Polarity and Rigidity

Brigitte and Herman Servatius

Worcester Polytechnic Institute

1. Edmond's Theorem

Given a hypergraph $H = (V, E)$.
$|E| \times |V|$ Matrix: $T(H, X)$

$$t_{ij} = \begin{cases} x_{ij} & v_j \in e_i \\ 0 & \text{otherwise} \end{cases}$$

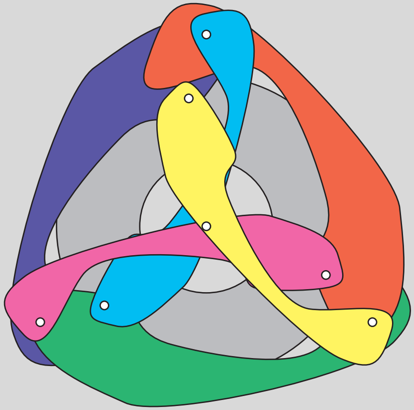
Theorem

The rows of $T(H, X)$ are independent if and only if $|E| \leq |V|$ and for each subset $E' \subseteq E$, $|E'| \leq |V'|$ where V' is the set of vertices supporting E'.

Theorem

The kernel of $T(H, X)$ is of dimension k if and only if $|E| \leq |V| - k$ and for each subset $E' \subseteq E$, $|E'| \leq |V'| - k$ where V' is the set of vertices supporting E'.

Example: The Fano Plane



$$\begin{bmatrix} 0 & x_{12} & x_{13} & 0 & 0 & x_{16} & 0 \\ x_{21} & 0 & x_{23} & 0 & x_{25} & 0 & 0 \\ x_{31} & x_{32} & 0 & x_{34} & 0 & 0 & 0 \\ 0 & 0 & x_{43} & x_{44} & 0 & 0 & x_{47} \\ 0 & x_{52} & 0 & 0 & x_{55} & 0 & x_{57} \\ x_{61} & 0 & 0 & 0 & 0 & x_{66} & x_{67} \\ 0 & 0 & 0 & x_{74} & x_{75} & x_{76} & 0 \end{bmatrix}$$

Whiteley's Idea

Start with the matrix of a regular hypergraph.

$$\begin{bmatrix} 0 & x_{12} & x_{13} & 0 & 0 & x_{16} & 0 \\ x_{21} & 0 & x_{23} & 0 & x_{25} & 0 & 0 \\ x_{31} & x_{32} & 0 & x_{34} & 0 & 0 & 0 \\ 0 & 0 & x_{43} & x_{44} & 0 & 0 & x_{47} \\ 0 & x_{52} & 0 & 0 & x_{55} & 0 & x_{57} \\ x_{61} & 0 & 0 & 0 & 0 & x_{66} & x_{67} \\ 0 & 0 & 0 & x_{74} & x_{75} & x_{76} & 0 \end{bmatrix}$$

Whiteley's Idea

Start with the matrix of a regular hypergraph where

$$|E| \leq |V| - 2.$$

The matrix with generic entries has a two dimensional kernel.

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & x_{12} & x_{13} & 0 & 0 & x_{16} & 0 \\ x_{21} & 0 & x_{23} & 0 & x_{25} & 0 & 0 \\ x_{31} & x_{32} & 0 & x_{34} & 0 & 0 & 0 \\ 0 & 0 & x_{43} & x_{44} & 0 & 0 & x_{47} \\ 0 & x_{52} & 0 & 0 & x_{55} & 0 & x_{57} \end{bmatrix}$$

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$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Scale the columns so there is a row of 1's (alters the matrix, but not its rank properties)

$$\begin{bmatrix} 0 & x_{12} & x_{13} & 0 & 0 & x_{16} & 0 \\ x_{21} & 0 & x_{23} & 0 & x_{25} & 0 & 0 \\ x_{31} & x_{32} & 0 & x_{34} & 0 & 0 & 0 \\ 0 & 0 & x_{43} & x_{44} & 0 & 0 & x_{47} \\ 0 & x_{52} & 0 & 0 & x_{55} & 0 & x_{57} \end{bmatrix}$$

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Since each row is supported by 3 columns, the original (generic) entries can be expressed in terms of the kernel.

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$$\begin{bmatrix} 0 & x_{12} & x_{13} & 0 & 0 & x_{16} & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \begin{vmatrix} x_3 & x_6 \\ 1 & 1 \end{vmatrix} & - \begin{vmatrix} x_2 & x_6 \\ 1 & 1 \end{vmatrix} & 0 & 0 & \begin{vmatrix} x_2 & x_3 \\ 1 & 1 \end{vmatrix} & 0 \end{bmatrix}$$

Edmond's Theorem
The k-plane ...
Whiteley's Theorem
Jackson Jordán ...
$\mathfrak{M}_2(K_{n,m})$
Rigid Line Structures

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\begin{bmatrix}
0 & \begin{vmatrix} x_3 & x_6 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} x_2 & x_6 \\ 1 & 1 \end{vmatrix} & 0 & 0 & \begin{vmatrix} x_2 & x_3 \\ 1 & 1 \end{vmatrix} & 0 \\
\begin{vmatrix} x_3 & x_5 \\ 1 & 1 \end{vmatrix} & 0 & -\begin{vmatrix} x_1 & x_5 \\ 1 & 1 \end{vmatrix} & 0 & \begin{vmatrix} x_1 & x_3 \\ 1 & 1 \end{vmatrix} & 0 & 0 \\
\begin{vmatrix} x_2 & x_4 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} x_1 & x_4 \\ 1 & 1 \end{vmatrix} & 0 & \begin{vmatrix} x_1 & x_2 \\ 1 & 1 \end{vmatrix} & 0 & 0 & 0 \\
0 & 0 & \begin{vmatrix} x_4 & x_7 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} x_3 & x_7 \\ 1 & 1 \end{vmatrix} & 0 & 0 & \begin{vmatrix} x_3 & x_4 \\ 1 & 1 \end{vmatrix} \\
0 & 0 & \begin{vmatrix} x_5 & x_7 \\ 1 & 1 \end{vmatrix} & 0 & -\begin{vmatrix} x_3 & x_7 \\ 1 & 1 \end{vmatrix} & 0 & \begin{vmatrix} x_3 & x_5 \\ 1 & 1 \end{vmatrix}
\end{bmatrix}
$$

If x_i's are given, an element of the kernel gives y_i's such that for every row of the matrix supported by $\{i, j, k\}$, the points (x_i, y_i), (x_j, y_j), and (x_k, y_k) are colinear.

There is a choice of x_i's so that the matrix entries are generic.

$\Longrightarrow$ there is a choice of x_i's so that Edmond's Theorem characterizes the independence.

$\Longrightarrow$ For any generic x_i's, Edmond's Theorem characterizes the independence.

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$$\begin{bmatrix}
0 & \begin{vmatrix} x_3 & x_6 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} x_2 & x_6 \\ 1 & 1 \end{vmatrix} & 0 & 0 & \begin{vmatrix} x_2 & x_3 \\ 1 & 1 \end{vmatrix} & 0 \\
\begin{vmatrix} x_3 & x_5 \\ 1 & 1 \end{vmatrix} & 0 & -\begin{vmatrix} x_1 & x_5 \\ 1 & 1 \end{vmatrix} & 0 & \begin{vmatrix} x_1 & x_3 \\ 1 & 1 \end{vmatrix} & 0 & 0 \\
\begin{vmatrix} x_2 & x_4 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} x_1 & x_4 \\ 1 & 1 \end{vmatrix} & 0 & \begin{vmatrix} x_1 & x_2 \\ 1 & 1 \end{vmatrix} & 0 & 0 & 0 \\
0 & 0 & \begin{vmatrix} x_4 & x_7 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} x_3 & x_7 \\ 1 & 1 \end{vmatrix} & 0 & 0 & \begin{vmatrix} x_3 & x_4 \\ 1 & 1 \end{vmatrix} \\
0 & 0 & \begin{vmatrix} x_5 & x_7 \\ 1 & 1 \end{vmatrix} & 0 & -\begin{vmatrix} x_3 & x_7 \\ 1 & 1 \end{vmatrix} & 0 & \begin{vmatrix} x_3 & x_5 \\ 1 & 1 \end{vmatrix}
\end{bmatrix}$$

There is a 2 dimensional space of trivial lifts.
For the Fano Plane we conclude that generically choosing the x-coordinates of a drawing, there is no non-trivial way to chose the y coordinates to represent 5 of the seven lines.
(We know a fifth and sixth line can be represented, but not generically.)

2. The k-plane matroids

Given: A Hypergraph: $(A, B; I)$
The *k-plane matroid* on I has has independent sets $I' \subseteq I$ defined via:

For all $I'' \subseteq I'$, we have

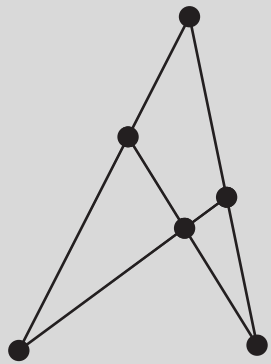
$$|I''| \leq |A(I'')| + k|B(I'')| - k$$

The 2-plane matroids

Given: A Hypergraph: $(A, B; I)$
The 2-*plane matroid* on I has has independent sets $I' \subseteq I$ defined via:

For all $I'' \subseteq I'$, we have

$$|I''| \leq |A(I'')| + 2|B(I'')| - 2$$



A: The lines
B: the points
I the incidence relation

The 2-plane matroids

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For all $I'' \subseteq I'$, we have

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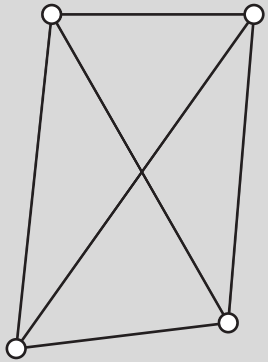


A: The points
B: the lines
I the incidence relation

3. Whiteley's Theorem

Given an incidence graph $G = (B, J; I)$ the following are equivalent:

(i) G has a realization as an independent (isostatic) identified body and joint framework in the plane.

(ii) G satisfies

$$2i \leq 3b + 2j - 3(=)$$

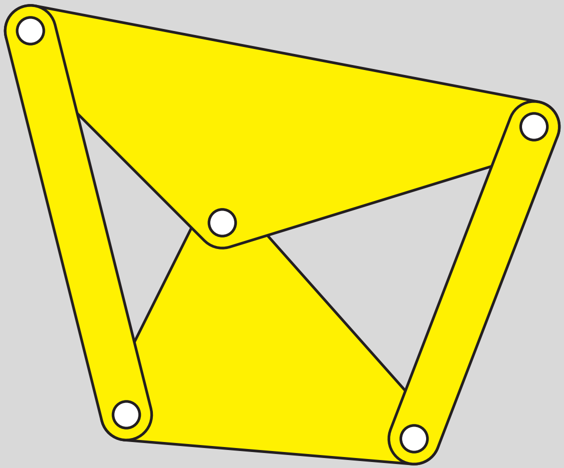
and, for every subset of bodies and induced subgraph of attached joints,

$$2i' \leq 3b' + 2j' - 3.$$

(iii) G has an independent (isostatic) realization as an identified body and joint framework in the plane such that each body has all its joints collinear.

The Problems

1. Characterizing Rigidity of Body and Pin Frameworks cannot be done by searching for isostatic sub-frameworks.



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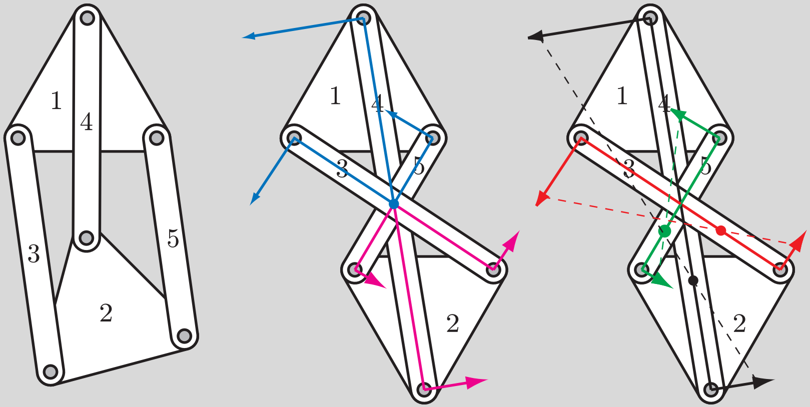
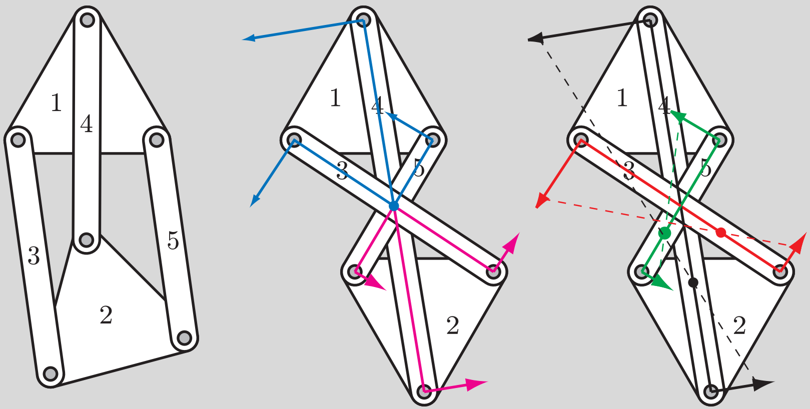
1. Characterizing Rigidity of Body and Pin Frameworks cannot be done by searching for isostatic sub-frameworks.
2. Adding pins generically may decrease the degree of freedom by 2, 1 or 0.



The Problems

1. Characterizing Rigidity of Body and Pin Frameworks cannot be done by searching for isostatic sub-frameworks.
2. Adding pins genericly my increase the degree of freedom by 2 or 0.
3. None of this handles the case of pining multiple bodies with one pin.



4. Jackson Jordán Theorem

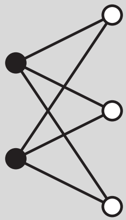
Let G be a multigraph. Then G has an infinitesimally rigid pincollinear body-and-pin realization if and only if $2G$ contains three edge-disjoint spanning trees.
(In other words the incidence count predicted by Whiteley via Edmonds)

5. $\mathfrak{M}_2(K_{n,m})$

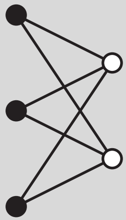
Question: What are the bases

Examples:

$\mathfrak{M}_2(K_{2,3})$
rank 6



$\mathfrak{M}_2(K_{3,2})$
rank 5



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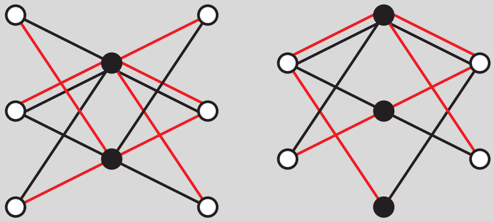
$\mathfrak{M}_2(K_{2,3})$ rank 6 $\quad$ $\mathfrak{M}_2(K_{3,2})$ rank 5



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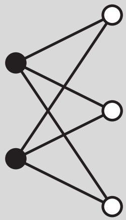
rank 6



$\mathfrak{M}_2(K_{3,2})$
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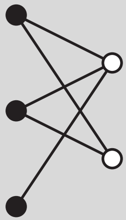

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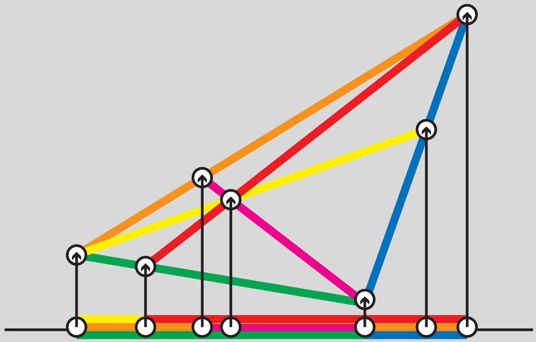
6. Rigid Line Structures



Theorem

If the 2-plane matroid of an incidence structure has rank $a + 2b-2$, then placing a points on any line in the plane with generic x-coordinates and joining them appropriately with b rigid bars gives a structure which is infinitesimally rigid in the plane.

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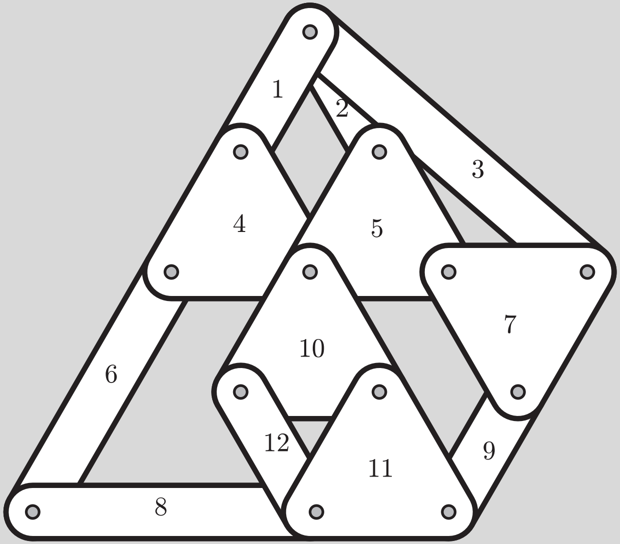

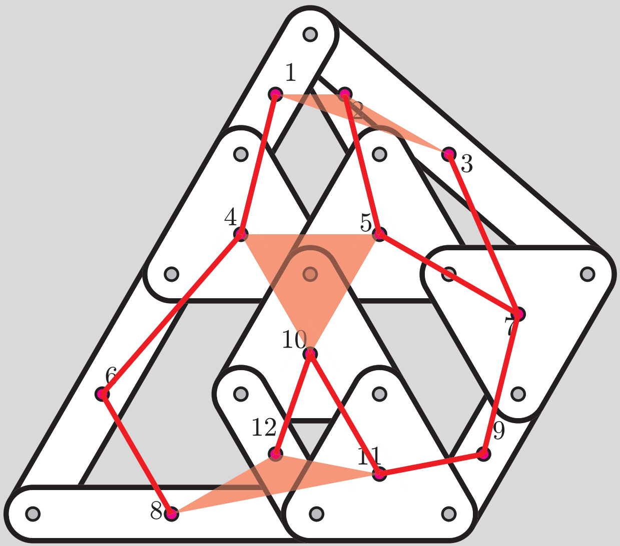
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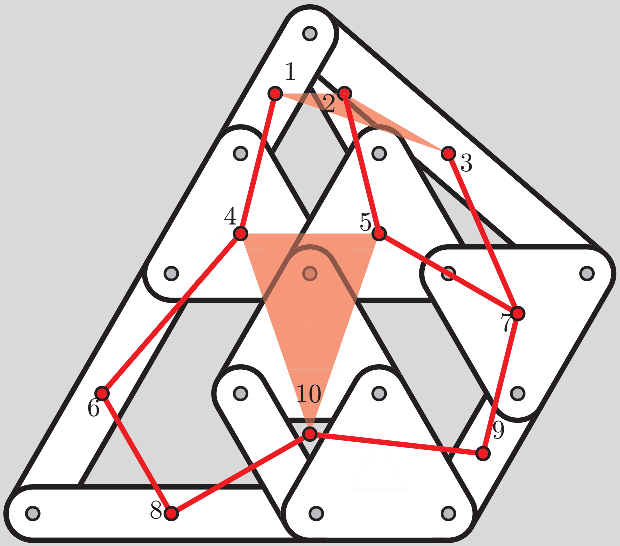
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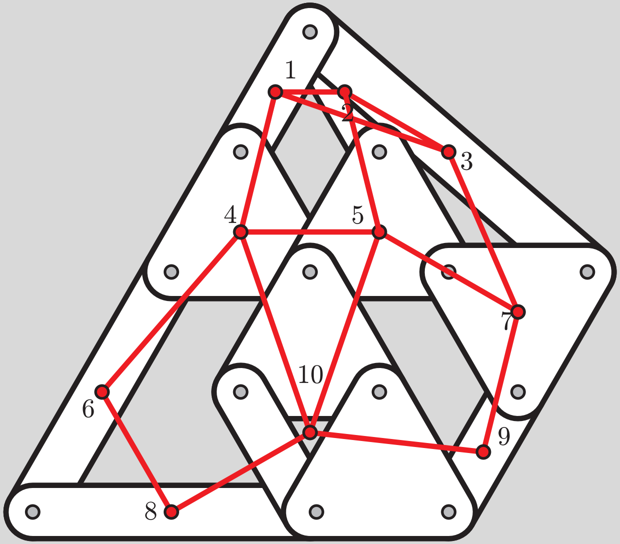
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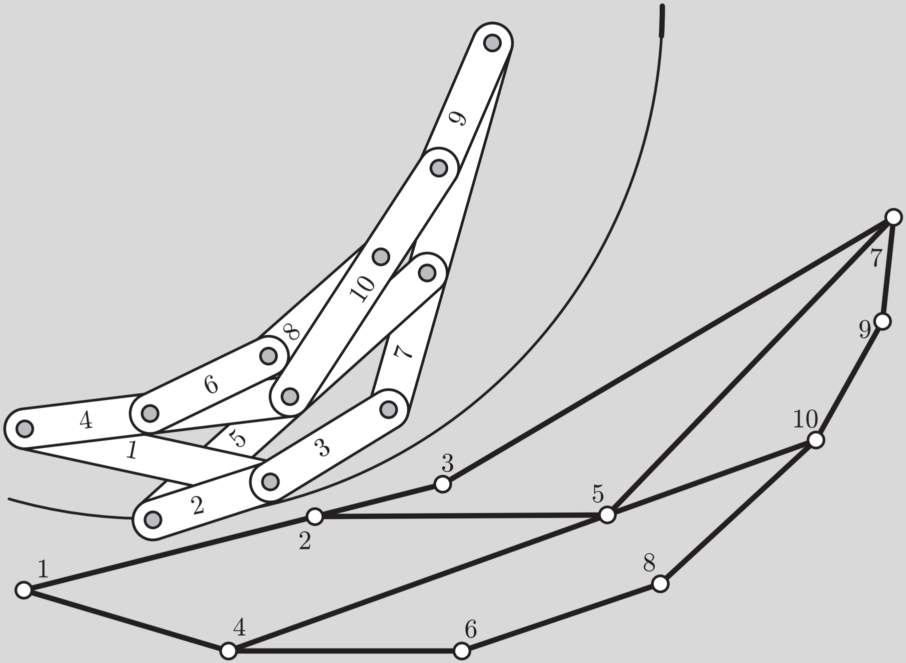
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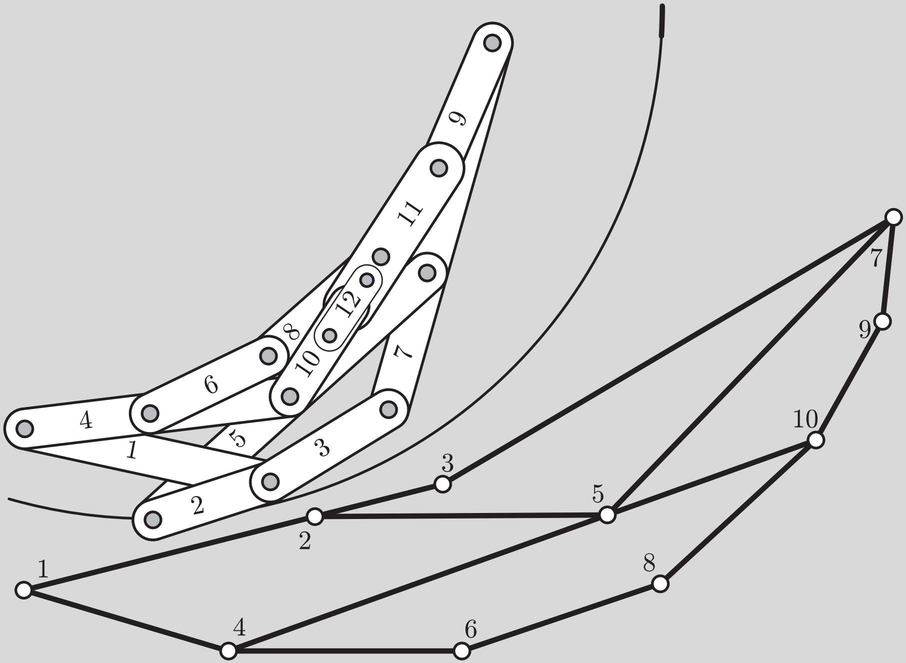
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