transitive permutation groups.

Semiregular elements

(B) but not (A):

Regular action of $H = (\mathbb{Z}_2)^2 = \{id, (12)(34), (13)(24), (14)(23)\}$ on $V = \{1, 2, 3, 4\}$. Each of the orbital graphs has a dihedral automorphism group intersecting in H; so H is 2-closed but not the automorphism group of a (di)graph.



Semiregular elements

(C) but not (B):

$AGL(1, p^2)$, for $p = 2^k - 1$ a Mersenne prime, acting on the set of $p(p + 1)$ lines of the affine plane $AG(2, p)$.

Results

- All transitive permutation groups of degree p^k or mp, for some prime p and $m < p$, have SE of order p (DM, '81).
- All cubic VTG have SA (DM, Scapellato, '93).
- All VTD of order $2p^2$ have SA of order p (DM, Scapellato, '93).
- All vertex-primitive graphs have SA (Giudici, '03).
- All vertex-quasiprimitive graphs have SA (Giudici, '03).
- All vertex-transitive bipartite graphs where only system of imprimitivity is the bipartition, have SA (Giudici, Xu, '07).
- Every 2-arc-transitive graph has SA (Xu, '07).
- Every ATG of prime valency has SA (Xu, '07).
- All quartic VTG have SA (Dobson, Malnič, DM, Nowitz, '07).

- All VTG of valency $p+1$ admitting a transitive $\{2,p\}$-group for p odd have SA (Dobson, Malnič, DM, Nowitz, '07).
- There are no elusive 2-closed groups of square-free degree (Dobson, Malnič, DM, Nowitz, '07).
- All ATG with valency pq, p, q primes, such that Aut(X) has a nonabelian minimal normal subgroup N with at least 3 vertex orbits, have SA (Xu, '08).
- Every VT, edge-primitive graph has SA (Giudici, Li, '09).
- All distance-transitive graphs have SA (Kutnar, Šparl, '09).

Semiregular automorphisms

The main steps towards a possible complete solution of the problem would have to consist of a proof of the existence of semiregular automorphisms in vertex-transitive graphs admitting a transitive solvable group.

Even for small valency graphs this is not easy. For example, valency 5 is still open.

TO THE COMMON BORDER BETWEEN SLOVAKIA AND SLOVENIA!

NA SPOLOČNÚ HRANICU MEDZI SLOVENSKOM A SLOVINSKOM!

NA SKUPNO MEJO MED SLOVAŠKO IN SLOVENIJO!