Gapfilling for the Symmetric Genus

Thomas W. Tucker (joint with Marston Conder)

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- (Burnside et al, 1895) acts preserving orientation with quotient the sphere

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Question

Are there gaps for γ, σ, σ^o?

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But all these groups have $\sigma = 1$, so not useful for symmetric genus.

Riemann-Hurwitz

If G acts on suface S preserving orientation with quotient surface S/G of genus g', then

$$2g - 2 = |G|((2g' - 2 + \Sigma(1/r_i - 1))$$

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Special case with three branch points and $g' = 0$ occurs all the time with factor $(1 - 1/p - 1/q - 1/r)$.
If acts not preserving orientation, it is basically the same but with $|G|/2$ instead of $|G|$ and branch points are of two forms (depending on whether locally D_r or Z_r). Most common case is again $g' = 0$ and three branch points corresponding to generating set u, v where $u^2 = 1, v^p = 1, [u, v]^r = 1$.

Symmetric genus: gapfilling for odd genus

Let
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Idea of proofs:We have $G/\langle x^2, y^2 \rangle \cong D_{2k}$. The only involutions are x^2, y^2, x^2y^2 for the first group and in addition $(xy)^k$ for the second group. In the first case, this means there are no reflections, since all involutions are squares and cannot reverse orientation, so only need to compare $\{r, s, t\}$ generating sets with $\{4, 4, 4k\}$. Second case, also must consider a reflective action with reflection $(xy)^k$, so a few more details.

Genus $g \equiv 0, 6, 12 \mod 18$

Let $G_{2,3k,3k} = \langle x, y : x^2 = y^{3k} = (xy)^{3k} = 1, [x, y^3] = 1\rangle$.

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Idea of Proof: We have $G/\langle y^3 \rangle \cong A_4$, so $|G| = 12k$. The abelianization is Z_{3k} so no subgroup of index two, so no orientation-reversing actions. Also, any $\{2, s, t\}$ generating set must have $s = t = 3k$.

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Theorem

For odd $k \neq 3, 9$, $\sigma(Z_k \times S_4) = 1 + 6k(0 - 1/2) = 3k + 1$.

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Idea of proof: The group has order $24k$. There is a reflective action with quotient S/G_0 the torus using the generating set $x = (1, (12)(34)), y = (0, (12)), z = (0, (13))$ with y, z the reflections. Notice the S_4 coordinates of x, y, z generate S_4 and since k is odd $\langle x \rangle$ contains both $(0, (12)(34))$ and $(1,)$. Note also that x and yz are in the index two subgroup $Z_k \times A_4$. Also $[x, y] = 1$ and $[x, z] = (0, (24)(13))$. The quotient graph in the torus has two vertices and two faces: $[x, y]$ and $[x, z]$ so one branch point of order $1/2$.

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Best orientation-preserving action in sphere is $2, k, 2k$ giving

$$g = 1 + 12k(1 - 1/2 - 1/k - 1/(2k)) = 6k - 17 \leq 3k + 1 \text{ for } k > 5.$$

For $k = 5$, the orders are at least $2k, 2k$ and it works. Reflective actions are more trouble and also allow $k = 9$.

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Idea of proof: $|G| = 27k$ and the obvious $3, 3, 3k, 3k$ generating set gives

$$2g - 2 = 27k(2 - 2/3 - 2/(3k)) \text{ so } g = 1 + 18k - 9.$$

When k is odd, there are no index two subgroups and we're done. For k even, since only involution is $(0, 0, 3k/2)$, we will need at least 6 branch points off the reflection circle, at least two of order $3k/2$, so

$$2g - 2 \geq (27k/2)(4 - 4/3 - 4/(3k)) \text{ so again } g \geq 1 + 18k - 9.$$

Genus $g \equiv 2 \bmod 18$

Let $G_{2,3k,[4]}$ be defined by

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Dividing about by the central $\langle y^3, [x,y]^2\rangle$, which has order $2k$ we get a standard presentation for S_4, so $|G| = 48k$. The given presentation gives a reflective action with $g = 9k - 7$. The abelianization is Z_{6k} so $|G'| = 8$. Since G' is generated by conjugates of $[x,y]$, we must have G' is the quaternions. Thus $[u,v]$ has order 4 for any two-element generating set u, v, so there is no better reflective action. Orientation-preserving actions eliminated the usual way using $G/G' \cong Z_{6k}$.

The Belolipetsky-Jones maps

The BJ maps are balanced, regular $\{6,6\}$ Cayley maps for $Z_2 \times Z_p$ based on a nontrivial root of $r^3 \equiv -1 \bmod p$ when $p \equiv 1 \bmod 6$. They can be generalized to $Z_2 \times Z_n$ where n is odd and all primes p dividing n satisfy $p \equiv 1 \bmod 6$ (use Chinese Remainder and standard elementary number theory mod p^e). It will not work otherwise, since what you really want is $r^2 - r + 1 \equiv 0 \bmod n$ and that is impossible mod p when $p = 5 \bmod 6$ (need -3 a square mod p). Note, the Euler charactestic is $(2n - 6n + 2n) = -2n$ so genus is $n + 1$.

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But the matrix $A = (0 - 1|11)$ does satisfy $A^2 - A + I = 0$ so you can do a BJ regular Cayley map over $Z_m \times Z_m$. Thus suppose that $k = nm^2$, where m, n are odd and all primes dividing n are 1 mod 6. Then

Theorem

There is a Z_6 split extension G of $Z_2 \times Z_n \times Z_m \times Z_m$, coming from a BJ-type regular map such that $\sigma(G) = k + 1$.

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Example: take any two primes p, q with $p \equiv q \equiv 5 \bmod 6$ and $pq \not\equiv 1 \bmod 18$. For example, $5.17 + 1, 5.23 + 1, 11.17 + 1, \cdots$.

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With $m = 2$, this gives all $g \equiv 1 \mod 4$, but also gives all g such that $g - 1$ is a multiple of a square but also does $g \equiv 1 \mod 9$ etc.