Dessins associated with Finite Projective Spaces and Frobenius Compatibility

Cristina Sarti

University of Frankfurt

June 30, 2009

Outline

Dessins d'enfants

Dessins d'enfants (=children drawings) are hypermaps on Riemann surfaces.

We say that a dessin on a surface X has signature $< p, q, r >$, if

$$\begin{aligned} p &:= \text{lcm of all valencies of the white vertices,} \\ q &:= \text{lcm of all valencies of the black vertices,} \\ 2r &:= \text{lcm of all valencies of the faces.} \end{aligned}$$

If all white vertices have the same valency p, if all black vertices have the same valency q, and if all faces have the same valency $2r$, we say that the dessin is uniform.

Finite Projective Spaces

$$\mathbb{P}^m(\mathbb{F}_n) = (\mathbb{F}_n^{m+1}\backslash\{0\})/\mathbb{F}_n^*, \qquad n = p^e,\ p \text{ prime},\ e \in \mathbb{N}\backslash\{0\}$$
$$\cong \mathbb{F}^*_{n^{m+1}}/\mathbb{F}^*_n.$$

$$\ell = \frac{n^{m+1}-1}{n-1}, \qquad q = \frac{n^m-1}{n-1}.$$

Each element of $\mathbb{F}^*_{n^{m+1}}/\mathbb{F}^*_n$ can be written as a power g^i of a generator $< g >$ of the multiplicative group.

Notation:

$$g^b = P_b \text{ (points)} \qquad g^w = h_w \text{ (hyperplanes)} .$$

Constructing the Dessins I

Incidence pattern of points P_b and hyperplanes h_w using bipartite graphs.

Table: Conventions

point	black vertex $\bullet$
hyperplane	white vertex $\circ$
incidence	joining edge —

Constructing the Dessins II

How do we know which point P_b is incident with which hyperplane h_w and vice versa? $\longrightarrow$ difference sets (SINGER 1938).

Definition

A (v, k, λ)-difference set $D = \{d_1, \cdots, d_k\}$ is a collection of k residues modulo v, such that for any residue $\alpha \not\equiv 0 \mod v$ the congruence

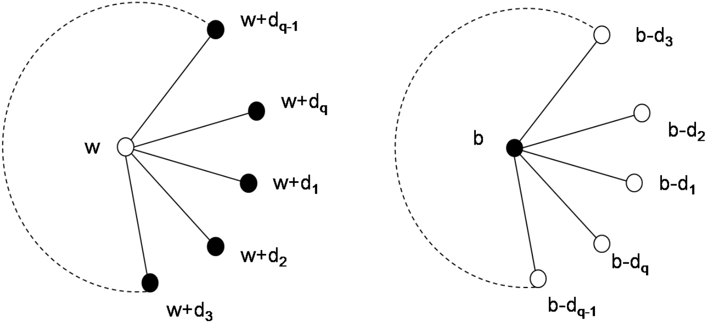
$$d_i - d_j \equiv \alpha \mod v$$

has exactly λ solution pairs (d_i, d_j) with d_i and d_j in D.

1. For projective spaces: $v = \ell$, $k = q$.
2. P_b and h_w are incident $\Leftrightarrow$ $b - w \equiv d_i \mod \ell$.
3. Any pair of points occur in λ different hyperplanes.

Constructing the Dessins III

Figure: Local incidence pattern



Incidence relation: $b - w \equiv d_i \mod \ell.$

Wada Dessins I

Construction of a uniform $< q, q, \ell >$ Wada dessin (STREIT - WOLFART 2001) if the Wada condition is satisfied, i.e. if

$$(d_{i+1} - d_i, \ell) = 1 \; \forall i \in \mathbb{Z}/q\mathbb{Z}$$

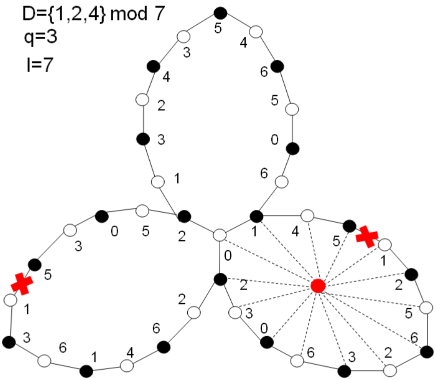
Wada property: each vertex of each color lies on the boundary of each cell.

$\longrightarrow$ Wada dessins are always uniform!

The difference set we choose for the construction is not unique. Multiplications $t \cdot D$ with integers $t \in \mathbb{Z}/\ell\mathbb{Z}$, $(t, \ell) = 1$ and shifts $D + s$ with $s \in \mathbb{Z}/\ell\mathbb{Z}$ are also permitted.

Wada Dessins II

Figure: The bipartite graph of $\mathbb{P}^2(\mathbb{F}_2)$ (Fano plane)



Embedding the bipartite graph in a Riemann surface (Klein's quartic in this case) $\Longrightarrow$ Wada dessin with signature $< 3, 3, 7 >$.

The Frobenius Automorphism

The Frobenius automorphism σ acts on the elements of $\mathbb{P}^m(\mathbb{F}_n) \cong \mathbb{F}^*_{n^{m+1}}/\mathbb{F}^*_n$, $n = p^e$:

$$\sigma : g^i \longmapsto g^{p \cdot i}\,, \quad < g > \text{ generator of } \mathbb{F}^*_{n^{m+1}}/\mathbb{F}^*_n\,.$$

Action on points and hyperplanes:

$$\sigma : \quad \begin{aligned} P_b &\longmapsto P_{b \cdot p} \\ h_w &\longmapsto h_{w \cdot p} \end{aligned}$$

(Recall: $g^b = P_b$, $g^w = h_w$)

σ is **not** always an automorphism of Wada dessins $\longrightarrow$ it depends on the elements of the difference set!

Frobenius Difference Sets I

Example(1a):
We construct Wada dessins $\mathcal{D}_1$ and $\mathcal{D}_2$ associated with $\mathbb{P}^4(\mathbb{F}_2)$, $\ell = 31$, $q = 15$ using the following difference sets:

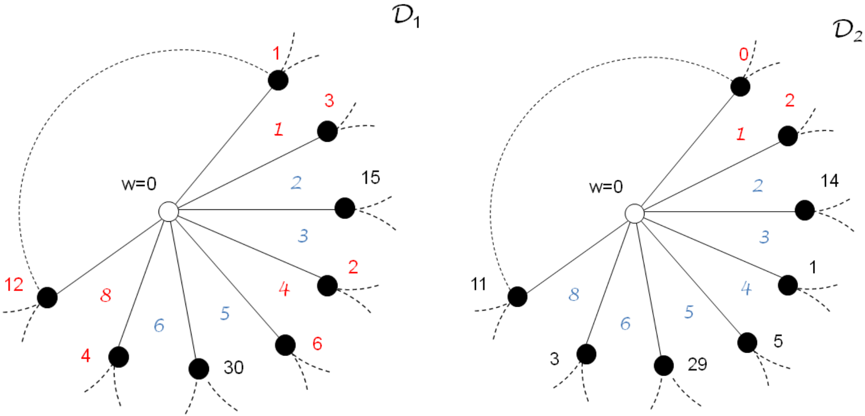
$$D_1 = \{1, 3, 15, 2, 6, 30, 4, 12, 29, 8, 24, 27, 16, 17, 23\} \mod 31\,,$$

$$\begin{aligned} D_2 &\equiv D_1 - 1 \mod 31 \\ &= \{0, 2, 14, 1, 5, 29, 3, 11, 28, 7, 23, 26, 15, 16, 23\} \mod 31 \end{aligned}$$

σ acts on the vertices (=indices of points and hyperplanes) by multiplication:

$$b \longmapsto b \cdot 2\,, \quad w \longmapsto w \cdot 2$$

Frobenius Difference Sets II



σ is an automorphism of $\mathcal{D}_1$ (rotation of the cells around $w = 0$ by an angle $\varphi = \frac{2\pi}{5}$) but not of $\mathcal{D}_2$.

Frobenius Difference Sets III

The Frobenius automorphism σ acts as an automorphism of $\mathcal{D}_1$ but not of $\mathcal{D}_2$.

Reason: D_1 is fixed under the action of σ, D_2 is not.

$$\sigma(D_1) = 2D_1 \equiv D_1 \mod 31\,,$$
$$D_1 = \{1, 3, 15, 2, 6, 30, 4, 12, 29, 8, 24, 27, 16, 17, 23\} \mod 31$$

$$\sigma(D_2) = 2D_2 \equiv D_2' \mod 31\,,$$
$$D_2 = \{0, 2, 14, 1, 5, 29, 3, 11, 28, 7, 23, 26, 15, 16, 22\} \mod 31$$
$$D_2' = \{0, 4, 28, 2, 10, 27, 6, 22, 25, 14, 15, 21, 30, 1, 13\} \mod 31$$

$$\text{e.g. } \sigma : 2 \mapsto 4 \text{ and } 4 \in D_2' \text{ but } 4 \notin D_2\,.$$

Frobenius Difference Sets IV

Necessary condition for the construction of Wada dessins for which σ is an automorphism:

The corresponding difference set D (Frobenius difference set) is fixed under the action of σ.

$\longrightarrow$ the condition is not sufficient!

Recall: Wada condition for elements d_{i+1}, d_i of the difference set also necessary

$$(d_{i+1} - d_i, \ell) = 1 \; \forall i \in \mathbb{Z}/q\mathbb{Z}\,.$$

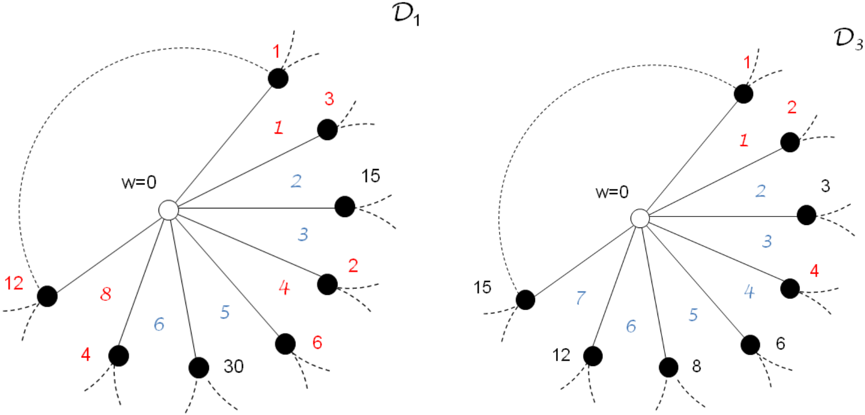
Frobenius Compatibility I

Example(1b):
We construct another dessin $\mathcal{D}_3$ associated with the projective space $\mathbb{P}^4(\mathbb{F}_2)$ using the difference set D_1 but with its elements ordered in a different way:

$$D_1 = \{1, 3, 15, 2, 6, 30, 4, 12, 29, 8, 24, 27, 16, 17, 23\} \mod 31\,,$$

$$D_3 = \{1, 2, 3, 4, 6, 8, 12, 15, 16, 17, 23, 24, 27, 29, 30\} \mod 31$$

Frobenius Compatibility II



σ is an automorphism of $\mathcal{D}_1$ but not of $\mathcal{D}_3$.

Frobenius Compatibility III

The Frobenius automorphism σ acts as an automorphism of $\mathcal{D}_1$ but not of $\mathcal{D}_3$.

$\longrightarrow$ it depends on the ordering of the elements of the difference set.

$$D_1 = \{1, 3, 15, 2, 6, 30, 4, 12, 29, 8, 24, 27, 16, 17, 23\} \mod 31\,,$$

$$D_3 = \{1, 2, 3, 4, 6, 8, 12, 15, 16, 17, 23, 24, 27, 29, 30\} \mod 31\,.$$

Frobenius Compatibility IV

Necessary and sufficient condition for the construction of Wada dessins for which σ is an automorphism:

D is fixed under the action of σ (Frobenius difference set) and its elements are ordered in a way 'compatible' with the action of σ (Frobenius compatibility).

Recall: Wada condition should also be satisfied.

Existence Conditions I

1. We always find, at least, one Frobenius difference set D_f.
   Geometry: the hyperplane h_0 is always fixed under the action of σ.

   Elements of $D_f \longrightarrow$ indices of the points on h_0 (Singer construction).
2. We cannot always order the elements of a Frobenius difference set in a Frobenius compatible way.
   $\longrightarrow D_f$ fixed, but σ divides its elements in orbits of different lengths.

Example(2): Wada dessin for $\mathbb{P}^3(\mathbb{F}_3)$, $\ell = 40$, $q = 13$.

Orbits of elements of the difference set D_f under the action of σ(=multiplication by the integer 3):

$$D_f : \quad \{21, 23, 29, 7\}, \{22, 26, 34, 38\}, \{25, 35\}, \{5, 15\}, \{0\}$$

'Nice' Consequences I

Projective spaces $\mathbb{P}^m(\mathbb{F}_n)$, $n = p^e$, ℓ points (hyperplanes), q hyperplanes through a point (q points through a hyperplane).

Let $C_f \cong Gal(\mathbb{F}_{n^{m+1}}/\mathbb{F}_p)$ be the cyclic group generated by σ. If C_f is a group of automorphism of a Wada dessin $\mathcal{D}$ associated with $\mathbb{P}^m(\mathbb{F}_n)$ then:

1. The full automorphism group can be explicitely determined as a group generated by C_f and C_ℓ.
   (C_ℓ= Singer group. C_ℓ acts permuting cyclically the vertices of each color on the boundary of the cells of $\mathcal{D}$.)

2. Applying suitable algebraic operations H ('mock' Wilson operations), we may always 'transform' Wada dessins $< q, q, \ell >$ in regular Wada dessins $< f, f, \ell >$.
   (regular dessin = transitive operation of the automorphism group on the edges.)

'Nice' Consequences II

3. The Riemann surface X defined by the dessin with signature $< q, q, \ell >$ is a ramified covering of the surface Y defined by the dessin with signature $< f, f, \ell >$. Ramification points are the black and the white vertices.

Example(3):
$\mathbb{P}^4(\mathbb{F}_2)$, $\ell = 31$, $q = 15$, $C_5 = Gal(\mathbb{F}_{2^5}/\mathbb{F}_2)$

$$\mathcal{D} :< 15, 15, 31 > \quad X \quad \longrightarrow \quad Y \quad H\mathcal{D} :< 5, 5, 31 >$$

Full automorphism group: $C_5 \ltimes C_{31}$.


Thank You!